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Zoology. — *Three successive layers of external cuticle in Sacculina leptodiae.* By H. BOSCHMA.

(Communicated at the meeting of December 21, 1946.)

KOSSMANN (1872), who found that the excrescenses of the external cuticle of the mantle may be characteristic for the species of Sacculinidae, also reported upon a case of moulting of this cuticle: in the species described by him as *Sacculina crucifera* two layers of external cuticle were present, an inner layer with well developed spinous excescences and a much thinner outer layer without excrescences. A quite similar double external cuticle was found in a specimen of *Sacculina rotundata* Miers (VAN KAMPEN and BOSCHMA, 1925; in this paper the specimen, a parasite of *Eriphia laevimana* Latr., erroneously was identified as *Sacculina pilosa* Kossm.). A third case of a cuticle with spinous excrescences covered by a much thinner cuticle without excrescences occurs in a parasite of *Carupa laeviuscula* Heller (BOSCHMA, 1928), the latter parasite was described as *Sacculina aculeata*, but as the excrescences of the definite external cuticle have not yet fully developed it is as yet uncertain whether this form really is to be regarded as a distinct species. Similar phenomena are known in a number of other species of the family. Two layers of external cuticle have been found in specimens of *Sacculina carcini* Thomps., *S. inflata* Leuck., and *Drepanorchis neglecta* (Fraisse); here the two layers possess excrescences consisting of papillae of corresponding shape (BOSCHMA, 1927). Of *Sacculina angulata*, *S. teretiuscula*, *S. anceps* and *Heterosaccus ruginosus* specimens were described with two layers of external cuticle; here the outer layer does not show distinct excrescences whilst the inner layer in some cases has distinct papillae, in other cases a little pronounced indication of a certain differentiation of its surface (BOSCHMA, 1931). Moreover in *Sacculina setosa* two layers of external cuticle were found, both of which bear small spinous excrescences of corresponding shape (BOSCHMA, 1933).

Among specimens of a parasite on the crab *Thalamita stimpsoni* A. M. E. there are two young stages which show that in this parasite there occur three successive layers of external cuticle, each with excrescences which in their shape and size are entirely different from the other. Provisionally this parasite may be named *Sacculina leptodiae*, as its characters closely correspond with those of this species.

Sacculina leptodiae was described as a parasite of *Xantho exaratus* (H. M. E.) by GUÉRIN–GANIVET (1911) on material from the Gulf of Aden and from the Comoro Islands; other specimens of this parasite were collected in the Red Sea, Zanzibar and the Andaman Islands. The parasite of *Thalamita stimpsoni*, which closely corresponds with *Sacculina leptodiae*

in the shape of the male organs, that of the colleteric glands, and that of the shape and size of the excrescences of the external cuticle, was found at Singapore and at the Aru Islands. In both forms the excrescences of the external cuticle consist of small groups of hyaline spines as a rule united on a common basal part. In the type specimen of *S. leptodiae* the excrescences vary in length from 50 to 80 μ, in the two other specimens reported upon by GUÉRIN–GANIVET the excrescences have a length of 25 to 45 μ (cf. BOSCHMA, 1936).

In full grown specimens on *Thalamita stimpsoni* the excrescences of the external cuticle correspond in every detail with those of the parasite of *Xantho exaratus*. In the material from the Aru Islands, collected during the Snellius expedition in 1929, these excrescences have a length of 30 to 70 μ. In the various specimens they have slighly different shapes and sizes, the excrescences of three specimens are represented in fig. 1. In one specimen the spines are rather short and thick (fig. *1a*), here the excrescences as a whole have a length of 35 to 50 μ. In a second specimen the spines are slightly longer, so that the excrescences as a whole have a length of 50 to 70 μ. In a third specimen the spines as a rule remain isolated, though they are arranged in groups on the surface of the external cuticle (fig. 1*c*), here they vary in length from 30 to 50 μ. In some parts of the cuticle of this specimen, however, the spines are united into distinct groups with common basal parts (fig. 1*d*, e), just as in the other specimens, generally the excrescences then are somewhat longer (50—60 μ). The measurements given above hold for the excrescences of the greater part of the mantle; in some regions, e. g., near the stalk and in the vicinity of the mantle opening, there may occur excrescences of smaller size.

Among the numerous parasites of *Thalamita stimpsoni* from the Aru Islands there are two young stages which are of special interest as their external cuticle is entirely different from that in the adult stages of the same parasite.

In the first specimen the external cuticle of the mantle possesses distinct excrescences which consist of small papillae which at their top bear a number of minute marginal spines (fig. 2). These excrescences have a height of $4\frac{1}{2}$ to $7\frac{1}{2}$ μ, their thickness amounts to 4 to 9 μ. The papillae may occur rather crowdedly or more sparsely distributed. The excrescences consist of the same kind of chitin as that of the main layers of the external cuticle.

Sections of the mantle show that under this cuticle, which has a thickness of about 4 to 8 μ, the excrescences of a new cuticle are developing. It is interesting that in some parts of the mantle this process has not yet started (fig. 3*a*), in other parts of the mantle the first traces of these new excrescences have just appeared (fig. 3*b*), and in still other parts the excrescences already have a fairly large size (fig. 3*c*). Consequently in this specimen we can study the gradual development of the excrescences up to a certain stage.

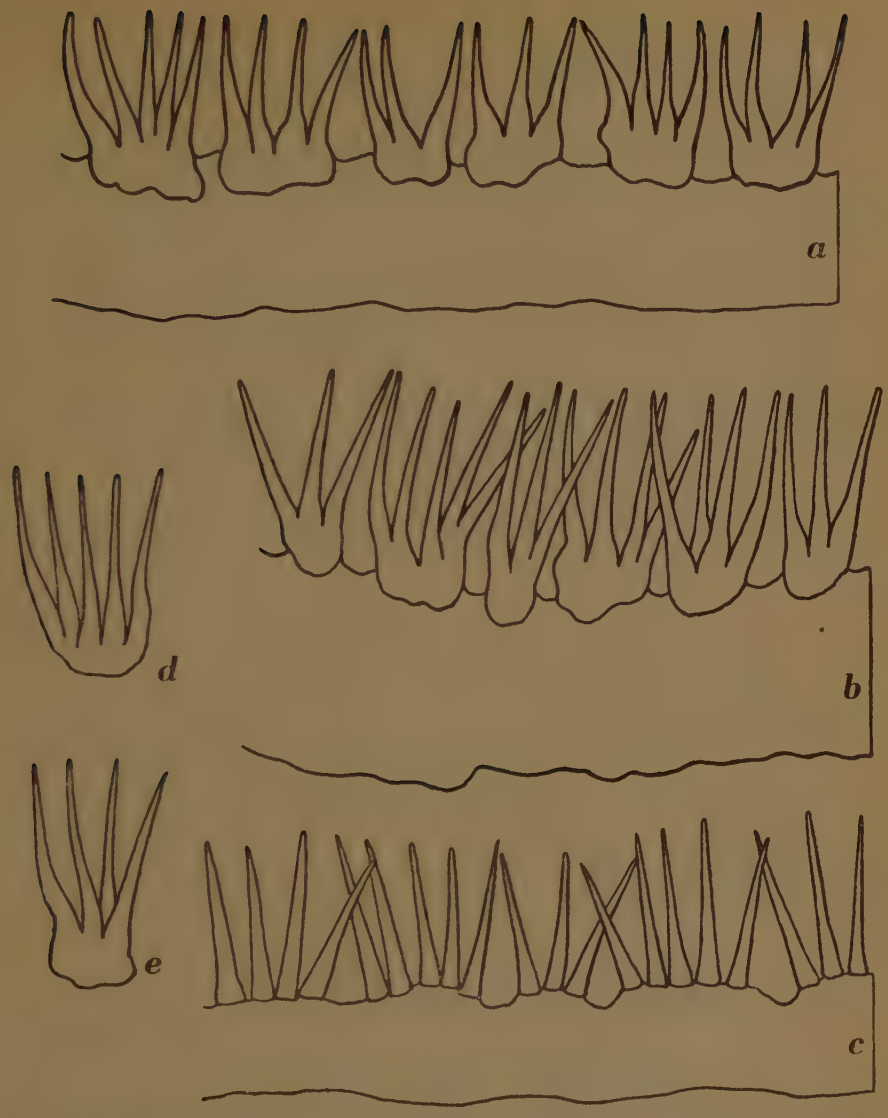

Fig. 1. *Sacculina leptodiae* from *Thalamita stimpsoni*. *a*—*c*, sections of the external cuticle of three specimens; *d*, *e*, excrescences of the external cuticle of the specimen of fig. 1 c. × 530.

In fig. 3*a* a complete section of the mantle has been drawn. This section shows the chief elements described by DELAGE (1884, pl. XXVII fig. 51 and other figures). The cells of the epithelium (*e*) adhere with their flat extremities perpendicularly to the external cuticle which has been secreted by them. Towards the middle region of the mantle the epithelium cells are prolonged into muscular or connective elements which are united with the corresponding elements of the other surface of the mantle. The cells of the epithelium are arranged in groups of a more or less conical shape, the

bundles of muscular or connective elements forming the tops of the cones. These bundles traverse the muscular layers in the middle region of the mantle, two of which (*m*) are visible in the figure, one in transverse, the other in more or less longitudinal section. The remaining elements of the

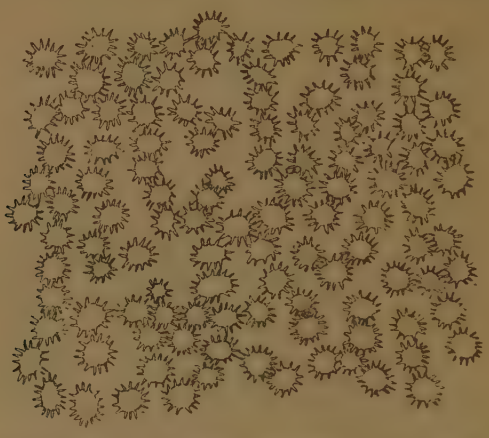

Fig. 2. *Sacculina leptodiae* from *Thalamita stimpsoni*, young specimen. Surface view of the excrescences of the external cuticle. × 530.

mantle (lacunae, etc.) are represented in the figure by dotting. The thin internal cuticle (*ci*), to which the innermost row of epithelium cells adheres, does not show any important particulars.

In fig. 3*b* and *c* the external parts of sections of the mantle are drawn.

Fig. 3*b* shows a region of the mantle in which the external cuticle (*ce*) is found at a slight distance from the epithelium cells (*e*). On the top of each of these cells there is a little cone of chitin (length about 6 μ), apparently of the same kind as that of the external cuticle, as it stains in the same manner. These cones (*ex*) form the beginning of the external cuticle as it is found in the adult animal. As yet the little cones form a continuous row of small spines, the only arrangement in groups is that of the cells of the epithelium.

A further development of the excrescences is shown in fig. 3*c*, where there is a considerable distance between the external cuticle (*ce*) and the cells of the epithelium (*e*), apparently on account of further growth of the excrescences (*ex*) of the new cuticle, which have a length of about 18 μ. Here the excrescences still consist of chitin of a comparatively weak kind, as they are stained in the same degree as the old external cuticle. In this figure again the arrangement in groups is apparent in the cells of the epithelium only.

The second young specimen too has two layers of external cuticle, but here the inner layer is covered with papillae with a peripheral row of minute spines (fig. 4). This inner layer (*ce*) has a thickness of 4 to 8 μ, its excrescences have a height of $4\frac{1}{2}$ to 8 μ and a thickness of 2 to $4\frac{1}{2}$ μ. In all important details these excrescences correspond with those of the outer

layer of cuticle in the former specimen. In this specimen the outer layer of the external cuticle (*cu*), apparently the original external cuticle, is

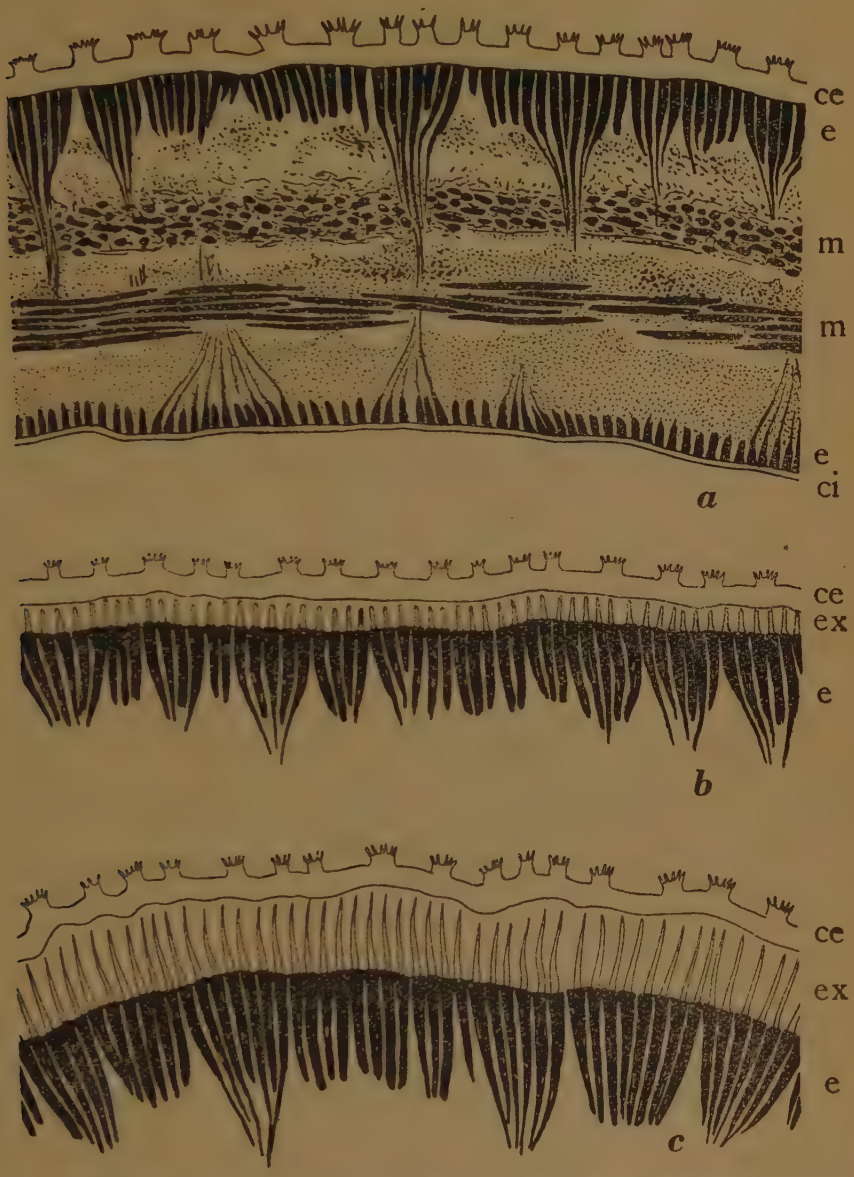

Fig. 3. *Sacculina leptodiae* from *Thalamita stimpsoni*, young specimen. *a*, section of the mantle; *b*, *c*, sections of the external part of the mantle. *ce*, second external cuticle; *ci*, internal cuticle; *e*, epithelial cells; *ex*, excrescences of the third external cuticle; *m*, muscles of the mantle. × 530.

extremely thin (1 to 1½ μ); it may be quite smooth (fig. 4*a*) or may possess minute excrescences in the shape of small pointed spines of a height of 1½ μ or less (fig. 4*b*).

The two young parasites undoubtedly belong to the same species as the specimens of which excrescences are represented in fig. 1. The excrescences indicated with *ex* in fig. 3*c* show already a similar arrangement as those

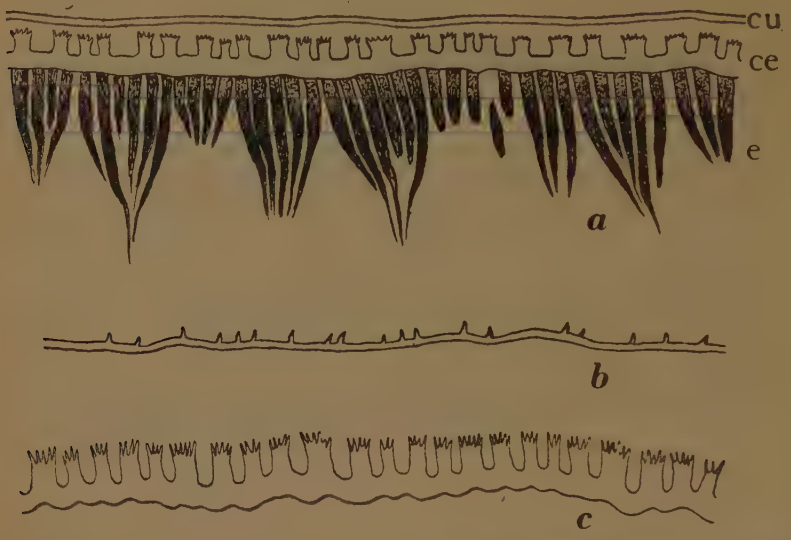

Fig. 4. *Sacculina leptodiae* from *Thalamita stimpsoni*, young specimen. *a*, section of the external part of the mantle; *b*, section of the first external cuticle; *c*, section of the second external cuticle. *ce*, second external cuticle; *cu*, first external cuticle; *e*, epithelial cells. × 700.

of fig. 1*c*. As yet they do not consist of the hyaline kind of chitin which is characteristic of the excrescences of *Sacculina leptodiae*, but during further development gradually the spines would have changed into this harder kind of chitin, a process which begins at the tops and proceeds towards the lower parts of the spines. In the specimens of *Sacculina rotundata* and *S. aculeata* mentioned above this process is already much farther advanced than in the specimen described here.

The occurrence of three successive layers of external cuticle in *Sacculina leptodiae* emphasizes the fact that immature specimens of species of Sacculinidae may show characters of this cuticle which erroneously might be mistaken for definite specific characters. Especially the excrescences of the second layer (fig. 3*a*, fig. 4*c*) are of a so well defined shape that they show more distinct peculiarities than the typical excrescences of many other species of the family.

LITERATURE.

BOSCHMA, H., 1927. Ueber europäische Formen der Gattung Sacculina. Zool. Jahrb., Syst., vol. 54.

————, 1928. The Rhizocephala of the Leiden Museum. Zool. Meded., vol. 11.

————, 1931. Rhizocephala. Papers from Dr. Th. Mortensen's Pacific Expedition 1914—16, LV. Vidensk. Medd. Dansk naturh. Foren., vol. 89.

BOSCHMA, H., 1933. New species of Sacculinidae in the Collection of the United States National Museum. Tijdschr. Ned. Dierk. Ver. (3), vol. 3.

————, 1936. Sur la Sacculina carpiliae et la Sacculina leptodiae de Guérin-Ganivet. Bull. Mus. Hist. Nat. Paris (2), vol. 8.

DELAGE, Y., 1884. Evolution de la Sacculine (Sacculina carcini Thomps.) Crustacé endoparasite de l'ordre nouveau des Kentrogonides. Arch. Zool. exp. gén. (2), vol. 2.

GUÉRIN–GANIVET, J., 1911. Contribution à l'Etude systématique et biologique des Rhizocéphales. Trav. scient. Lab. Zool. et Physiol. marit. Concarneau, vol. 3.

KAMPEN, P. N. VAN and H. BOSCHMA, 1925. Die Rhizocephalen der Siboga-Expedition. Siboga Exp., monogr. XXXI bis.

KOSSMANN, R., 1872. Beiträge zur Anatomie der schmarotzenden Rankenfüssler. Inaug. Diss. Würzburg (also in: Arb. zool.-zoot. Inst. Würzburg, vol. 1, 1874).

Geophysics. — *On the propagation of seismic waves.* II. By J. G. SCHOLTE. (Communicated by Prof. J. D. VAN DER WAALS Jr.)

(Communicated at the meeting of December 21, 1946.)

C. The reflected transversal waves.

The third part of (6) is

$$U_\psi = \frac{Z}{\pi\mu} k \sqrt{\frac{2}{\pi h r}} \int \frac{n \sin^{3/2}\alpha \cos\alpha \cos^2\beta \cos 2\beta \{(P_1+Q_2-R_2)-(P_2+Q_1-R_1)\}}{N(n^2\cos^2 2\beta + \sin 2\alpha \sin 2\beta)} \frac{e^{-ikr\sin\beta-2i\psi}}{1+e^{-2i\psi}} d\beta$$

$$W_\psi = \frac{Z}{\pi\mu} k \sqrt{\frac{2}{\pi h r}} \int \frac{\sin^{1/2}\alpha \cos\alpha \cos\beta \cos 2\alpha \cos 2\beta \{(P_1+Q_2-R_2)-(P_2+Q_1-R_1)\}}{2N(n^2\cos^2 2\beta + \sin 2\alpha \sin 2\beta)} \frac{e^{-ikr\sin\beta-2i\psi}}{1+e^{-2i\psi}} d\beta.$$

In the same way as in section B we obtain with $\frac{e^{-2i\psi}}{1+e^{-2i\psi}} = \sum_{s=0}^{\infty} (-e^{-2i\psi})^s$ an expansion of the integrands into a series of exponential functions with exponents $= ik\sqrt{r^2+(2sd)^2}\cos(\beta-\theta)$. The integration can now be carried out in the complex β plane by the same method as in the preceding section.

In every point of the free surface the wave system $A_\psi(k)$ exists:

$$U_\psi(k) \text{ or } W_\psi(k) = \\ = \frac{Z}{\pi\mu} \sqrt{\frac{n}{\sin\theta}} \frac{e^{i(\nu t - k\sqrt{r^2+4s^2d^2})}}{\sqrt{r^2+4s^2d^2}} \left\{ \sum_{j=0,2,4\ldots} \frac{F_0^{(2j)}}{2^j j!} \left(\frac{i}{k\sqrt{r^2+4s^2d^2}} \right)^j \right\}. \quad (14)$$

At distances $r > \frac{2sd}{\sqrt{n^2-1}}$ the system $A_\psi(h)$, connected with the branch point $\cos\alpha = 0$, appears:

$$U_\psi(h) \text{ or } W_\psi(h) = \\ = i\frac{Z}{\pi\mu} \left\{ \frac{F(\cos\alpha)-F(-\cos\alpha)}{\cos\alpha} \right\}_{\cos\alpha=0} \frac{n\sqrt{n^2-1}}{\sin^{3/2}\theta \sin^{3/2}(\theta-\beta_1)} \frac{e^{i(\nu t - hr - 2ksd\sqrt{1-1/n^2})}}{h(r^2+4s^2d^2)} \quad (15)$$

where $\sin\beta_1 = 1/n$.

Other waves, associated with the branch points $\cos\alpha' = 0$ and $\cos\beta' = 0$ (the systems $A_\psi(h')$ and $A_\psi(k')$) and with the poles of the integrand, come into existance if certain points of the free surface are passed. These points however do not coincide with the points of emergence of the corresponding longitudinal waves. The RAYLEIGH waves $A_\varphi(R)$ for

instance reach the surface at the minimum distance $r = 2d/\sqrt{\sin^2 \alpha_0 - 1}$, but the RAYLEIGH waves $A_\psi(R)$ are only observed at distances larger than $2d/\sqrt{\sin^2 \beta_0 - 1}$.

D. The longitudinal-transversal waves.

The function $\frac{e^{-i\varphi - i\psi}}{(1+e^{-2i\varphi})(1+e^{-2i\psi})}$ appearing in the fourth part of (6) represents a series of waves which have traversed the depth of the layer an uneven number of times (say s_1) as a longitudinal wave and an uneven number of times (s_2) as a transversal wave. The corresponding expression in the last part of (6) denotes the waves which travel an even number of times (at least twice) both as a transversal and as a longitudinal wave between the two boundaries of the layer. It is obvious that these two systems, together with the three cases already dealt with in sections *A*, *B* and *C* involve every possible wave which, starting at the surface, reaches another point of this surface.

As the exponent in both cases is of the form

$$-ihr\sin\alpha - ihs_1 d\cos\alpha - iks_2 d\cos\beta,$$

where s_1 and s_2 are entire numbers, its saddle point is determined by

$$-r\cos\alpha + s_1 d\sin\alpha + s_2 d\sin\beta\frac{\cos\alpha}{\cos\beta} = 0$$

or

$$r/d = s_1 \operatorname{tg}\alpha_s + s_2 \operatorname{tg}\beta_s.$$

The exponent being equal to $-id(s_1 h/\cos\alpha_s + s_2 k/\cos\beta_s)$ if $\alpha = \alpha_s$ the equation of the line of steepest descent is:

Real part of $(r/d\sin\alpha + s_1\cos\alpha + n s_2\cos\beta) = s_1/\cos\alpha_s + n s_2/\cos\beta_s$ which can be reduced to

$$R\left\{\frac{s_1 \sin^2 {}^1/_2(\alpha-\alpha_s)}{\cos\alpha_s} + \frac{s_2 \sin^2 {}^1/_2(\beta-\beta_s)}{\cos\beta_s}\right\} = 0.$$

It can be easily shown that with increasing value of r this curve meets every singular point of the integrand with the exception of $\alpha = 0$ and $\alpha = {}^1/_2\pi$.

Without entering into more detailed calculations which proceed on the same lines as in the preceding sections, it will be evident that the two last parts of (6) yield the wave systems: $A_{\varphi\psi}(hk)$, contributed by α_s; $A_{\varphi\psi}(k)$, $A_{\varphi\psi}(h')$ and $A_{\varphi\psi}(k')$, contributed by the branch points and the surface waves $A_{\varphi\psi}(R)$ and $A_{\varphi\psi}(\alpha_N^i)$. Only the first one exists in every point of the free surface (provided $hr \gg 1$).

§ 4. *The points of emergence.*

The direct waves $A_0(h)$, $A_0(k)$, $A_0(R)$ and the reflected waves, obtained by the saddle-point integration, $A_\varphi(h)$, $A_\psi(k)$, $A_{\varphi\psi}(hk)$ are registered at every point of the surface. The other reflected systems, associated with

the singular points, only occur if r is larger than certain minimum values. Assuming $V' > V > \mathfrak{V}' > \mathfrak{V}$ these minima are for waves which have suffered only one reflection at the plane $z = d$:

$r = 2d(n^2-1)^{-1/2}$ for $A_\varphi(k)$ and $A_\psi(h)$

$r = 2d(V'^2/V^2-1)^{-1/2}$ for $A_\varphi(h')$, $r = 2d(V'^2/\mathfrak{V}^2-1)^{-1/2}$ for $A_\psi(h')$

$r = 2d(V^2/\mathfrak{V}'^2-1)^{-1/2}$ for $A_\varphi(k')$, $r = 2d(\mathfrak{V}'^2/\mathfrak{V}^2-1)^{-1/2}$ for $A_\psi(k')$

$r = 2d(\sin^2\alpha_0-1)^{-1/2}$ for $A_\varphi(R)$, $r = 2d(\sin^2\beta_0-1)^{-1/2}$ for $A_\psi(R)$

and $r = d(\operatorname{tg}\alpha_s + \operatorname{tg}\beta_s)$ for the $A_{\varphi\psi}$ systems, where

$$\sin\alpha_s = \frac{1+\sqrt{8n^2+1}}{4n} \text{ for } A_{\varphi\psi}(k)$$

$$(\operatorname{tg}\alpha_s + \operatorname{tg}\beta_s)\, V/\mathfrak{V}' + \sqrt{V^2/\mathfrak{V}^2 - V^2/\mathfrak{V}'^2} = 1/\cos\alpha_s + n/\cos\beta_s \text{ for } A_{\varphi\psi}(h')$$

$$\frac{\cos(\alpha_s-\beta_s)}{\sin\alpha_s} = \frac{1}{\sin\alpha_0} \text{ for } A_{\varphi\psi}(R).$$

Assuming $V/\mathfrak{V} = V'/\mathfrak{V}' = \sqrt{3}$ and $V'/V = \sqrt{2}$, we obtain the following results: if $r < 0{,}856\,d$ only the direct waves and $A_\varphi(h)$, $A_\psi(k)$, $A_{\varphi\psi}(h\,k)$ exist. In the points $r \geqq 0{,}856\,d$ the wave system $A_\psi(R)$ can be observed; in $r \geqq 0{,}894\,d$ the system $A_\psi(h')$ reaches the surface. This wave starts as a transversal wave, travels along the interface as a surface wave and reaches the surface again as a transversal wave. In $r \geqq 0{,}912\,d$ the system $A_{\varphi\psi}(R)$ occurs; if $r \geqq 1{,}252\,d$ we observe also $A_\varphi(R)$; $A_\varphi(k)$ and $A_\psi(h)$ are registered if $r \geqq 1{,}414\,d$; $A_{\varphi\psi}(h')$ if $r \geqq 1{,}447\,d$; $A_\varphi(h')$ and $A_\psi(k')$ if $r \geqq 2\,d$; $A_{\varphi\psi}(k')$ and $A_{\varphi\psi}(k)$ if $r \geqq 2{,}309\,d$; $A_\varphi(k')$ if $r \geqq 2{,}828\,d$.

The wave systems discussed in this and the preceding paragraph as well as the primary disturbance T_{zz} are stationary; as the motion of the free surface during an earthquake is obviously transitory these calculations have no direct bearing on the registration of a seismograph. It is however possible to generate stationary waves by using a vibration machine; some time after this apparatus is started the motion of the surface has become stationary and the above results can be used. By identifying the points of emergence and by comparison of the amplitudes of the different waves new data, useful to seismic investigation of small area's, may be obtained.

§ 5. *The first impulses.*

If the disturbing action is not stationary — usually it is only different from zero during a short time — it can always be regarded as a superposition of stationary and periodical disturbations with frequencies ν and amplitudes which are a function $F(\nu)$ of the frequency, by means of a FOURIER integral

$$T_{zz} = Z\int_{-\infty}^{+\infty} F(\nu)\, e^{i\nu t}\, d\nu.$$

The corresponding movement of the surface will be found by applying the operator $\int F(\nu)\,d\nu$ to the results of § 3; we obtain in this way a series of integrals which can be evaluated by integration in the complex ν-plane. We choose the path of integration on the negative side of the real axis.

The integrands consist of $F(\nu)$, the amplitude function $G(\nu)$ and an exponential function $e^{i\nu(t-\tau)}$:

$$I=\int_{-\infty}^{+\infty} F(\nu)\cdot G(\nu)\, e^{i\nu(t-\tau)}\, d\nu$$

the quantity $\tau=r/V$ in the case of the direct longitudinal waves, etc.

Denoting the largest value of $|\nu|$ at which $F(\nu)$ or $G(\nu)$ are singular by $\bar{\nu}$ we can expand $F(\nu)\cdot G(\nu)$ in a series $\Sigma a_n \nu^{-n}$ if $|\nu|>\bar{\nu}$. Hence the integrals are equal to

$$I=\int \Sigma a_n \nu^{-n}\cdot e^{i\nu(t-\tau)}\, d\nu$$

if we change the chosen path of integration into a semi-circle with radius $|\nu|>\bar{\nu}$. With increasing radius these integrals become equal to zero if $t<\tau$. As the integrals are obviously not equal to zero if $t>\tau$ the travel time of these waves is τ.

For instance: the travel time τ of the wave $s=1$ of the system $A_\gamma(h')$ is $\frac{r}{V'}+\frac{2d\sqrt{1-m_1^2}}{V}$ or $\frac{r-2d\,\mathrm{tg}\,\alpha_2}{V'}+\frac{2d}{V\cos\alpha_2}$ (where $\sin\alpha_2=V/V'$); this wave traverses the layer in the direction α_2 with velocity V, runs along the boundary plane $z=d$ as a surface wave with velocity V' and finally travels upwards in the direction α_2 with velocity V. The time τ is equal to the travel time of the direct longitudinal waves if $r=2d\sqrt{\frac{V'+V}{V'-V}}$; at more distant points r the first impulse registered by a seismograph is caused by $A_\gamma(h')$, $s=1$. Because of this property this wave has been the subject of many investigations, mainly carried out by V. SCHMIDT [6]) ("geführte Welle"). As its amplitude is proportional to r^{-2} this wave is however not very important at large distances; the major part of the energy released in the hearth is carried on by the surface waves proper with amplitudes proportional to r^{-1}.

As a large part of seismological field work is concerned with first impulses it will be of some interest to calculate the amplitudes of the fastest waves.

The value of the integrals I when $t>\tau$ can be calculated by adding the same integral but integrated along a semi circle with infinite radius in the upper part of the ν plane, on which the integrand is zero. The

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6) O. V. SCHMIDT, Ann. de Phys. V. 19, 891—912 (1934).

two integrals together form one integral along a closed curve around the point $\nu = \infty$:

$$I = \oint \Sigma a_n \nu^{-n} \cdot e^{i\nu(t-\tau)}\, d\nu \quad \text{or} \quad I = 2\pi i \sum \frac{a_n \cdot i^n (t-\tau)^n}{n!} \ldots \quad (16)$$

The amplitude of the first impulse is, assuming $F(\nu) \backsim 1/\nu$ for large values of ν: if $r < 2d\sqrt{\frac{V'+V}{V'-V}}$:

$$U \approx i \frac{Z}{\mu} \frac{2n^2\sqrt{n^2-1}}{(n^2-2)^3} \frac{V}{r^2}\left(t - \frac{r}{V}\right) \text{ and } W \approx i \frac{Z}{\mu} \frac{n^2}{(n^2-2)^2} \frac{V}{r^2}\left(t - \frac{r}{V}\right) \quad (17)$$

and if $r > 2d\sqrt{\frac{V'+V}{V'-V}}$:

$$\left.\begin{aligned} U &\approx -i \frac{Z}{\mu} \cdot 2\sqrt{n^2-m_1^2} \cdot \frac{V}{r^2}\left(t - \frac{r}{V'} - \frac{2d\sqrt{1-m_1^2}}{V}\right) \cdot K \text{ and} \\ W &\approx -i \frac{Z}{\mu} \cdot \frac{n^2-2m_1^2}{m_1} \frac{V}{r^2}\left(t - \frac{r}{V'} - \frac{2d\sqrt{1-m_1^2}}{V}\right) \cdot K. \end{aligned}\right\} \quad (18)$$

Where K is the value of

$$\frac{2\sin 2\alpha \cos 2\beta}{(n^2\cos^2 2\beta + \sin 2\alpha \sin 2\beta)^2}\left\{\frac{\sin\theta\cos\alpha}{\sin(\theta-\alpha)}\right\}^{3/2} \cdot \frac{P_1 P_2 - (Q_1+R_1)(Q_2+R_2)}{(P_1+Q_1+R_1)^2 \cos\alpha'}$$

for $\alpha' = \arccos 0$.

Substituting the numerical values used in § 4 we obtain

$$U/W = \frac{2\sqrt{n^2-1}}{n^2-2} \text{ or } 2\sqrt{2} \text{ if } r < 4{,}82\, d,$$

and

$$U/W = \frac{2\sqrt{n^2/m_1^2-1}}{n^2/m_1^2-2} \text{ or } {}^1/_2\sqrt{5} \text{ if } r > 4{,}82\, d.$$

This marked discontinuity of U/W can perhaps be used in the analysis of seismograms.

Similar calculations can be carried out for every wave found in § 3; it is evident that the saddle point wave systems $A_\varphi(h)$, $A_\psi(k)$ and $A_{\varphi\psi}(hk)$ consist of waves which travel s times up and down between $z = 0$ and $z = d$ with velocities V and $\mathfrak{V}$. Their amplitudes decrease with increasing s.

The series (16) converges very slowly at larger values of $t-\tau$ and becomes therefore unusable some time after the arrival of the first impuls imparted by the wave system. The movement can then only be calculated if the primary disturbance, characterised by $F(\nu)$, is known.

However the motion caused by these waves is only important at small values of $t-\tau$ as these vibrations are followed by a series of surface

waves with much larger amplitudes. Instead of continuing the examination of the V and $\mathfrak{V}$ waves we therefore start with the calculation of the major parts of the movement transmitted by the surface waves.

§ 6. *The surface waves.*

The RAYLEIGH wave $A_0(R)$ has the same amplitude as in the case of a homogeneous medium without a layer. At various points r other RAYLEIGH waves, belonging to the wave systems $A_\varphi(R)$, $A_{\varphi\psi}(R)$ and $A_\psi(R)$, emerge; this happens at the same moment as that on which the direct RAYLEIGH wave reaches these points. Although the amplitude of all surface waves is proportional to r^{-1} the total RAYLEIGH movement caused by these waves rapidly diminishes in comparison with the other waves at increasing value of r. This can be proved in the following way: with increasing r the number of residue terms (§ 3) due to the pole α_0 increases as more and more lines of integration are carried beyond the point α_0. The total sum of all residue terms is equal to zero as the original expressions (3) for L/N and M/N are finite at $\alpha = \alpha_0$. Hence the total RAYLEIGH movement at any point r can just as well be obtained by adding up the residue terms of those integrals whose paths of integration have not yet met the point α_0. As the residue terms are proportional to exp. $(-2shd\,|\cos\alpha_0|)$ and $s \approx r/2d$ this movement decreases exponentially with r. The movement is therefore discernible only at small values of r. Further calculations can be carried out by means of the method of LAMB, as the travel time τ is independent of ν.

The travel time of the surface waves α_N^l being a function of ν this method is of no use at the investigation of these waves; again applying the saddle point method we change the path of integration into the line of steepest descent. The saddle point ω is determined by (wave system A_φ):

$$\frac{\partial}{\partial\nu}\left\{\nu t - \nu\frac{r}{V}\sin\alpha_N^l - 2\nu\frac{sd}{V}\cos\alpha_N^l\right\} = 0;$$

introducing the group velocity

$$C = \frac{d\nu}{d\,(h\sin\alpha_N^l)} \quad \text{or} \quad C = \frac{V}{\sin\alpha_N^l + \nu\cos\alpha_N^l\dfrac{d\alpha_N^l}{d\nu}}$$

this equation can easily be reduced to

$$t_\omega = \frac{r - 2sd\,\mathrm{tg}\,\alpha_N^l}{C} + \frac{2sd}{V\cos\alpha_N^l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

α_N^l is a function of ν satisfying $N(\alpha, \nu) = 0$; substituting $\nu = \omega$ in the above equation we obtain the travel time t_ω of the frequency ω. The vibration ω is therefore propagated in the direction $\alpha_N^l(\omega)$ with longitudinal velocity, is s times reflected at the plane $z = d$, retaining the velocity V,

and covers the remaining distance $r-2sd\,\mathrm{tg}\,\alpha_N^i$ along the free surface with the group velocity C.

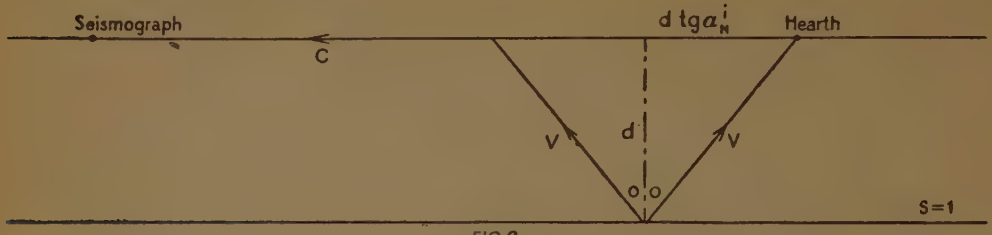

FIG. 2

The amplitude of the vibration with frequency ω can be obtained in the usual way; in the neighbourhood of $\nu=\omega$ the exponent can be expanded into

$$i\omega(t-\tau_\omega)-i\frac{(\nu-\omega)^2}{2!}\left(\frac{d^2\nu\tau}{d\nu^2}\right)_\omega-i\frac{(\nu-\omega)^3}{3!}\left(\frac{d^3\nu\tau}{d\nu^3}\right)_\omega-\ldots$$

Integrating along the tangent $\nu=\omega+ue^{-1/4\pi i}$ to the line of steepest descent we get in first approximation

$$I_\omega\approx F(\omega)\,G(\omega)\,e^{i\omega(t-\tau_\omega)-1/4\pi i}\int_{-u_0}^{+u_0}e^{-1/2\,u\left(\frac{d^2\nu\tau}{d\nu^2}\right)_\omega}du$$

or

$$I_\omega\approx\frac{F(\omega)\,G(\omega)}{\sqrt{\left(\frac{d^2\nu\tau}{d\nu^2}\right)_\omega}}\sqrt{2\pi}\;e^{i\omega(t-\tau_\omega)-1/4\pi i}\quad\ldots\ldots\quad(20)$$

The function $\frac{d^2\nu\tau}{d\nu^2}$ can easily be reduced to

$$-\frac{r-2sd\,\mathrm{tg}\,\alpha_N^i}{C^2}\frac{dC}{d\nu}-\frac{2sd}{\nu V\cos^3\alpha_N^i}\left(\frac{V}{C}-\sin\alpha_N^i\right)^2;$$

as $G(\omega)\backsim r^{-1/2}$ the amplitudes of these waves are proportional to r^{-1} if $r\gg sd$.

The above approximation is no longer valid if $e^{-i\frac{(\nu-\omega)^2}{2!}\left(\frac{d^2\nu\tau}{d\nu^2}\right)_\omega}$ is not the dominating factor of the integrand; this circumstance arises for instance when ω satisfies the equation $d^2\nu\tau/d\nu^2=0$, or

$$\nu\frac{dC}{d\nu}=-\frac{\cos\theta}{\sin(\theta-\alpha_N^i)}\frac{\left(1-\frac{C}{V}\sin\alpha_N^i\right)^2}{\cos^2\alpha_N^i}\quad\text{where}\quad\theta=\mathrm{arc\,tg}\,\frac{r}{2sd}\quad\ldots\quad(21)$$

As $(d\nu\tau/d\nu)_\omega$ is the travel time of the frequency ω this equation determines the frequency ω_m which is propagated with either a maximum or a minimum velocity; hence the time of arrival of this frequency marks either the beginning or the end of a wave train which has suffered s reflections at the plane $z=d$.

The amplitude of this tremor is:

$$I_{\omega_m} \approx \frac{F(\omega_m)\, G(\omega_m)}{\sqrt[3]{\frac{1}{3!}\left(\frac{d^3 \nu\tau}{d\nu^3}\right)_{\omega_m}}} \frac{\Gamma(1/3)}{\sqrt{3}} e^{i\omega_m(t-\tau_{\omega_m})} \quad . \quad . \quad . \quad . \quad (22)$$

This amplitude decreases more slowly ($\backsim r^{-5/6}$) with increasing r as the amplitude of the other frequencies ($\backsim r^{-1}$); consequently the frequency ω_m is in distant points predominant. As the roots ω_m of equation (21) are a function of Θ the period and the velocity associated with the maximum amplitude are also a function of the distance r.

It is a well known fact that both the period and the velocity of the waves with the largest amplitudes increase with increasing epicentral distance [5]). Equation (21) shows that these two phenomena are interrelated: an increase or decrease of the velocity involves the same change of the period. This can be easily shown: if $\Theta = {}^1/_2\,\pi$ this equation is $dC/d\nu = 0$; the vibrations $\omega_m(\infty)$ observed at large distances are therefore propagated with maximum or minimum velocity. As the right hand side of the equation is negative if $\Theta \neq {}^1/_2\,\pi$ the major frequencies ω_m observed at smaller distances are higher then $\omega_m(\infty)$ if C has a minimum value, and are lower then $\omega_m(\infty)$ if C shows a maximum. In view of the rather complicated character of the equation $N = 0$ it is impossible to determine the function $C(\nu)$ without a very laborious numerical computation.

§ 7. *Summary.*

In this paper the movement caused by a normal pressure, periodical in t and uniformly exerted on the surface of a mono-stratified medium, has been calculated. By means of a FOURIER integration this simple pressure distribution can be transformed into a pressure concentrated in a point. The corresponding surface movement is found by application of the saddle point integration method.

It appears that at epicentral distances comparable to the depth of the layer the same waves occur as in the case of a homogeneous medium; in addition to these waves several kinds of reflected wave systems reach the surface. With increasing distance to the hearth the movement consists more and more of generalised RAYLEIGH waves, which are dependent on both media.

A second FOURIER integration changes the periodical normal pressure into a pressure existing only during a finite time; the movement caused by this disturbation has been investigated. The amplitudes of the seismologically important first impulses are calculated and the travel times of the major parts of the movement have been determined.

It follows that this method is quite generally applicable to the propagation of seismic waves in any horizontally stratified medium.

In conclusion I wish to express my thanks to Prof. Van der Waals for his kind interest taken in this paper.

Mathematics. — *Sur l'extension de la condition de Legendre du Calcul des Variations aux intégrales multiples à plusieurs fonctions inconnues.* By LÉON VAN HOVE. (Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meeting of December 21, 1946.)

1. Considérons l'intégrale du Calcul des Variations

$$I = \int_D F(x_\alpha, z_j, p_{j\alpha})\, dx_1 \ldots dx_\mu \qquad (\alpha = 1, \ldots \mu; j = 1, \ldots n). \quad (1.1)$$

Les x_α sont les μ variables indépendantes; les z_j désignent n fonctions inconnues de ces variables, ayant les $p_{j\alpha}$ comme dérivées premières $\frac{\partial z_j}{\partial x_\alpha}$. L'intégrale I est étendue à un domaine fermé D de l'espace $x_1, \ldots x_\mu$. Pour la facilité, nous supposerons ce domaine limité par un nombre fini d'hypersurfaces ayant en chaque point un hyperplan tangent qui varie d'une manière continue avec son point de contact. Supposons encore que F soit une fonction de classe c_2 [1]) en tous ses arguments, pour les valeurs des x_α qui correspondent aux points de D et pour toutes les valeurs des z_j et des $p_{j\alpha}$. Toutes les grandeurs figurant dans (1. 1) sont réelles.

L'extension de la condition de Legendre à une intégrale multiple I contenant plusieurs fonctions inconnues ($n > 1$ et $\mu > 1$) a été étudiée par HADAMARD. Il a obtenu le résultat suivant [2]): si une extrémale V_0 de I réalise un minimum de cette intégrale, les coordonnées x_α, z_j, $p_{j\alpha}$ de chacun de ses éléments de contact rendent la forme biquadratique $F_{p_{j\alpha} p_{k\beta}} \xi_j \xi_k \eta_\alpha \eta_\beta$ [3]) semi-définie positive; c'est-à-dire que l'on a

$$F_{p_{j\alpha} p_{k\beta}} \xi_j \xi_k \eta_\alpha \eta_\beta \geqq 0 \text{ pour toutes les valeurs des } \xi_j \text{ et des } \eta_\alpha. \quad (1.2)$$

Inversement, supposons que pour l'élément de contact en un point P_0

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[1]) Nous disons qu'une fonction des x_α est de classe c_ϱ dans D quand ses dérivées partielles, jusqu'à l'ordre ϱ inclusivement, existent et sont continues en tous les points de ce domaine.

[2]) HADAMARD, J., Sur quelques questions du Calcul des Variations. Bull. Soc. Math. de France, **30**, 253—256 (1902).

Pour l'établissement de la condition nécessaire de minimum (1. 2) quelles que soient les valeurs de n et de μ, voir par exemple BOERNER, H., Math. Zeitschrift, **46**, 729 (1940).

La condition suffisante, qui se rattache à une remarque de CLEBSCH (Journal für reine und ang. Math., **56**, 122—148 (1859)), peut s'établir par la méthode exposée plus loin (n°. 3).

[3]) Les indices latins varient de 1 à n, les indices grecs de 1 à μ; il faut toujours sommer par rapport aux indices répétés dans les produits.

d'une extrémale V_0 de classe c_1 [4]), le faisceau de formes quadratiques

$$F_{p_{j\alpha}\, p_{k\beta}}\, \omega_{j\alpha}\, \omega_{k\beta} + \lambda_{j\alpha,k\beta}\, (\omega_{j\alpha}\, \omega_{k\beta} - \omega_{k\alpha}\, \omega_{j\beta}) \quad . \quad . \quad . \quad (1.3)$$

de paramètres $\lambda_{j\alpha,k\beta}$ contienne une forme définie positive. Alors V_0 réalise le minimum de I parmi toutes les variétés V de classe D_1 dans D [5]) qui appartiennent à un voisinage faible suffisamment petit de V_0, admettent même contour que cette extrémale et n'en sont distinctes que dans un voisinage suffisamment petit de P_0 (minimum faible local).

HADAMARD a posé le problème suivant, de nature algébrique: supposons que la forme biquadratique $F_{p_{j\alpha}\, p_{k\beta}}\, \xi_j\, \xi_k\, \eta_\alpha\, \eta_\beta$ soit définie positive, c'est-à-dire que l'on ait

$$\left.\begin{array}{r} F_{p_{j\alpha}\, p_{k\beta}}\, \xi_j\, \xi_k\, \eta_\alpha\, \eta_\beta > 0 \text{ sauf quand } \xi_1 = \ldots = \xi_n = 0 \\ \text{ou quand } \eta_1 = \ldots = \eta_\mu = 0 \end{array}\right\} \quad . \quad . \quad (1.4)$$

Peut-on en déduire que le faisceau (1.3) contient une forme quadratique définie positive?

Ce problème a été étudié par plusieurs auteurs [6]). On doit à TERPSTRA sa solution complète [7]):

a. quand n ou μ vaut 2, le caractère défini positif de la forme biquadratique entraîne l'existence de formes quadratiques définies positives dans le faisceau (1.3) [8]);

b. il n'en est pas ainsi quand n et μ dépassent 2.

Il résulte de ce théorème algébrique de TERPSTRA que si n ou μ vaut 2, la condition (1.4) est suffisante pour le minimum faible local de I. Par contre, on ne peut rien conclure quand n et μ dépassent 2.

Notre but est de montrer par une méthode nouvelle n'utilisant pas le faisceau (1.3), que l'inégalité (1.4) est une condition suffisante de minimum faible local, quelles que soient les valeurs de n et de μ.

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[4]) Nous représentons tout système de fonctions $z_j\ (x_\alpha)$ par une variété V à μ dimensions de l'espace $x_1, \ldots x_\mu, z_1, \ldots z_n$, définie par les équations $z_j = z_j(x_\alpha)$. Nous disons que V est d'une certaine classe quand toutes les fonctions $z_j(x_\alpha)$ sont de cette classe. C'est dans ce sens que nous parlons d'une extrémale de classe C_1.

[5]) Une fonction $f(x_1, \ldots x_\mu)$ est de classe D_ϱ dans le domaine D lorsqu'elle y est continue et que D se décompose en un nombre fini de domaines partiels fermés dans chacun desquels f est de classe C_ϱ. On suppose les domaines partiels séparés les uns des autres par un nombre fini d'hypersurfaces ayant en chaque point un hyperplan tangent qui varie d'une manière continue avec son point de contact. Pour la définition d'une variété V de classe D_1, voir [4])

[6]) FINSLER, P., Enseign. Math., **26**, 319 (1927).
ALBERT, A. A., Bull. Amer. Math. Soc., **44**, 250—252 (1938).
REID, W. T., ibidem, 437—440.

[7]) TERPSTRA, F. J., Die Darstellung biquadratischer Formen als Summen von Quadraten mit Anwendung auf die Variationsrechnung. Math. Ann., **116**, 166—180 (1938).

[8]) Cette proposition a été retrouvée par une méthode différente, par HESTENES et MAC SHANE (Trans. Amer. Math. Soc., **47**, 501—512 (1940)).

2. *Un théorème de calcul intégral.*

Considérons, pour des valeurs quelconques de n et de μ, l'intégrale

$$I_0 = \int_D A_{k\alpha, j\beta}\, \zeta_{k\alpha}\, \zeta_{j\beta}\, dx_1 \ldots dx_\mu,$$

où les $A_{k\alpha, j\beta}$ sont des constantes réelles et où les $\zeta_{j\alpha}$ sont les dérivées $\frac{\partial \zeta_j}{\partial x_\alpha}$ des fonctions réelles $\zeta_j(x_1, \ldots x_\mu)$. Nous allons établir le théorème suivant:

Pour que I_0 ne prenne aucune valeur négative quand les fonctions $\zeta_1, \ldots \zeta_n$ sont de classe D_1 dans le domaine D et s'annulent à sa frontière, il faut et il suffit que la forme biquadratique $A_{j\alpha, k\beta}\, \xi_j\, \xi_k\, \eta_\alpha\, \eta_\beta$ soit semi-définie positive.

La condition est nécessaire d'après les résultats d' HADAMARD. Pour montrer qu'elle est suffisante, posons $\zeta_j = 0$ hors de D. Les fonctions ζ_j étant nulles à la frontière de D, elles sont ainsi prolongées en des fonctions de classe D_1 dans tout l'espace $x_1, \ldots x_\mu$. La transformée de Fourier de ζ_j est la fonction complexe de variables réelles

$$\chi_j(x_1, \ldots x_\mu) = \left(\frac{1}{2\pi}\right)^{\frac{\mu}{2}} \int_D e^{i x_\gamma y_\gamma}\, \zeta_j(y_1, \ldots y_\mu)\, dy_1 \ldots dy_\mu.$$

La transformée de $\zeta_{j\alpha}$ s'écrit, à l'aide d'une intégration par parties,

$$\left(\frac{1}{2\pi}\right)^{\frac{\mu}{2}} \int_D e^{i x_\gamma y_\gamma}\, \zeta_{j\alpha}(y_1, \ldots y_\mu)\, dy_1 \ldots dy_\mu =$$

$$- i x_\alpha \left(\frac{1}{2\pi}\right)^{\frac{\mu}{2}} \int_D e^{i x_\gamma y_\gamma}\, \zeta_j(y_1, \ldots y_\mu)\, dy_1 \ldots dy_\mu = - i x_\alpha\, \chi_j.$$

Une propriété bien connue des transformations de Fourier donne alors

$$\int_D \zeta_{j\alpha}\, \zeta_{k\beta}\, dx_1 \ldots dx_\mu = \int_{(x)} (-i x_\alpha\, \chi_j)\, \overline{(-i x_\beta\, \chi_k)}\, dx_1 \ldots dx_\mu.\ ^{9}) \quad (2.1)$$

La seconde intégrale est étendue à tout l'espace $x_1, \ldots x_\mu$ et est absolument convergente. Posons maintenant $\chi_j = \chi'_j + i\chi''_j$, les fonctions χ'_j et χ''_j étant réelles, et égalons le premier membre de (2.1) à la partie réelle du second:

$$\int_D \zeta_{j\alpha}\, \zeta_{k\beta}\, dx_1 \ldots dx_\mu = \int_{(x)} x_\alpha\, x_\beta\, (\chi'_j\, \chi'_k + \chi''_j\, \chi''_k)\, dx_1 \ldots dx_\mu.$$

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[9]) La barre $-$ indique le nombre complexe conjugué.

Comme les $A_{j\alpha,k\beta}$ sont des constantes, l'expression de I_0 devient

$$I_0 = \int_{(x)} (A_{j\alpha,k\beta}\, \chi'_j\, \chi'_k\, x_\alpha\, x_\beta + A_{j\alpha,k\beta}\, \chi''_j\, \chi''_k\, x_\alpha\, x_\beta)\, dx_1 \ldots dx_\mu,$$

ce qui montre que la condition énoncée est suffisante. Notre théorème est ainsi établi.

Quand n ou μ vaut 2, on peut aussi le démontrer en déduisant du théorème de TERPSTRA un théorème analogue pour les formes semi-définies. Quand n ou μ vaut 1, le caractère suffisant de la condition énoncée est évident.

3. *Le minimum faible local.*

Revenons à l'intégrale (1. 1) et considérons une extrémale V_0 d'équations $z_j = z_j^0(x_\alpha)$ de classe c_1 dans le domaine D. Soit un point $x_1^0, \ldots x_\mu^0$ intérieur à D, et désignons par $F^0_{p_{j\alpha}p_{k\beta}}$ la valeur de la dérivée $F_{p_{j\alpha}p_{k\beta}}$ pour l'élément de contact à V_0 au point P_0 dont les coordonnées sont x_α^0, $z_j^0(x_\alpha^0)$. Supposons la forme biquadratique $F^0_{p_{j\alpha}p_{k\beta}}\, \xi_j\, \xi_k\, \eta_\alpha\, \eta_\beta$ définie positive. Il existe alors un nombre positif w pour lequel la forme biquadratique

$$F^0_{p_{j\alpha}p_{k\beta}}\, \xi_j\, \xi_k\, \eta_\alpha\, \eta_\beta - w\, \xi_j\, \xi_j\, \eta_\alpha\, \eta_\alpha \quad . \quad . \quad . \quad . \quad . \quad (3.1)$$

soit semi-définie positive.

Considérons une variété V d'équations $z_j = z_j^0(x_\alpha) + \zeta_j(x_\alpha)$ de classe D_1 dans D, qui ait même contour que l'extrémale V_0 et qui n'en soit distincte que dans un voisinage de P_0; les fonctions ζ_j sont donc nulles sur la frontière de D et en tous les points de ce domaine situés hors d'un domaine Δ qui contient le point $x_1^0, \ldots x_\mu^0$ en son intérieur [10]).

Supposons de plus que la variété V appartienne au voisinage faible ε de V_0; les fonctions ζ_j et leurs dérivées $\zeta_{j\alpha}$ vérifient donc en tous les points de D les inégalités

$$|\zeta_j| < \varepsilon, |\zeta_{j\alpha}| < \varepsilon.$$

Cela étant, calculons par la formule de TAYLOR l'accroissement

$$\Delta I = \int_D \left\{ F\left(x_\alpha, z_j^0 + \zeta_j, \frac{\partial z_j^0}{\partial x_\alpha} + \zeta_{j\alpha}\right) - F\left(x_\alpha, z_j^0, \frac{\partial z_j^0}{\partial x_\alpha}\right) \right\} dx_1 \ldots dx_\mu.$$

Il suffit d'étendre l'intégrale à Δ. Comme l'extrémale V_0 annule la variation première, il vient

$$\Delta I = \frac{1}{2} \int_\Delta \{ F^1_{p_{j\alpha}p_{k\beta}}\, \zeta_{j\alpha}\, \zeta_{k\beta} + 2 F^1_{p_{j\alpha}z_k}\, \zeta_{j\alpha}\, \zeta_k + F^1_{z_j z_k}\, \zeta_j\, \zeta_k \}\, dx_1 \ldots dx_\mu,$$

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[10]) On peut supposer Δ limité par un nombre fini d'hypersurfaces ayant en chaque point un hyperplan tangent qui varie d'une manière continue avec son point de contact.

où l'indice [1]) indique qu'il faut prendre comme arguments dans les dérivées secondes de F des expressions de la forme suivante:

$$x_\alpha,\ z_j^0 + \theta(x_\beta)\,\zeta_j,\ \frac{\partial z_j^0}{\partial x_\alpha} + \theta(x_\beta)\,\zeta_{j\alpha}, \qquad (0 < \theta(x_\beta) < 1).$$

On trouve facilement un nombre positif M, indépendant des fonctions ζ_j, tel que

$$\left| \int_\Delta (2\,F^1_{p_{j\alpha} z_k}\,\zeta_{j\alpha}\,\zeta_k + F^1_{z_j z_k}\,\zeta_j\,\zeta_k)\,dx_1 \ldots dx_\mu \right| \leqq$$

$$\leqq \frac{w}{4} \int_\Delta \zeta_{j\alpha}\,\zeta_{j\alpha}\,dx_1 \ldots dx_\mu + M \int_\Delta \zeta_j\,\zeta_j\,dx_1 \ldots dx_\mu.$$

D'autre part, L étant le diamètre de Δ, les fonctions ζ_j vérifient l'inégalité connue

$$\int_\Delta \zeta_j\,\zeta_j\,dx_1 \ldots dx_\mu \leqq \frac{L^2}{2} \int_\Delta \zeta_{j\alpha}\,\zeta_{j\alpha}\,dx_1 \ldots dx_\mu \ ^{11})$$

Si L est suffisamment petit, on aura donc

$$\left| \int_\Delta (2\,F^1_{p_{j\alpha} z_k}\,\zeta_{j\alpha}\,\zeta_k + F^1_{z_j z_k}\,\zeta_j\,\zeta_k)\,dx_1 \ldots dx_\mu \right| \leqq \frac{w}{2} \int_\Delta \zeta_{j\alpha}\,\zeta_{j\alpha}\,dx_1 \ldots dx_\mu.$$

Si ε et L sont suffisamment petits, on aura aussi

$$\left| \int_\Delta (F^1_{p_{j\alpha} p_{k\beta}}\,\zeta_{j\alpha}\,\zeta_{k\beta} - F^0_{p_{j\alpha} p_{k\beta}}\,\zeta_{j\alpha}\,\zeta_{k\beta})\,dx_1 \ldots dx_\mu \right| \leqq \frac{w}{4} \int_\Delta \zeta_{j\alpha}\,\zeta_{j\alpha}\,dx_1 \ldots dx_\mu.$$

Ces inégalités entraînent la suivante

$$\Delta I \geqq \frac{1}{2} \int_\Delta \left(F^0_{p_{j\alpha} p_{k\beta}}\,\zeta_{j\alpha}\,\zeta_{k\beta} - \frac{3\,w}{4}\,\zeta_{j\alpha}\,\zeta_{j\alpha} \right) dx_1 \ldots dx_\mu.$$

Mais les $F^0{}_{p_{j\alpha} p_{k\beta}}$ sont des constantes et la forme biquadratique (3. 1) est semi-définie positive. Comme les ζ_j sont nulles à la frontière de Δ, le théorème du no. 2 donne

$$\Delta I \geqq \frac{w}{8} \int_\Delta \zeta_{j\alpha}\,\zeta_{j\alpha}\,dx_1 \ldots dx_\mu,$$

et l'accroissement ΔI est positif sauf quand les fonctions ζ_j sont identiquement nulles. Nous avons ainsi démontré, pour des valeurs quelconques de n et de μ, la proposition suivante:

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[11]) Voir par exemple HADAMARD, J., Leçons sur le Calcul des Variations, 1910. Note au bas de la p. 466. Il suffit d'appliquer l'inégalité établie à cet endroit en prenant x_1 comme variable d'intégration et d'intégrer ensuite par rapport à $x_2, \ldots x_\mu$.

Si la forme biquadratique $F_{p_{j\alpha}p_{k\beta}\xi_j\xi_k\eta_\alpha\eta_\beta}$ *est définie positive pour l'élément de contact au point* P_0 *d'une extrémale* V_0 *de classe* C_1, *toute variété* V *de classe* D_1, *distincte de* V_0, *mais de même contour, donne à* I *une valeur plus grande que l'extrémale, pourvu qu'elle appartienne à un voisinage faible suffisamment petit de* V_0 *et qu'elle se confonde avec* V_0 *hors d'un voisinage suffisamment petit de* P_0.

Ainsi se trouve établi, d'une manière générale, que pour le minimum faible local la condition d'HADAMARD au sens large (1. 2) est une condition nécessaire et que la même condition au sens strict (1. 4) est suffisante.

Mathematics. — *Ueber den Aussagen- und den engeren Prädikatenkalkül.* II. By J. RIDDER. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of December 21, 1946.)

Zusammenhang und Eigenschaften der beiden Axiomensysteme.

§ 6. Aus den Sätzen 1—4, 5bis, 6, 7, 12 einerseits und den Sätzen 30, 34—37, 39—41 andererseits, nebst den Definitionen, Satz 8, der Folgerung aus Satz 29 und Satz 31, folgt unmittelbar, *dass das Axiomensystem I—VI mit der Einsetzungsregel E und dem Schlussschema S völlig gleichwertig ist mit dem Axiomensystem* 1°—7° *mit der Einsetzungsregel E**. Beim ersten System sind das Produkt, die Implikation und die Inklusion abgeleitete Begriffe, beim zweiten System gilt das von der Implikation und der Assertion.

Das zweite Axiomensystem zeigt, dass der erweiterte Aussagenkalkül als eine Art BOOLE'scher Algebra zu betrachten ist, deren Elemente abzählbar unendlich viele sind, und in der nur die in ihren Axiomen enthaltenen Schlussschemata (und die aus diesen ableitbaren) erlaubt sind.

§ 7. *Die beiden einander gleichwertigen Axiomensysteme sind widerspruchsfrei* in dem Sinne, dass $X \doteq \bar{\nu}$ (oder $X = \nu$) in den Systemen nicht ableitbar ist.

Um diese Widerspruchsfreiheit einzusehen fügen wir den (nach Annahme endlich oder abzählbar unendlich vielen) elementaren Kalkülformeln $A, B, \ldots, A_1, \ldots, A_n, \ldots$ arithmetische Werte zu, welche nur 0 oder 1 sein können. $A + B$ ($\mathfrak{A} + \mathfrak{B}$) ordnen wir den Wert 0 zu, falls A ($\mathfrak{A}$) und B ($\mathfrak{B}$) der Wert 0 zugeordnet ist; sonst den Wert 1. A' ($\mathfrak{A}'$) sei der Wert 1 zugeordnet, falls A ($\mathfrak{A}$) 0 zugeordnet ist, und umgekehrt.

Welche Werte X, Y und Z auch zugeordnet sind, in allen Fällen ist den in den Axiomen I—IV im Vorderglied vorkommenden Kalkülformeln derselbe Wert (und zwar 1) zugeordnet.

Wendet man auf die in $\mathfrak{A} \doteq \bar{\nu}$ vorkommende Kalkülformel $\mathfrak{A}$ die Einsetzungsregel E an, wobei sie in $\mathfrak{B}$ übergehen soll, und ist $\mathfrak{A}$ immer der Wert 1 zugeordnet (wie auch die in ihr auftretenden elementaren Aussagen bewertet sind), so ist klar, dass dasselbe von $\mathfrak{B}$ gilt.

Schliesslich führt Anwendung des Schlussschemas S von $\mathfrak{A} \doteq \bar{\nu}$ und $(\mathfrak{A} \to \mathfrak{B}) \doteq \bar{\nu}$, wobei $\mathfrak{A}$ und $\mathfrak{A} \to \mathfrak{B}$ immer der Wert 1 zugeordnet sein soll, zu einem Theorem $\mathfrak{B} \doteq \bar{\nu}$, wobei auch $\mathfrak{B}$ dieser Wert zukommt.

Weist man noch λ den Wert 0, ν den Wert 1 zu, so liefern auch die Vorderglieder der Axiome V und VI immer den Wert 1.

Daraus folgt schon unsere Behauptung.

Ein zweites Beweisverfahren, welches das vorige als Spezialfall umfasst, ist folgendes. Wir betrachten eine Menge M von endlich vielen diskreten Elementen [14]). Den elementaren Mengen können als „Werte" die Teilmengen von M hinzugefügt werden (M selbst und die leere Menge mit eingeschlossen) [15]). Betrachten wir das zweite Axiomensystem. $A + B$ ($\mathfrak{A} + \mathfrak{B}$) ordnen wir den „Wert" $T_1 + T_2$ (Summe der Mengen T_1 und T_2) zu, falls A ($\mathfrak{A}$) die Menge T_1 und B ($\mathfrak{B}$) die Menge T_2 zugeordnet ist; $A \cdot B$ ($\mathfrak{A} \cdot \mathfrak{B}$) sei dann die Produktmenge $T_1 \cdot T_2$ zugeordnet. A' ($\mathfrak{A}'$) sei die Komplementärmenge $C(T)$ oder $M - T$ als „Wert" zugeordnet, falls A ($\mathfrak{A}$) T zugeordnet ist. $\mathfrak{A} \subset \mathfrak{B}$ soll dann gelten, wenn die $\mathfrak{A}$ zugeordnete Menge immer eine Teilmenge der $\mathfrak{B}$ zugeordneten Menge ist, wie auch den in $\mathfrak{A}$ und $\mathfrak{B}$ vorkommenden elementaren Kalkülformeln Teilmengen von M zugeordnet sind. Schliesslich sei λ die leere Menge, ν die Menge M selbst zugeordnet.

Es ist sofort klar, dass die Axiome 1°—7° sich in richtige Behauptungen über die Teilmengen von M übersetzen. Ist $\mathfrak{A} \subset \mathfrak{B}$ ableitbar, so ist jede $\mathfrak{A}$ zugeordnete Teilmenge von M Teil der $\mathfrak{B}$ gleichzeitig zugeordneten Menge; dies bleibt so für $\mathfrak{D} \subset \mathfrak{E}$, wenn diese Inklusion mittels der Einsetzungsregel E^* aus der ersten hervorgeht.

Daraus folgt, dass $X \doteq \bar{\nu}$ oder $X = \nu$ oder „$X \subset \nu$ und $\nu \subset X$" nicht ableitbar ist. Denn jeder „Wert" von X, welcher nicht gleich der vollen Menge M ist, gibt eine unrichtige Behauptung über endliche Mengen [16]).

Eine leichte Abänderung des zweiten Beweisverfahrens wird uns später nützlich sein. Man hätte für die elementaren Kalkülformeln als „Werte" nur die n ($\geqq 1$) aus einem einzelnen Punkt aufgebauten Teilmengen von M zulassen können; natürlich gibt es auch dann zu jeder Teilmenge T von M Kalkülformeln, welche T als „Wert" zulassen.

Das zweite Beweisverfahren zeigt, dass der erweiterte Aussagenkalkül, und somit auch der RUSSELL–WHITEHEAD'sche Aussagenkalkül, *ein 2^n-wertiger Kalkül* ist; der Namen von zweiwertigem Kalkül für den Kalkül von RUSSELL und WHITEHEAD ist somit nicht ganz zutreffend.

§ 8. *Jedes der Axiome I—VI ist, unter Annahme von Einsetzungsregel E und Schlussschema S, von den übrigen unabhängig.*

Axiom V lässt sich mit Regel E und Schlussschema S nicht aus den übrigen Axiomen ableiten; wäre dies wohl möglich, somit $(\lambda' + U) \doteq \bar{\nu}$

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[14]) Zum ersten Verfahren kommt man durch Betrachtung einer aus einem einzelnen Element aufgebauten Menge; diese Menge sei dann 1, die leere Menge 0 genannt.

[15]) Enthält M n Elemente, so ist ihre Anzahl 2^n.

[16]) Ausgehend vom ersten Axiomensystem genügen die im Texte angenommenen Zuordnungen, welche sich auf die Disjunktion, das Komplement, λ und ν beziehen. Für die Theoreme $\mathfrak{A} \doteq \bar{\nu}$ (und nur für diese) wird dann $\mathfrak{A}$ immer die volle Menge M zugeordnet sein. $\mathfrak{A} \subset \mathfrak{B}$, d.h. das Theorem $(\mathfrak{A}' + \mathfrak{B}) \doteq \bar{\nu}$, liefert dann wieder, dass bei jeder Zuordnung von „Werten" zu den elementaren Kalkülformeln die $\mathfrak{A}$ zugeordnete Menge eine Teilmenge der $\mathfrak{B}$ zugeordneten Menge sein wird.

ableitbar, so müsste, da die Rolle einer grossen lateinischen Buchstaben in Einsetzungsregel E und in den Axiomen I—IV, VI sich durch λ übernehmen lässt, auch $(X' + U) \doteq \bar{\nu}$ ableitbar sein; die Bewertung von § 7 Anfang zeigt jedoch, dass dies unmöglich ist (man ordne X den Wert 1, U den Wert 0 zu).

In analoger Weise lässt sich die Unabhängigkeit des Axioms VI von den übrigen Axiomen, Regel E und Schema S beweisen.

Dass jedes der Axiome I—IV, unter Annahme von Regel E und Schema S, von den übrigen und den Axiomen V und VI unabhängig ist, zeigen vier Bewertungen in H.–A., Kap. 1, § 13, falls man dabei noch für λ bzw. nur den Wert 1 (mod 4), 1,1 und 1, für ν bzw. nur den Wert 0 (mod 4), 0,0 und 0 zulässt.

Das Schlussschema S ist, unter Annahme der Einsetzungsregel E, von den Axiomen I—VI unabhängig. Alle Axiome enthalten nämlich Kalkülformeln der Gestalt $\mathfrak{A}' + \mathfrak{B}$, und Regel E liefert nur wieder Formeln dieser Art, so dass ein Theorem $(X + X') \doteq \bar{\nu}$ sich nicht ableiten lässt; dies wäre jedoch wohl der Fall, wenn das Schema S sich anwenden liesse.

Die Einsetzungsregel E ist von den Axiomen I—VI und dem Schlussschema S unabhängig. Denn das Schlussschema S allein liefert nicht die Möglichkeit zu einem Theorem zu gelangen, welches von den in den Axiomen enthaltenen verschieden ist.

§ 9. Ebenso wie im R.–W. Aussagenkalkül werden in unserem erweiterten Aussagenkalkül *die Theoreme* $\mathfrak{A} \doteq \bar{\nu}$ *dadurch charakterisiert, dass in der (immer vorhandenen) konjunktiven Normalform von* $\mathfrak{A}$ *in jeder Disjunktion mindestens eine elementare Aussage zugleich mit ihrem Komplement auftritt* [17]).

Der erweiterte Kalkül ist, ebenso wie der von R. u. W., *vollständig in dem Sinne, dass durch die Hinzufügung eines nicht ableitbaren Ausdrucks* $\mathfrak{A} \doteq \bar{\nu}$ (diese gehöre somit nicht zu den Theoremen im Sinne der Definition von § 1) *zu dem System von Axiomen stets* $X \doteq \bar{\nu}$ *ableitbar wird* [18]).

§ 10. Definition. $\mathfrak{A} \Rightarrow \mathfrak{B}$ ist eine andere Schreibweise von $\mathfrak{A}' \cdot \mathfrak{B}$.

$X \to Y$ und $X \Rightarrow Y$ stehen somit einander dual gegenüber gemäss dem Dualitätsprinzip (§ 3), α) [19]).

Aus diesem Dualitätsprinzip folgt:

Satz 42. *Ist* $\mathfrak{A} \doteq \bar{\nu}$ *ein Theorem im Sinne von* § 1, *so auch* $\mathfrak{B}' \doteq \bar{\nu}$; *dabei gehe Kalkülformel* $\mathfrak{B}$ *aus Kalkülformel* $\mathfrak{A}$ *dadurch hervor, dass:* α) $+$ *durch* $\cdot$, *und umgekehrt,* β) λ *durch* ν, *und umgekehrt,* γ) $\to$ *durch*

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[17]) Vergl. Hilbert–Ackermann, loc. cit. 1), S. 12, 31. Die Theoreme unseres erweiterten Aussagenkulküls, welche λ und ν nicht enthalten, und die Theoreme des R.–W. Aussagenkalküls bilden somit dieselbe Klasse.

[18]) Vergl. Hilbert–Ackermann, loc. cit. 1), S. 35.

[19]) Definiert man $X \Leftrightarrow Y$ als eine andere Schreibweise von $(X' \cdot Y) + (Y' \cdot X)$, so gilt dasselbe von $X \longleftrightarrow Y$ und $X \Leftrightarrow Y$.

$\rightarrow$, *und umgekehrt, ersetzt wird; ev. vorkommende Akzenten sollen ungeändert bleiben.*

Beweis. $\mathfrak{A} \doteq \bar{\nu}$ oder auch $\mathfrak{A} = \nu$, also nach dem Dualitätsprinzip: $\mathfrak{B} = \lambda$, oder $\mathfrak{B}' = \nu$ (Satz 10, 11, 15) oder $\mathfrak{B}' \doteq \bar{\nu}$.

Auch bei Beschränkung auf Theoreme, welche λ und ν nicht enthalten, d.h. bei Beschränkung auf die Theoreme des R.–W. Aussagenkalküls, führt Satz 42 über das in HILBERT–ACKERMANN, loc. cit. 1), Kap. 1, § 5 vorkommende Dualitätsprinzip hinaus [20]).

Das Axiomensystem 1°—7° mit Einsetzungsregel E^* steht dual gegenüber dem aus den Axiomen 1°, 2°, 3°, 5°, 6°, 7° und Satz 22 mit Einsetzungsregel E^* gebildeten, gleichwertigen System.

Das Axiomensystem I—VI mit Einsetzungsregel E und Schlussschema S steht dual gegenüber folgendem System, auf welchem sich der verallgemeinerte Aussagenkalkül ebenfalls aufbauen lässt.

Einsetzungsregel E_0, aus E hervorgehend durch Aenderung von $\mathfrak{R} + \mathfrak{S}$ in $\mathfrak{R} \cdot \mathfrak{S}$.

Schlussschema S_0. Sind

$$\mathfrak{A}' \doteq \bar{\nu} \text{ und } [\mathfrak{A} \rightarrow \mathfrak{B}]' \doteq \bar{\nu}$$

Theoreme, so ist auch

$$\mathfrak{B}' \doteq \bar{\nu}$$

ein Theorem.

Axiom I_0. $[(X \cdot X) \rightarrow X]' \doteq \bar{\nu}$.

Axiom II_0. $[X \rightarrow (X \cdot Y)]' \doteq \bar{\nu}$.

Axiom III_0. $[(X \cdot Y) \rightarrow (Y \cdot X)]' \doteq \bar{\nu}$.

Axiom IV_0. $[(Y \rightarrow Z) \rightarrow \{(X \cdot Y) \rightarrow (X \cdot Z)\}]' \doteq \bar{\nu}$.

Die undefinierten Grundverknüpfungen sind hier *und* und *nicht-(Komplement von)*, bzw. angedeutet durch $\cdot$ und ein Akzent; $\mathfrak{A} \rightarrow \mathfrak{B}$ wird definiert als eine andere Schreibweise von $\mathfrak{A}' \cdot \mathfrak{B}$. Die Definition eines Theorems weicht insofern von der Definition des § 1 ab, dass Regel E_0, Schema S_0 und die Axiome I_0—VI_0 Regel E, Schema S und die Axiome I—VI ersetzen sollen.

Axiom $V_0 \cdot [\nu \rightarrow X]' \doteq \bar{\nu}$.

Axiom $VI_0 \cdot [X \rightarrow \lambda]' \doteq \bar{\nu}$.

§ 11. Der erweiterte Aussagenkalkül ist als *Aussagenkalkül einer elementaren Wahrscheinlichkeitsrechnung* zu betrachten, und hat damit auch

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[20]) Das beweist man wie folgt. Es sei $(\mathfrak{A} \leftrightarrow \mathfrak{B}) \doteq \bar{\nu}$ ein Theorem, wobei $\mathfrak{A}$ und $\mathfrak{B}$ λ und ν nicht enthalten, und in $\mathfrak{A}$ und $\mathfrak{B}$ nur $+$, ., und das Akzent als Operationszeichen vorkommen, wie dies von HILBERT und ACKERMANN vorausgesetzt wird. Dann folgt aus Satz 42, dass $(\overline{\mathfrak{A}} \leftrightarrow \overline{\mathfrak{B}})' \doteq \bar{\nu}$ ein Theorem ist, wobei $\overline{\mathfrak{A}}$ und $\overline{\mathfrak{B}}$ aus $\mathfrak{A}$ bzw. $\mathfrak{B}$ durch Vertauschung von $+$ und . hervorgehen. Also ist $|[(\overline{\mathfrak{A}})' \cdot \overline{\mathfrak{B}}] + [(\overline{\mathfrak{B}})' \cdot \overline{\mathfrak{A}}]|' = \nu$, oder $|[\overline{\mathfrak{A}} + (\overline{\mathfrak{B}})'] \cdot [\overline{\mathfrak{B}} + (\overline{\mathfrak{A}})']| = \nu$, oder schliesslich $(\overline{\mathfrak{A}} \leftrightarrow \overline{\mathfrak{B}}) \doteq \bar{\nu}$; das ist das duale Theorem im Sinne von HILBERT und ACKERMANN.

als mehrwertiger Kalkül eine inhaltliche Deutung; den elementaren Aussagen A, B, ..., A_1, ..., A_n, ... (deren Anzahl nach Annahme endlich oder abzählbar unendlich sein soll) seien als „Werte" die n Mengen zugeordnet, deren jede aus einem einzelnen von n einander ausschliessenden elementaren Ereignissen besteht ($n \geqq 2$). λ sei die leere, also die keines der elementaren Ereignisse enthaltende Menge als einzig möglichen „Wert" hinzugefügt, ν möge die Menge aller elementaren Ereignisse als einzigen „Wert" haben. Ist die Menge p von elementaren Ereignissen einer der möglichen „Werte" von $\mathfrak{A}$, die Menge q von elementaren Ereignissen einer der möglichen „Werte" von $\mathfrak{B}$, so sei die Menge $p + q$ einer der möglichen „Werte" von $\mathfrak{A} + \mathfrak{B}$, und $p \cdot q$ einer der möglichen „Werte" von $\mathfrak{A} \cdot \mathfrak{B}$; die Menge der zu $\mathfrak{A}'$ gehörenden „Werte" enthalte alle und nur alle diejenigen Mengen von elementaren Ereignissen, deren Komplement in bezug auf die Menge aller n elementaren Ereignisse einen „Wert" von $\mathfrak{A}$ liefert.

Ein Theorem $\mathfrak{A} \doteq \bar{\nu}$ hat die inhaltliche Bedeutung, dass „Wert" von $\mathfrak{A}$ nur die Menge aller n elementaren Ereignisse ist, wie auch die in $\mathfrak{A}$ vorkommenden elementaren Kalkülformeln (zulässige) „Werte" zugeordnet sind.

Die in § 7 angedeutete Abänderung des zweiten Beweisverfahrens für die Widerspruchsfreiheit des erweiterten Aussagenkalküls zeigt unmittelbar die Möglichkeit obiger Interpretation.

Der erweiterte engere Prädikatenkalkül.

§ 12. Was den Prädikatenkalkül betrifft, können wir uns nach dem Vorigen kurz fassen.

Einen ersten Aufbau erhält man dadurch, dass man neben den Axiomen I—VI und Schlussschema S einführt: 1° eine Abänderung der Einsetzungsregel E (und somit nebenbei der Definition der Kalkülformeln) gemäss H.–A., S. 54 (1—5), mit der Hinzufügung, dass auch λ und ν Kalkülformeln sind, und S. 56, 57 (α 1), 2), 3)); 2° die Umbenennungsregel δ) in H.–A., S. 57; 3° die beiden folgenden Axiome: VII. $[(x)F(x) \to F(y)] \doteq \bar{\nu}$, VIII. $[F(y) \to (Ex)F(x)] \doteq \bar{\nu}$; 4° die Schlussschemata: $S_1 \cdot$ ist $[\mathfrak{A} \to \mathfrak{B}(x)] \doteq \bar{\nu}$, wobei die freie Variable x nur in $\mathfrak{B}$ vorkommt, ein Theorem, so auch $[\mathfrak{A} \to (x)\mathfrak{B}(x)] \doteq \bar{\nu}$, $S_2 \cdot$ ist $[\mathfrak{B}(x) \to \mathfrak{A}] \doteq \bar{\nu}$ ein Theorem, wobei die freie Variable x nur in $\mathfrak{B}$ vorkommt, so ist auch $[(Ex)\mathfrak{B}(x) \to \mathfrak{A}] \doteq \bar{\nu}$ ein Theorem.

Wie man *einen völlig aequivalenten Aufbau* durch Erweiterung des von den Axiomen 1°—7° und Einsetzungsregel E^* gebildeten Systems erhält, ist fast evident; der Wortlaut der Regel E^* in ihrer abgeänderten Form und der der neu hinzukommenden Umbenennungsregel liegen auf der Hand; als Axiome kommen hinzu: Ax. 8° $[(x)F(x)] \subset F(y)$, Ax. 9° $F(y) \subset (Ex)F(x)$, als Schlussschemata: $S_1^* \cdot$ lässt sich schreiben $\mathfrak{A} \subset [\mathfrak{B}(x)]$, wobei die freie Variable x nur in $\mathfrak{B}$ vorkommt, so auch

$\mathfrak{A} \subset [(x)\mathfrak{B}(x)]$, S_2^* · lässt sich schreiben $[\mathfrak{B}(x)] \subset \mathfrak{A}$, wobei die freie Variable x nur in $\mathfrak{B}$ vorkommt, so auch $[(Ex)\mathfrak{B}(x)] \subset \mathfrak{A}$.

Das Dualitätsprinzip behält seine Gültigkeit im erweiterten Prädikatenkalkül, wenn man zu den angegebenen Vertauschungen hinzufügt: δ) jedes Allzeichen ( ) soll durch ein Seinszeichen $(E\ )$, und umgekehrt, ersetzt werden.

Dadurch lässt sich Satz 42 ausbreiten zu:

Satz 42*. *Ist $\mathfrak{A} \doteq \bar{\nu}$ ein Theorem des erweiterten Prädikatenkalküls, so auch $\mathfrak{B}' \doteq \bar{\nu}$; dabei gehe Kalkülformel $\mathfrak{B}$ aus Kalkülformel $\mathfrak{A}$ dadurch hervor, dass: α) $+$ durch $\cdot$, und umgekehrt, β) λ durch ν, und umgekehrt, γ) $\rightarrow$ durch $=\!\!\rightarrow$, und umgekehrt, schliesslich: δ) jedes Allzeichen durch ein Seinszeichen, und umgekehrt, ersetzt wird.*

Auch bei Beschränkung auf Theoreme, welche λ und ν nicht enthalten, d.h. bei Beschränkung auf die Theoreme des (nicht erweiterten) engeren Prädikatenkalküls, führt Satz 42* über das in HILBERT–ACKERMANN, loc. cit. 1), Kap. 3, § 8 vorkommende Dualitätsprinzip hinaus [21]).

In dem erweiterten Prädikatenkalkül lässt sich jede Kalkülformel auf eine pränexe Normalform bringen, und gilt auch die durch den SKOLEM'schen Satz gelieferte Verschärfung dieses Resultates [22]).

§ 13. *Die beiden einander gleichwertigen Axiomensysteme des erweiterten Prädikatenkalküls sind widerspruchsfrei* in dem Sinne, dass $X \doteq \bar{\nu}$ (oder $X = \nu$) in den Systemen nicht ableitbar ist [23]).

Um die Widerspruchsfreiheit einzusehen fügen wir den (endlich oder abzählbar unendlich vielen) elementaren Kalkülformeln

$A, B, \ldots, A_1, \ldots, A_n, \ldots, A(x), \ldots, A_n(x), \ldots, A(x, y), \ldots, A_n(x, y), \ldots$

arithmetische Werte zu, welche nur 0 oder 1 sein können. Bei der Bewertung einer zusammengesetzten Kalkülformel werden Seins- und Allzeichen fortgelassen; $\mathfrak{A} + \mathfrak{B}$ ordnen wir den Wert 0 zu, falls $\mathfrak{A}$ und $\mathfrak{B}$ der Wert 0 zugeordnet ist; sonst den Wert 1. $\mathfrak{A}'$ sei der Wert 1 zugeordnet, falls $\mathfrak{A}$ 0 zugeordnet ist, und umgekehrt.

Welche Werte $X, Y, Z, F(x)$ auch zugeordnet sind, in allen Fällen ist den in den Axiomen I—IV, VII, VIII im Vorderglied vorkommenden Kalkülformeln der Wert 1 zugeordnet.

Die Einsetzungsregel und die Umbenennungsregel führen immer von Theoremen $\mathfrak{A} \doteq \bar{\nu}$, bei welchen $\mathfrak{A}$ immer der Wert 1 zugeordnet ist, zu Theoremen $\mathfrak{B} \doteq \bar{\nu}$, bei denen dasselbe von $\mathfrak{B}$ gilt.

Schliesslich führt das Schlussschema S_1 von $[\mathfrak{A} \rightarrow \mathfrak{B}(x)] \doteq \bar{\nu}$, wobei $\mathfrak{A} \rightarrow \mathfrak{B}(x)$ immer der Wert 1 zugeordnet ist, zu $[\mathfrak{A} \rightarrow (x)\mathfrak{B}(x)] \doteq \bar{\nu}$, wobei auch $\mathfrak{A} \rightarrow (x)\mathfrak{B}(x)$ immer dieser Wert zugeordnet ist; die Schemata S und S_2 haben die analogen Eigenschaften.

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[21]) Zum Beweise vergleiche man das Verfahren in Fussn. 20.

[22]) Siehe H.–A., loc. cit. 1), S. 67, 68.

[23]) Anwendung des Dualitätsprinzips führt zu zwei weiteren gleichwertigen Systemen. Vergl. auch § 10.

Dadurch dass man λ den Wert 0, ν den Wert 1 zuweist, liefern die Vorderglieder der Axiome V und VI immer den Wert 1.

Daraus folgt obige Behauptung.

Auch das zweite Beweisverfahren von § 7 oder die zugehörige Abänderung hätten sich hier anwenden lassen.

Sie zeigen: 1° *dass der erweiterte engere Prädikatenkalkül auch ein 2^n-wertiger Kalkül ist* (n eine natürliche Zahl), 2° *dass er sich intuitiv als engerer Prädikatenkalkül einer elementaren Wahrscheinlichkeitsrechnung deuten lässt* [24]).

§ 14. *Die Axiome I—VIII, die Einsetzungsregeln α 1), α 2), α 3), die Umbenennungsregel δ) und die Schlussschemata S, S_1 und S_2 sind voneinander unabhängig.*

Den Beweis können wir unterdrücken; man hat nur die Betrachtungen von § 8 Anfang und von H.–A., loc. cit. 1), Kap. 3, § 8 zu benutzen.

Das soeben genannte System von Axiomen u.s.w. ist auch vollständig im GÖDEL*'schen Sinne* [25]) ebenso wie das in H.–A., Kap. 3 für den engeren Prädikatenkalkül gegebene System; die Ableitung ist fast dieselbe [26]). Auch in dem erweiterten engeren Prädikatenkalkül lässt sich das Entscheidungsproblem lösen bei Kalkülformeln, welche nur einstellige Prädikate enthalten, sowohl wenn wir den Kalkül 2-wertig wie wenn wir ihn 2^n-wertig ($n \geqq 2$) auffassen [27]).

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[24]) Vergl. § 11.

[25]) Der hierbei angewandte Begriff der „identischen" Formel (siehe H.–A., loc. cit. 1), S. 55, 56) setzt Zweiwertigkeit der Aussagen und Prädikate voraus. Wie der Begriff zu verallgemeinern ist, wenn man n-wertige elementare Aussagen und -Prädikate (und dabei höchstens 2^n-wertige zusammengesetzte Aussagen und -Prädikate) zulässt (n eine natürliche Zahl $\geqq 2$), liegt auf der Hand (vgl. auch §§ 7, 11 u. 13); nennen wir identische Formeln im neuen Sinne (n)-identische Formeln, so bleiben die in H.–A., S. 78 kursiv gedruckten Resultate gültig bei Benutzung von (n)-identischen Formeln ($n \geqq 2$) an Stelle der identischen. Auch ist es möglich schon die elementaren Formeln des Prädikatenkalküls als 2^n-wertig aufzufassen ($n \geqq 2$) und den Begriff der identischen Formeln in Uebereinstimmung damit so zu modifizieren, dass das GÖDEL'sche Resultat erhalten bleibt.

[26]) Vergl. H.–A., loc. cit. 1), Kap. 3, § 10.

[27]) Vergl. H.–A., loc. cit. 1), Kap. 3, § 12.

Mathematics. — *On the theory of simultaneous linear and quadratic representation*. Part I. By F. VAN DER BLIJ. (Communicated by Prof. J. G. VAN DER CORPUT.)

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§ 1. *Introduction.*

We consider the system of Diophantic equations

$$\left.\begin{aligned} \sum_{i,k=1}^{r} s_{ik}\, x_i\, x_k &= n, & \mathbf{x}'\,\mathbf{S}\,\mathbf{x} &= n, \\ \sum_{i=1}^{r} \sigma_i\, x_i &= m, & \mathbf{x}'\,\mathbf{s} &= m, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1.1)$$

where the matrix S is a positive (definite), symmetric, integral one. Some special cases of this system have been considered long before now. We find in DICKSON's History of the Theory of Numbers [4][1]) Vol. II, p. 259 a numerical example given by REGIOMONTANUS (1436—1476). Here we will also remark that PALL considered some special cases in 1931 [12] and [13].

It is quite easy to eliminate one of the x_l from the equations. Then the linear equation may be replaced by a linear congruence. (Vid. KLOOSTERMAN [7] Zusatz bei der Korrectur.)

In some special cases it will be easy to eliminate this congruence, too. In order to obtain the number of the solutions of the system (1.1), we must calculate the number of the solutions of a quadratic Diophantic equation $\mathbf{y}'\,\mathbf{T}\,\mathbf{y} = N$.

From the arithmetic theory of the quadratic forms given by SIEGEL [14], we can deduce the average value of this number if T runs through a complete system of representatives of all classes of the genius of T. If the genius of T contains one class only, the number of the solutions is found at once.

We will be able to calculate the number of the solutions of

$$\left.\begin{aligned} \sum_{l=1}^{r} x_l^2 &= n, \\ \sum_{l=1}^{r} x_l &= m, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.2)$$

if $r = 3$[2]), 4, 5 or 8 using only well-known results concerning the re-

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[1]) In this way we refer to the bibliography at the end of this part. (pag. 40).

[2]) Vid. DE BRUIJN [3].

presentation of numbers by quadratic forms:

If $r = 3$ we consider $x^2 + 3y^2$ vid. DICKSON [5], pag. 80.
If $r = 4$ we consider $x^2 + y^2 + z^2$ vid. BACHMANN [1], pag. 137.
If $r = 5$ we consider $x^2 + y^2 + z^2 + 5t^2$ vid. LIOUVILLE [8].
If $r = 8$ we consider $x_1^2 + x_2^2 + \ldots + x_7^2$ vid. DICKSON [6].
If $r = 6$ or 7 we can calculate the number of the solutions using the theory of SIEGEL [14].

Thus the formulae first given by KLOOSTERMAN [7] and BRONKHORST [2] are deduced in a new, somewhat more simple way.

If $r = 9$ we will show that the number of the solutions can not be deduced with this theory.

We study the number of the representations of integers by sums of (generalized) polygon numbers, as it was done by STREEFKERK [15], in a new way, too.

At last we study the representation of integers by those cubic forms which can be written as a product of a linear and a quadratic form. In the cases of some quite special forms, of which $x^3 + y^3 + z^3 - 3xyz$ is the most simple example; these formulae are multiplicative ones.

At the end of this introduction I will express my grateful thanks to Dr. H. D. KLOOSTERMAN for his kind help by the compiling of this paper.

§ 2. *Matrices.*

We will use hereafter the well-known matrix notation. If the contrary is not expressely mentioned we will suppose the matrices to be *integral* ones, that is to say matrices of which all elements are rational integers. We denote these matrices by fat types; matrices of one column only are called vectors and are denoted by fat small letters.

The transposed of a matrix A is denoted by A′. The transposed of a column a is a row a′. If the matrix S is a square one the product $\mathbf{x}' \mathbf{S} \mathbf{x}$ is a quadratic form and $\mathbf{x}' \mathbf{s}$ is a linear form in the elements of $\mathbf{x}$. The scalar product $\mathbf{a}'\mathbf{b} = \mathbf{b}'\mathbf{a}$ is a number, the product $\mathbf{a}\,\mathbf{b}'$ is a square matrix of r rows and r columns (or of degree r), if the vectors a and b contain each r elements.

The letters N and n are only used for matrices (vectors) of which all elements vanish. The letter E will denote the unity matrix. The matrix which we obtain by writing the columns of a matrix B at the righthand side of those of the matrix A is denoted by (A B). Analogical meanings have the expressions

$$\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix} \text{ and } \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}.$$

Henceforth the letter S is restricted to symmetric positive matrices of degree r and the determinant of these matrices is denoted by S. The letter s is restricted to a *primitive* vector of r elements, that is to say a vector such that its elements have no common divisor except 1. A vector of which all elements are equal 1 is denoted by e.

Further we denote

$$\mathfrak{S}^* = \begin{bmatrix} \mathfrak{S} & \mathfrak{s} \\ \mathfrak{s}' & 0 \end{bmatrix};$$

thus

$$S^* = |\mathfrak{S}^*| = -\mathfrak{s}'\,\bar{\mathfrak{S}}\,\mathfrak{s}, \text{ where } \bar{\mathfrak{S}}\,\mathfrak{S} = S\,\mathfrak{E}. \quad . \quad . \quad . \quad . \quad (2.1)$$

§ 3. *Arithmetics.*

Let N be a positive rational integer, a and b be rational integers, p be a prime, and α be a non negative integer, we use the following symbols:

$a \mid N$	a is a divisor of N.
$a \nmid N$	a is not a divisor of N.
$p^\alpha \parallel N$	p^α is the highest power of p divising N, thus $p^\alpha \mid N$ and $p^{\alpha+1} \nmid N$.
(a, b)	the greatest common divisor of a and b.
$a \equiv b \pmod N$ $a \equiv b \ (N)$	the numbers (vectors) a and b are congruent to the modulus N that is to say their difference is divisible by N.
$\sum_{x \bmod N}$	the sum over x runs through a complete system of residues modulo N.
$\sum'_{x \bmod N}$	the sum over x runs through a reduced system of residues modulo N (only those x satisfying $(x, N) = 1$).
$\sum_{\substack{x=1 \\ x \equiv a\,(N)}}^{\infty}$	the sum over x runs through all positive integers which satisfy the congruence $x \equiv a \pmod N$.
$\prod_{p}$	the product must be extended over all prime numbers p.
$\prod_{p \mid N}$	the product must be extended over all prime divisors p of N.
$\prod_{p \nmid N}$	the product must be extended over all prime numbers p which are not a divisor of N.
$\left(\frac{a}{b}\right)$	is the well-known symbol of JACOBI from the theory of quadratic residues, if b is odd; $\left(\frac{a}{b}\right) = 0$ if $(a, b) > 1$.

We will use the abbreviations:

$$e(x) = e^{2\pi i x};$$

$$I_n = i^{\frac{(N-1)^2}{4}}, \text{ if } N \text{ is odd.}$$

We also use the following formulae concerning Gaussian sums and Ramanuyan sums:

$$\sum_{x \bmod N} e\left(\frac{a x^2}{N}\right) = \left(\frac{a}{N}\right) I_n \sqrt{N} \quad \text{if } N \text{ is odd and } (a, N) = 1 \ldots \quad (3.1)$$

$$\sum_{x \bmod 2^k} e\left(\frac{a x^2}{2^k}\right) = \left.\begin{array}{ll} 0 & \text{if } k = 1 \\ e\left(\frac{a}{8}\right)\left(\frac{2}{a}\right)^{k+1} 2^{\frac{k+1}{2}} & \text{if } k \geqq 2 \end{array}\right\} \text{ and } a \text{ is odd.} \quad (3.2)$$

$$\sum_{x \bmod p^\lambda}{}' e\left(\frac{a x}{p^\lambda}\right) = \left.\begin{array}{ll} 0 & \text{if } p^\mu \| a \text{ and } \mu < \lambda - 1; \\ -p^{\lambda-1} & \text{if } p^\mu \| a \text{ and } \mu = \lambda - 1; \\ p^\lambda - p^{\lambda-1} & \text{if } p^\mu \| a \text{ and } \mu > \lambda - 1. \end{array}\right\} \quad (3.3)$$

At last we consider if p is an odd prime and $(a, p) = 1$:

$$\sum_{x \bmod p^\lambda} \left(\frac{x}{p}\right) e\left(\frac{a x}{p^\lambda}\right). \quad (3.4)$$

This sum vanishes unless $\lambda = 1$.
The sum

$$\sum_{x \bmod 2^\lambda} \left(\frac{2}{x}\right) e\left(\frac{a x}{2^\lambda}\right), \quad (3.5)$$

where $(a, 2) = 1$, vanishes unless $\lambda = 3$.

§ 4. *First general theory.*

We consider the following system of Diophantic equations

$$\left.\begin{array}{l} \mathbf{x}' \mathrm{S}\, \mathbf{x} = n, \\ \mathbf{x}'\, \mathbf{s} \quad = m, \end{array}\right\} \quad (4.1)$$

where n and m are given integers of which n is a positive one. We may replace (4.1) by, $(\lambda \neq 0)$,

$$\left.\begin{array}{l} \lambda\, \mathbf{x}' \mathrm{S}\, \mathbf{x} + \mu\, \mathbf{x}'\, \mathbf{s}\, \mathbf{s}'\, \mathbf{x} = \lambda n + \mu m^2, \\ \mathbf{x}'\, \mathbf{s} = m. \end{array}\right\} \quad (4.2)$$

Now we chose λ and μ in such a way that the matrix $\lambda \mathrm{S} + \mu\, \mathbf{s}\, \mathbf{s}'$ is a singular one. It is quite obvious that the vector $\mathbf{x} = \overline{\mathrm{S}}\, \mathbf{s}$ must then satisfy the equation

$$\lambda\, \mathbf{x}' \mathrm{S}\, \mathbf{x} + \mu\, \mathbf{x}'\, \mathbf{s}\, \mathbf{s}'\, \mathbf{x} = 0,$$

so we can chose $\lambda = S^*$ and $\mu = S$.
So (4.2) can be written as

$$\left.\begin{array}{l} S^*\, \mathbf{x}' \mathrm{S}\, \mathbf{x} + S\, \mathbf{x}'\, \mathbf{s}\, \mathbf{s}'\, \mathbf{x} = S^* n + S m^2, \\ \mathbf{s}'\, \mathbf{x} = m. \end{array}\right\} \quad (4.3)$$

The matrix $S^* \mathrm{S} + S\, \mathbf{s}\, \mathbf{s}'$ has rank $r - 1$.

Let now $\bar{S}\,\mathfrak{s} = \delta\,\mathfrak{r}$ with a posive integer δ and a primitive vector $\mathfrak{r}$. Then there exists a matrix V_1 such that the square matrix $V = (V_1\,\mathfrak{r})$ has determinant unity.

Replacing $\mathfrak{x}$ by $V_1\,\mathfrak{y} + \mathfrak{r}\,\zeta$ in (4. 3) we get

$$S^*\,\mathfrak{y}'\,V_1'\,S\,V_1\,\mathfrak{y} + S\,\mathfrak{y}'\,V_1'\,\mathfrak{s}\,\mathfrak{s}'\,V_1\,\mathfrak{y} = S^* n + S\,m^2 \quad . \quad . \quad (4.\,4^1)$$

$$\mathfrak{y}'\,V_1'\,\mathfrak{s} - \zeta\,\delta^{-1}\,S^* = m. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.\,4^2)$$

We can deduce an integral value ζ from (4. 4^2) if and only if the congruence

$$\mathfrak{y}'\,V_1'\,\mathfrak{s} \equiv m \pmod{\delta^{-1}\,S^*} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.\,5)$$

is satisfied.

We deduce from (4. 4^1)

$$S\,(\mathfrak{y}'\,V_1'\,\mathfrak{s})^2 \equiv S\,m^2 \pmod{S^*},$$

if we denote $\delta_0 = (S, S^*)$ this becomes

$$(\mathfrak{y}'\,V_1'\,\mathfrak{s})^2 \equiv m^2 \pmod{\delta_0^{-1}\,S^*}. \quad . \quad . \quad . \quad . \quad . \quad (4.\,6)$$

Henceforth we shall suppose $\delta = \delta_0$.

Now we consider under what suppositions we may deduce the congruence (4. 5) from (4. 6). In the following special cases it is quite easy:

I. $\delta^{-1}\,S^*$ is either a power of a prime or the double of a power of a prime, and m is not divisible by this prime; the number of the solutions of the congruence (4. 5) is half the number of the solutions of the congruence (4. 6).

II. $\delta^{-1}\,S^*$ is either prime or the double of a prime and m is divisible by this prime. Then all solutions of the congruence (4. 6) do satisfy the congruence (4. 5), too.

§ 5. *Second general theory.*

We consider once more the quadratic equation of (4. 3).

The quadratic form $\mathfrak{x}'\,(S^*\,S + S\,\mathfrak{s}\,\mathfrak{s}')\,\mathfrak{x}$ can be written as the sum of $r - 1$ squares of integral linearly independant linear forms, each multiplied with a rational number. We use the following lemma (vid. E. PASCAL Repertorium der Höheren Analysis I 1, pag. 120).

If the symmetric matrix $Q^{(r)} = (q_{ik})$ is of rank $r - 1$, we have

$$\sum_{i,k=1}^{r} q_{ik}\,x_i\,x_k = \sum_{i=1}^{r-1} \frac{y_i^2}{Q_{i-1}\,Q_i}, \quad . \quad . \quad . \quad . \quad . \quad (5.\,1)$$

where

$$y_i = \sum_{k=1}^{r-1} Q_{i-1}^{(i,k)}\,x_k. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.\,2)$$

Here we wrote by abbreviation:

$$Q_l = \begin{vmatrix} q_{11} & \dots & q_{1l} \\ \vdots & & \vdots \\ q_{l1} & \dots & q_{ll} \end{vmatrix}; \qquad Q_l^{(k,e)} = \begin{vmatrix} q_{11} & \dots & q_{1l} & q_{1e} \\ \vdots & & \vdots & \vdots \\ q_{l1} & \dots & q_{ll} & q_{le} \\ q_{k1} & \dots & q_{kl} & q_{ke} \end{vmatrix} \quad . \quad (5.3)$$

$$Q_0 = 1 \qquad ; \qquad Q_l^{(k,e)} = q_{ke} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.4)$$

To each solution x of (4. 1) corresponds an integral solution y of

$$\sum_{l=1}^{r-1} \frac{y_l^2}{Q_{l-1}\, Q_l} = S^* n + S\, m^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.5)$$

Those integral solutions of (5. 5), such that integral numbers $x_1, x_2, \dots x_r$ can be found satisfying

$$\left.\begin{aligned} \sum_{k=1}^{r} Q_{l-1}^{(l,k)} x_k = y_l, \qquad & \text{where } l = 1, 2, \dots r-1, \\ \mathbf{x}'\,\mathbf{s} = m; \qquad & \end{aligned}\right\} \quad . \quad . \quad (5.6)$$

correspond with a solution of the Diophantic system (4. 1).

§ 6. *Special systems.*

Before considering a quite special system, we will now deduce how much the formulae of the preceding paragraphs can be simplified if we consider the special system:

$$\left.\begin{aligned} \sum_{l=1}^{r} s_l\, x_l^2 = n, \\ \sum_{l=1}^{r} s_l\, x_l = m. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.1)$$

By abbreviation we write

$$\sigma = \sum_{l=1}^{r} s_l.$$

First we consider the formulae of § 4.
We may write

$$\bar{\mathbf{S}}\,\mathbf{s} = S\,\mathbf{e}$$

so $\delta = S$ and we may chose

$$\mathbf{V}_1 = \begin{pmatrix} \mathbf{E} \\ \mathbf{n} \end{pmatrix}.$$

Further we have

$$S^* = -S\,\sigma.$$

The formulae (4. 4[1]) and (4. 5) can be written as

$$\sigma \sum_{i=1}^{r-1} s_i y_i^2 - \Big(\sum_{i=1}^{r-1} s_i y_i\Big)^2 = \sigma n - m^2 \quad . \quad . \quad . \quad . \quad . \quad (6.2)$$

and

$$\sum_{i=1}^{r-1} s_i y_i \equiv m \pmod{\sigma}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.3)$$

The formulae of § 5 become for the system (6. 1)

$$Q_{i-1}^{(i,k)} = S^i (-\sigma)^{i-1} s_1 s_2 \ldots s_i, s_k; \quad . \quad . \quad . \quad . \quad . \quad (6.4)$$

and

$$Q_i = -S^i (-\sigma)^{i-1} s_1 s_2 \ldots s_i (s_{i+1} + s_{i+2} + \ldots + s_r).$$

Thus

$$y_i = -S^i (-\sigma)^{i-1} s_1 s_2 \ldots s_i \Big\{(s_{i+1} + \ldots + s_r) x_i - \sum_{k=i+1}^{r} s_k x_k\Big\}.$$

In order to simplify these formulae we write

$$y_i = S^i (-\sigma)^{i-1} s_1 s_2 \ldots s_i z_i.$$

The form at the righthand side of (6. 1) is now

$$\sum_{i=1}^{r-1} \frac{-S \sigma s_i z_i^2}{(s_i + s_{i+1} + \ldots + s_r)(s_{i+1} + \ldots + s_r)}$$

and the equation (5. 5) can be written as:

$$\sum_{i=1}^{r-1} \frac{s_i z_i^2}{(s_i + \ldots s_r)(s_{i+1} + \ldots s_r)} = \frac{\sigma n - m^2}{\sigma} \quad . \quad . \quad . \quad . \quad (6.5)$$

From (5. 6) we deduce the following congruences to be satisfied by z_i in order to obtain a corresponding integral solution of (6. 1):

$$\left.\begin{aligned} &m \equiv z_1 \pmod{\sigma}, \\ &z_i \equiv z_{i+1} \pmod{s_{i+1} + \ldots + s_r}, \text{ where } i = 1, 2, \ldots r-2. \\ &z_{r-1} \equiv 0 \pmod{s_r}. \end{aligned}\right\} \quad (6.6)$$

§ 7. *Some quite special cases.*

In this paragraph we restrict us to the system

$$\left.\begin{aligned} \sum_{i=1}^{r} x_i^2 &= n, \\ \sum_{i=1}^{r} x_i &= m, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.1)$$

where $r = 2^\lambda$.

The trivial identity

$$2(x_0^2 + x_1^2) = (x_0 + x_1)^2 + (x_0 - x_1)^2 \quad . \quad . \quad . \quad . \quad . \quad (7.2)$$

is the first one in a series of analogical identities in 2^λ variables $x_0, x_1, \ldots x_{2^\lambda - 1}$, where $\lambda = 1, 2 \ldots$.

The next one is

$$\left.\begin{aligned} 4(x_0^2 + x_1^2 + x_2^2 + x_3^2) &= (x_0 + x_1 + x_2 + x_3)^2 + \\ + (x_0 - x_1 + x_2 - x_3)^2 &+ (x_0 + x_1 - x_2 - x_3)^2 + (x_0 - x_1 - x_2 + x_3)^2. \end{aligned}\right\} \quad (7.3)$$

In order to find a general formula we introduce the matrix notation. The identity

$$2^\lambda \sum_{i=0}^{2^\lambda - 1} x_i^2 = \sum_{i=0}^{2^\lambda - 1} y_i^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.4)$$

where $\mathbf{y} = \mathbf{V}\mathbf{x}$, can be written as

$$2^\lambda \mathbf{E} = \mathbf{V}' \mathbf{E} \mathbf{V}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.5)$$

The elements v_{ik} of $\mathbf{V}$ are all ± 1, in order to determine them we consider the two-adic representation of the indices i and k. Let

$$i = \sum_{\sigma=0}^{\lambda-1} i_\sigma 2^\sigma, \quad k = \sum_{\sigma=0}^{\lambda-1} k_\sigma 2^\sigma, \quad (\text{here } i, k = 0, 1, \ldots 2^\lambda - 1) \quad . \quad (7.6)$$

where the numbers i_σ and k_σ are either 0 or 1.

We define

$$v_{ik} = (-1)^{\sum_{\sigma=0}^{\lambda-1} i_\sigma k_\sigma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.7)$$

Now we must prove

$$\mathbf{V}' \mathbf{V} = 2^\lambda \mathbf{E},$$

that is to say we must calculate the sum

$$\sum_{k=0}^{2^\lambda - 1} v_{ik} v_{kj} = \sum_{\varrho=0}^{\lambda-1} \sum_{k_\varrho=0}^{1} \prod_{\sigma=0}^{\lambda-1} (-1)^{k_\sigma (i_\sigma + j_\sigma)}.$$

If there exists a τ with $i_\tau \neq j_\tau$, thus with $i_\tau + j_\tau = 1$, we have

$$\sum_{k=0}^{2^\lambda - 1} v_{ik} v_{kj} = \sum_{\substack{\varrho=0 \\ \varrho \neq \tau}}^{\lambda-1} \sum_{k_\varrho=0}^{1} (1 + (-1)^{i_\tau + j_\tau}) \prod_{\substack{\sigma=0 \\ \sigma \neq \tau}}^{\lambda-1} (-1)^{k_\sigma (i_\sigma + j_\sigma)},$$

and

$$\sum_{k=0}^{2^\lambda - 1} v_{ik} v_{kj} = 0.$$

If such a τ does not exist we have $i = j$ and

$$\sum_{k=0}^{2^\lambda - 1} v_{ik} v_{ki} = 2^\lambda.$$

So we proved the matrix $W = (w_{ik})$ with $w_{ik} = 2^{-\frac{\lambda}{2}} v_{ik}$ to be an orthogonal one.

Now we consider the equation

$$\sum_{i=1}^{2^\lambda - 1} y_i^2 = 2^\lambda n - m^2. \quad \ldots\ldots\ldots \quad (7.8)$$

With each integral solution of the system (7. 1) corresponds an integral solution of (7. 8). With each integral solution of the equation (7. 8) which satisfies the congruences

$$\mathbf{V}\,\mathbf{y} \equiv 0 \pmod{2^\lambda}, \quad \text{where } y_0 = m, \quad \ldots\ldots \quad (7,9)$$

corresponds an integral solution of the system (7. 1).

§ 8. *Necessary conditions.*

We shall try to find the necessary and sufficient conditions to be satisfied in order to obtain a real solution of the equations

$$\left.\begin{aligned} \mathbf{x}'\,\mathbf{S}\,\mathbf{x} &= n, \\ \mathbf{x}'\,\mathbf{S} &= m. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (8.1)$$

This condition is $\Delta = S^* n + S m^2 \leqslant 0$.

The matrix S being positive we have $S^* = -\mathbf{s}'\,\mathbf{S}\,\mathbf{s} < 0$, and for every real solution $\mathbf{x}$ of (8. 1)

$$S^* \left(\mathbf{x} + m \frac{\overline{\mathbf{S}}\,\mathbf{s}}{S^*}\right)' \mathbf{S} \left(\mathbf{x} + m \frac{\overline{\mathbf{S}}\,\mathbf{s}}{S^*}\right) \leqslant 0.$$

Thus

$$S^*\,\mathbf{x}'\,\mathbf{S}\,\mathbf{x} + 2\,S\,m\,\mathbf{x}'\,\mathbf{s} - m^2\,S \leqslant 0,$$

$$S^* n + S m^2 \leqslant 0 \quad \ldots\ldots\ldots \quad (8.2)$$

If $\Delta = 0$, there is one real solution only, which is given by $\mathbf{x} = -m \frac{\overline{\mathbf{S}}\,\mathbf{s}}{S^*}$.

Henceforth we shall suppose $\Delta < 0$.

If we have for every i in the interval $1 \leqslant i \leqslant r$ the congruence

$$s_{ii} \equiv \sigma_i \pmod 2, \quad \text{where } \mathbf{S} = (s_{ik}) \text{ and } \mathbf{s} = (\sigma_i); \quad \ldots \quad (8.3)$$

it may be seen readily that a necessary condition to be satisfied in order to obtain an integral solution of the system (8. 1) is given by

$$n \equiv m \pmod 2; \quad \ldots\ldots\ldots \quad (8.4)$$

since

$$n = \mathbf{x}'\,\mathbf{S}\,\mathbf{x} = \sum_{i,k} s_{ik}\,x_i\,x_k \equiv \sum_i s_{ii}\,x_i^2 \equiv \sum_i s_{ii}\,x_i \pmod 2$$

and

$$m = \mathbf{x}'\,\mathbf{s} = \sum_i \sigma_i\,x_i.$$

BIBLIOGRAPHY.

1. P. BACHMANN, Die Arithmetik der quadratischen Formen. I. (Leipzig 1898).
2. P. BRONKHORST, Over het aantal oplossingen van het stelsel Diophantische vergelijkingen $\sum_{i=1}^{s} x_i^2 = n$, $\sum_{i=1}^{s} x_i = m$ voor $s = 6$ en $s = 8$. Thesis, Groningen 1943.
3. N. G. DE BRUIJN, Over het aantal oplossingen van het stelsel
$$x^2 + y^2 + z^2 = n,\ x + y + z = m.$$
Nieuw Archief voor Wisk. (2) **22**, 53 (1943).
4. L. E. DICKSON, History of the Theory of Numbers. (Carnegie Institution), Vol. I (1919), II (1920), III 1923).
5. L. E. DICKSON, Introduction to the theory of numbers (Chicago 1929).
6. L. E. DICKSON, Studies in the theory of numbers (Chicago 1939).
7. H. D. KLOOSTERMAN, Simultane Darstellung zweier ganzen Zahlen als einer Summe von ganzen Zahlen und deren Quadratsumme. Math. Ann. **116**, 319—364 (1942).
8. LIOUVILLE, Sur la forme $x^2 + y^2 + z^2 + 5t^2$; Sur la forme $x^2 + 5(y^2 + z^2 + t^2)$. Journal de Math. (2) **9**, 1—12 and 19—22 (1864).
9. W. MAGNUS, Ueber die Anzahl der in einen Geschlecht enthaltenen Klassen von positiv-definite quadratische Formen. Math. Ann. **114**, 465 (1937).
10. H. MINKOWSKI, Grundlagen für eine Theorie der quadratische Formen mit ganzzahlige Koeffizienten. Gesammelte Abhandlungen I. Nr I, S. 3—144.
11. H. MINKOWSKI, Untersuchungen über quadratische Formen. Bestimmung der Anzahl verschiedener Formen, welche ein gegebenes Genus enthalt. Inaugaral Diss. Gesammelte Abhandlungen I. Nr. IV, S. 157—202.
12. G. PALL, Simultaneous quadratic and linear representation. Quart. Journal Math. Oxford series **2**, 136—143 (1931).
13. G. PALL, Simultaneous representation in a quadratic and linear form. Duke Math. Journ. **8**, 173—180 (1941).
14. C. L. SIEGEL, Analytische Theorie der quadratische Formen. I. Annals of Math. (2) **36**, 527—606 (1935).
15. H. STREEFKERK, Over het aantal oplossingen van de Diophantische vergelijking
$$U = \sum_{i=1}^{s} (A x_i^2 + B x_i + C).$$
Thesis, Amsterdam (V.U.) 1943.

Mathematics. — *On the theory of simultaneous linear and quadratic representation.* Part II. By F. VAN DER BLIJ. (Communicated by Prof. J. G. VAN DER CORPUT.)

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§ 9. *The binary system and some abbreviations.*

In this part we shall develop the results of the preceding § 7, if $r = 2^\lambda$ and $\lambda = 1, 2$ or 3.

The case $\lambda = 1$ is quite trivial. The system

$$\left.\begin{aligned} x_0^2 + x_1^2 &= n, \\ x_0 + x_1 &= m, \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (9.1)$$

has 2, 1 or 0 solutions in rational integers, according to $2n - m^2$ is a positive square, $2n - m^2 = 0$ or $2n - m^2$ is not a square of a rational integer. In order to obtain analogeous formulae for the number of the solutions of the system (1.2) if $r = 4$, 6 or 8, we introduce now the following abbreviations:

$$\mu = r - 3,$$
$$\Delta = rn - m^2,$$
$$2^\alpha \parallel \Delta,$$
$$\Delta = 2^\alpha \Delta_3 \Delta_4 = 2^\alpha \Delta_1, \text{ where}$$
$$\Delta_3 = p_1^{2\alpha_{p_1}-1} p_2^{2\alpha_{p_2}-1} \ldots \qquad \alpha_{p_l} > 0;$$
and
$$\Delta_4 = q_1^{2\beta_{q_1}} q_2^{2\beta_{q_2}} \ldots \qquad \beta_{q_l} > 0;$$

and the numbers p_l and q_l are primes.

To each prime divisor p of Δ_3 we define $A_p = (1 - p^{-\mu\alpha_p})\ (1 - p^{-\mu})^{-1}$.

To each prime divisor q of Δ_4 we define

$$B_q = (1 - q^{-\mu\beta_q})\ (1 - q^{-\mu})^{-1} \left(1 - \left(\frac{D}{q}\right) q^{-\frac{\mu+1}{2}}\right) + q^{-\mu\beta_q}.$$

At last we define D from $\Delta = (-1)^{\frac{\mu+1}{2}}\ \Delta_4 D$.

Hereafter we will suppose the congruence

$$n \equiv m \pmod 2$$

to be satisfied.

§ 10. *The quaternary system.*

Now we consider $\lambda = 2$ thus

$$\left.\begin{aligned} x_0^2 + x_1^2 + x_2^2 + x_3^2 &= n, \\ x_0 + x_1 + x_2 + x_3 &= m. \end{aligned}\right\} \quad \ldots\ldots \quad (10.1)$$

We may replace this system by

$$y_1^2 + y_2^2 + y_3^2 = 4n - m^2, \quad \ldots\ldots \quad (10.2)$$

accompanied with the following congruences to the modulus 4,

$$\begin{aligned} y_1 + y_2 + y_3 + m &\equiv 0, \\ y_1 - y_2 + y_3 - m &\equiv 0, \\ y_1 + y_2 - y_3 - m &\equiv 0, \\ y_1 - y_2 - y_3 + m &\equiv 0. \end{aligned}$$

We replace these by

$$\left.\begin{aligned} y_1 + y_2 + y_3 + m &\equiv 0 \pmod 4, \\ y_1 \equiv y_2 \equiv y_3 &\equiv m \pmod 2. \end{aligned}\right\} \quad \ldots\ldots \quad (10.3)$$

Now we deduce from

$$m^2 + y_1^2 + y_2^2 + y_3^2 = 4n$$

immediately

$$m \equiv y_1 \equiv y_2 \equiv y_3 \pmod 2.$$

If m and n are even, we can deduce from

$$m^2 + y_1^2 + y_2^2 + y_3^2 \equiv 0 \pmod 8$$

that

$$\left(\frac{m}{2}\right)^2 + \left(\frac{y_1}{2}\right)^2 + \left(\frac{y_2}{2}\right)^2 + \left(\frac{y_3}{2}\right)^2 \equiv 0 \pmod 2$$

and

$$\frac{m}{2} + \frac{y_1}{2} + \frac{y_2}{2} + \frac{y_3}{2} \equiv 0 \pmod 2$$

or

$$m + y_1 + y_2 + y_3 \equiv 0 \pmod 4.$$

If m and n are odd, we deduce that either $m + y_1 + y_2 + y_3 \equiv 0$ or $m - y_1 - y_2 - y_3 \equiv 0 \pmod 4$.

Of each pair solutions of (10.2) just one satisfies the congrences (10.3).

Thus we found:

Theorem 1. *The number of the solutions of the system* (10.1) *equals if* $n \equiv m \equiv 0$ (2) *the number* $A_3(\Delta)$ *of the representations of* Δ *as a sum of three squares, if* $n \equiv m \equiv 1$ (2) *it equals* $\frac{1}{2} A_3(\Delta)$.

For the number $A_3(\triangle)$, for which are given many formulae depending on the class number of binary quadratic forms, can be given the following formula (deduced from SIEGEL [14], using MAGNUS [9])

$$A_3(\triangle)=\frac{16}{\pi}\,\alpha_2^{(3)}(\triangle)\,\triangle^{\frac{1}{2}}\prod_{p\mid\triangle_3}A_p\prod_{q\mid\triangle_4}B_q\sum_{\substack{k=1\\(k,2D)=1}}^{\infty}\left(\frac{D}{k}\right)k^{-1}; \quad . \quad (10.4)$$

where

$$\alpha_2^{(3)}(\triangle)=\varkappa\,2^{-\left[\frac{\alpha}{2}\right]-1},$$

and

$\varkappa=3$ if $\alpha\equiv 0\,(2)$ and $\triangle_1\equiv 1\,(4)$ or $\alpha\equiv 1\,(2)$.
$\varkappa=2$ if $\alpha\equiv 0\,(2)$ and $\triangle_1\equiv 3\,(8)$.
$\varkappa=0$ if $\alpha\equiv 0\,(2)$ and $\triangle_1\equiv 7\,(8)$.

§ 11. *The system in eight variables.*

If $\lambda=3$ we consider the system

$$\left.\begin{aligned}x_0^2+x_1^2+\ldots+x_7^2&=n,\\ x_0+x_1+\ldots+x_7&=m.\end{aligned}\right\}\quad .\quad .\quad .\quad .\quad .\quad .\quad (11.1)$$

We replace this by the equation

$$y_1^2+y_2^2+\ldots+y_7^2=8n-m^2. \quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad (11.2)$$

With the following congruences modulo 8:

	m		y_1		y_2		y_3		y_4		y_5		y_6		y_7
+		+		+		+		+		+		+		+	
+		−		+		−		+		−		+		−	
+		+		−		−		+		+		−		−	
+		−		−		+		+		−		−		+	
+		+		+		+		−		−		−		−	
+		−		+		−		−		+		−		+	
+		+		−		−		−		−		+		+	
+		−		−		+		−		+		+		−	

$\equiv 0\,(\mathrm{mod}\,8)$. (11.3)

These congruences can be replaced by

$$\left.\begin{aligned}&m+y_1+y_2+y_3+y_4+y_5+y_6+y_7\equiv 0\,(\mathrm{mod}\,8),\\ &m+y_2+y_4+y_6\equiv 0\,(\mathrm{mod}\,4),\\ &m+y_3+y_4+y_7\equiv 0\,(\mathrm{mod}\,4),\\ &m+y_1+y_6+y_7\equiv 0\,(\mathrm{mod}\,4),\\ &m\equiv y_1\equiv y_2\equiv y_3\equiv y_4\equiv y_5\equiv y_6\equiv y_7\,(\mathrm{mod}\,2).\end{aligned}\right\}\quad .\quad (11.4)$$

It may be seen readily that all the solutions of the equation (11.2) satisfy the last congruence to modulus 2.

Now we proceed systematically.

Case I.

If m is even, we introduce $m = 2\mu$ and $y_i = 2\eta_i$.

Then we consider the solutions of

$$\eta_1^2 + \eta_2^2 + \ldots + \eta_7^2 = 2n - \mu^2; \quad \ldots\ldots \quad (11.5)$$

which satisfy the congruences

$$\begin{aligned} \mu + \eta_1 + \eta_2 + \ldots + \eta_7 &\equiv 0 \pmod 4, \\ \mu + \eta_2 + \eta_4 + \eta_6 &\equiv 0 \pmod 2, \\ \mu + \eta_3 + \eta_4 + \eta_7 &\equiv 0 \pmod 2, \\ \mu + \eta_1 + \eta_6 + \eta_7 &= 0 \pmod 2. \end{aligned}$$

I. a. Let $m \equiv 0\,(4)$ so μ is even, then, η_i satisfying (11.5) are either all even or four of the numbers η_i are odd and the remaining three are even.

I. a. 1. All η_i are even. Now we search the number of the solutions of

$$\bar{\eta}_1^2 + \bar{\eta}_2^2 + \ldots + \bar{\eta}_7^2 = \tfrac{1}{2} n - \bar{\mu}^2,$$

satisfying

$$\bar{\eta}_1 + \bar{\eta}_2 + \ldots + \bar{\eta}_7 + \bar{\mu} \equiv 0\,(2).$$

If $n \equiv 0\,(4)$, these are the solutions of

$$x_1^2 + x_2^2 + \ldots + x_7^2 = \tfrac{1}{16}(8n - m^2).$$

If $n \not\equiv 0\,(4)$ such solution don't exist.

I. a. 2. Among the numbers η_i there are four odd and three even ones. Now there are $\binom{7}{4} = 35$ possible combinations. We research systematically these combinations, calculating η_1, η_2, η_3 and η_5 by given fixed values of η_4, η_6 and η_7 from

$$\begin{aligned} \eta_1 + \eta_6 + \eta_7 &\equiv 0\,(2), \\ \eta_3 + \eta_4 + \eta_7 &\equiv 0\,(2), \\ \eta_2 + \eta_4 + \eta_6 &\equiv 0\,(2). \end{aligned}$$

η_4	η_6	η_7	$\rightarrow$	η_1	η_2	η_3	η_5
1	1	1		0	0	0	1
0	1	1		0	1	1	0
1	0	1		1	1	0	0
1	1	0		1	0	1	0
1	0	0		0	1	1	1
0	1	0		1	1	0	1
0	0	1		1	0	1	1
0	0	0		—	—	—	—

mod 2 . (11.6)

Thus we find that only $\frac{1}{8}$ of the number of these possible combinations equals the number of those which satisfy these congruences to modulus 2.

Now we research which of these solutions satisfy the congruence $\eta_1 + \eta_2 + \ldots + \eta_7 \equiv 0 \pmod 4$.

All solutions satisfy this congruence to modulus 2. From one solution $\eta_1, \eta_2 \ldots \eta_7$ we deduce 2^7—1 other solutions, to know $\pm \eta_1, \pm \eta_2, \ldots \pm \eta_7$. In order to find the asked number we must divide the just found number by 2.

I. b. Now let $m \equiv 2\,(4)$ so μ is odd. From

$$\Sigma \eta_i^2 + \mu^2 \equiv 0\,(4)$$

we deduce that either all η_i are odd or four of the numbers η_i are even and the remaining three are odd.

I. b. 1. We suppose all η_i to be odd. So we must count the number of the solutions of

$$\eta_1^2 + \eta_2^2 + \ldots + \eta_7^2 = 2n - \mu^2$$

satisfying the congruence

$$\eta_1 + \eta_2 + \ldots + \eta_7 + \mu \equiv 0 \pmod 4.$$

All η_i and μ being odd we deduce that one of each pair solutions

$$\eta_1, \eta_2, \ldots, \eta_7 \text{ and } -\eta_1, -\eta_2, \ldots -\eta_7$$

satisfies the congruence.

I. b. 2. Four of the numbers η_i are even ones and three are odd ones. From the table below we deduce once more that only 7 of the 35 combinations satisfy the congruences

$$\eta_1 + \eta_6 + \eta_7 \equiv 1\,(2),$$
$$\eta_3 + \eta_4 + \eta_7 \equiv 1\,(2),$$
$$\eta_2 + \eta_4 + \eta_6 \equiv 1\,(2).$$

η_4	η_6	η_7	→	η_1	η_2	η_3	η_5
0	0	0		1	1	1	0
0	1	1		1	0	0	0
1	0	1		0	0	1	0
1	1	0		0	1	0	0
1	0	0		1	0	0	1
0	1	0		0	0	1	1
0	0	1		1	1	0	0
1	1	1		—	—	—	—

(mod 2). (11.7)

We can deduce once more that one of each pair

$$\eta_1, \eta_2, \ldots, \eta_7 \text{ and } -\eta_1, -\eta_2, \ldots -\eta_7,$$

satisfies the congruence

$$\eta_1 + \eta_2 + \ldots + \eta_7 + \mu \equiv 0 \pmod 4.$$

Case II.

Let m be odd, then all y_i and n are also odd.

We consider the solutions of, ($y_i = 2\eta_i + 1$ and $m = 2\mu + 1$),

$$\sum_{i=1}^{7} \frac{\eta_i^2 + \eta_i}{2} + \frac{\mu^2 + \mu}{2} = n-1, \quad \ldots \quad (11.8)$$

satisfying

$$\left.\begin{aligned} &\eta_1 + \eta_2 + \ldots + \eta_7 + \mu \equiv 0 \,(4),\\ &\eta_1 + \eta_6 + \eta_7 + \mu \equiv 0 \,(2),\\ &\eta_3 + \eta_4 + \eta_7 + \mu \equiv 0 \,(2),\\ &\eta_2 + \eta_4 + \eta_6 + \mu \equiv 0 \,(2). \end{aligned}\right\} \quad \ldots \quad (11.9)$$

Again we consider couples of 2^7 solutions $\pm y_1, \pm y_2, \ldots \pm y_7$.

First we consider those solutions in a fixed couple which satisfy

$$\left.\begin{aligned} &\eta_2 + \eta_3 + \eta_4 + \eta_5 \equiv 0 \;(2),\\ &\eta_1 + \eta_6 + \eta_7 + \mu \equiv 0 \;(2),\\ &\eta_3 + \eta_4 + \eta_7 + \mu \equiv 0 \;(2),\\ &\eta_2 + \eta_4 + \eta_6 + \mu \equiv 0 \;(2). \end{aligned}\right\} \quad \ldots \quad (11.10)$$

If we fix arbitrary η_2, η_3 and η_4 we can deduce [3]) η_1, η_5, η_6, and η_7 in such a way that the congruences (11. 10) are satisfied. So there are 2^3 solutions in each couple which satisfy (11. 10). It may be seen readily that each of these eight couples contains either all even elements η_i and μ, or all odd elements or four even and four odd ones.

Thus we have

$$\Sigma \eta_i^2 + \mu^2 \equiv 0 \pmod 4$$

and from

$$\Sigma (\eta_i^2 + \eta_i) + \mu^2 + \mu = 2(n-1)$$

we deduce

$$\Sigma (\eta_i^2 + \eta_i) + \mu^2 + \mu \equiv 0 \pmod 4,$$

thus

$$\Sigma \eta_i + \mu \equiv 0 \pmod 4.$$

So we proved that each couple contains just 2^3 solutions which satisfy (11. 9).

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[3]) If we change y_i in $-y_i$ we must replace η_i by $-\eta_i - 1$.

Summary.

Let $r(N)$ denote the number of the solutions of $x_1^2 + x_2^2 + \ldots + x_7^2 = N$.
Let $R(N)$ denote the number of the solutions of $x_1^2 + x_2^2 + \ldots + x_7^2 = N$, where all numbers x_l must be odd ones.

Thus we have found for the number of the solutions of (10.1) the formulae

If $m \equiv 0\,(4)$; $n \equiv 0\,(4)$: $\frac{9}{10} r\left(\frac{1}{16}(8n-m^2)\right) + \frac{1}{10} r\left(\frac{1}{4}(8n-m^2)\right)$.

If $m \equiv 0\,(4)$; $n \equiv 2\,(4)$: $\frac{1}{10} r\left(\frac{1}{4}(8n-m^2)\right) - \frac{1}{10} r\left(\frac{1}{16}(8n-m^2)\right)$.

If $m \equiv 2\,(4)$; $n \equiv 0\,(4)$: $\frac{1}{10} r\left(\frac{1}{4}(8n-m^2)\right) + \frac{2}{5} R\left(\frac{1}{4}(8n-m^2)\right)$.

If $m \equiv 2\,(4)$; $n \equiv 2\,(4)$: $\frac{1}{10} r\left(\frac{1}{4}(8n-m^2)\right)$.

If $m \equiv n \equiv 1\,(2)$: $\frac{1}{16} R(8n-m^2)$.

Now we use the well-known formulae concerning $r(N)$ and $R(N)$ (DICKSON [6]):

$$r(\Delta) = \bar{a}_2(\Delta)\, \Delta^{\frac{5}{2}} \phi(\Delta);$$

$$R(\Delta) = 2^4\, \Delta^{\frac{5}{2}} \phi(\Delta).$$

where $\Delta = 2^\alpha \Delta_1$ and $(\Delta_1, 2) = 1$ and

$$\bar{a}_2(\Delta) = 9 \cdot 2^5 + \lambda \cdot 2^3 \qquad \text{if } \alpha < 2.$$

$$\bar{a}_2(\Delta) = 2^8 \cdot \frac{35}{31} - \frac{63}{31.32} \cdot 2^{-\frac{5}{2}\alpha + \frac{25}{2}} \qquad \text{if } \alpha \equiv 1\,(2),\ \alpha \geqq 3.$$

$$\bar{a}_2(\Delta) = 2^8 \cdot \frac{35}{31} - \frac{1}{31.32} 2^{-\frac{5}{2}\alpha + 15} + \varkappa\, 2^{-\frac{5}{2}\alpha + 3} \qquad \text{if } \alpha \equiv 0\,(2),\ \alpha \geqq 2.$$

And

$$\lambda = -8 \text{ and } \varkappa = -4 \text{ if } \Delta_1 \equiv 1\,(4).$$
$$\lambda = 1 \text{ and } \varkappa = 5 \text{ if } \Delta_1 \equiv 7\,(8).$$
$$\lambda = -1 \text{ and } \varkappa = 3 \text{ if } \Delta_1 \equiv 3\,(8).$$

Here we wrote by abbreviation

$$\phi(\Delta) = \pi^{-3} \prod_{p \mid \Delta_0} A_p \prod_{q \mid \Delta_1} B_q \sum_{\substack{k=1 \\ (k,2D)=1}}^{\infty} \left(\frac{D}{k}\right) k^{-3}.$$

A_p and B_q were defined in § 9.
Using all these results we are lead to the following

Theorem 2. *The number of the solutions of the system* (11. 1) *equals*

$$f(n,m)\frac{\Delta^{\frac{1}{2}}}{\pi^3}\prod_{p\mid\Delta_3}A_p\prod_{q\mid\Delta_4}B_q\sum_{\substack{k=1\\(k,2D)=1}}^{\infty}\left(\frac{D}{k}\right)k^{-3};$$

where $f(n,m)$ *is defined by*

$f(n,m)=1$ if $n\equiv m\equiv 1\ (2)$.

$f(n,m)=1+\frac{1}{8}(-1)^{\frac{n}{2}}$ if $n\equiv m\equiv 0\ (2)$ but not $n\equiv m\equiv 0\ (4)$.

$f(n,m)=\frac{35}{32}$ if $n\equiv 4\ (8)\ m\equiv 0\ (4)$.

$f(n,m)=\frac{37}{32}+\frac{1}{128}(-1)^{\frac{n}{8}}$ if $n\equiv 0\ (8)\ m\equiv 4\ (8)$.

$f(n,m)=\frac{37}{32}+\frac{1}{31.32}\left(1-63.2^{-\frac{5}{2}\alpha+\frac{25}{2}}\right)$

if $n\equiv m\equiv 0\ (8)\ \alpha\equiv 1\ (2)$.

$=\frac{37}{32}+\frac{1}{31.32}\left(1-2^{-\frac{5}{2}\alpha+15}\right)+\varkappa 2^{-\frac{5}{2}\alpha+3}$

if $n\equiv m\equiv 0\ (8)\ \alpha\equiv 0\ (2)$,

where

$\varkappa=5$ if $\Delta_1\equiv 7\ (8)$,

$\varkappa=3$ if $\Delta_1\equiv 3\ (8)$,

$\varkappa=-4$ if $\Delta_1\equiv 1\ (4)$.

Mathematics. — *Comparison of some direct Methods for Computing Determinants and inverse Matrices.* By E. BODEWIG. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of December 21, 1946.)

During the last years the most suitable numerical calculation of a determinant and of the inverse of a given matrix has been studied in several papers, but in many cases the exact number of necessary additions and multiplications has not been given so that the methods cannot be compared with each other. BANACHIEWICZ did not even split the process into additions and multiplications, but considered the whole process necessary for the application of formula (2) below as "one operation". (See lit. 3, p. 56). Yet the difficulty of this "operation" varies considerably with the order of the corner determinant that has been split off. So his judgement on the appropriateness of his method is precocious (and false). In most of these papers a relation discovered by SCHUR (lit. 1) already in 1917, has been rediscovered and each new author claims to the priority.

The aim of this paper is to *decide the question of priority* and further to *compare two methods with regard to the work of calculation.* We shall see that for the calculation of a *determinant the usual method is preferable to that of* SCHUR–BANACHIEWICZ, while in the case of calculation of *the inverse matrix the contrary is true.*

Methods of Computing a determinant.

Let the determinant a of the matrix $\mathfrak{a} = (a_{ik})$ have the order n. The most primitive method of writing a as a function of its n^2 elements and computing the terms can only be applied — special cases excepted — for $n = 2$ or 3. Practically we have only two methods to our disposal.

Method I uses the two elementary propositions on determinants: CRAMER's *theorem* which develops the determinant in terms of the elements of a row, and the *proposition of combination* according to which the value of a determinant remains unchanged if to a row an arbitrary linear combination of the other rows is added.

By adding a multiple of the first row to the other rows such that the first term in every row vanishes (apart from the first row, of course), we obtain

$$a = a_{11} A_{11},$$

where A_{11} is the minor of a_{11} in the transformed determinant and thus has the order $n - 1$. In this manner the degree of the determinant is diminished successively until it becomes equal to 2.

I denote by combination-theorem both CRAMER's theorem and the above mentioned proposition of combination.

Method II. A deeper theorem has been given by SCHUR (lit. 1) in the course of certain investigations. Let the matrix **a** be decomposed arbitrarily into partial matrices: **P**, **Q**, **R**, **S**. Then he established the relation:

$$\mathbf{a} = \begin{pmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{R} & \mathbf{S} \end{pmatrix} = \begin{pmatrix} \mathbf{P} & \mathbf{O} \\ \mathbf{R} & \mathbf{E} \end{pmatrix} \begin{pmatrix} \mathbf{E} & \mathbf{P}^{-1}\mathbf{Q} \\ \mathbf{O} & \mathbf{S}_1 \end{pmatrix}, \text{ where } \mathbf{S}_1 = \mathbf{S} - \mathbf{R}\mathbf{P}^{-1}\mathbf{Q} . \quad (1)$$

Passing from the matrices to the determinants we obtain:

$$a = \det P \cdot \det S_1 . \quad (2)$$

Let **P** have the order p, thus $\mathbf{S}_1$ the order $n - p$. Then a is decomposed into the product of two determinants of lower orders which are, of course, computed more easily than the original one of order n.

This fundamental relation seems to have remained completely unknown. The reason for this may be that it was given by SCHUR in a paper not devoted to determinants at all. So it was not until 20 years later, viz. in 1937, that is has been rediscovered by BANACHIEWICZ (lit. 2 and lit. 3, p. 51). That BANACHIEWICZ wrote it in the "cracovian-notation" (where rows are multiplied by rows), does not matter.

It was also BANACHIEWICZ who derived from (1) two methods for computing determinants: the *method of condensation* and the *method of enlargement*. The first consists of treating the matrix $\mathbf{S}_1$ occurring in (1) in the same manner as **a**, viz. by separating a left corner above and condensing according to $\mathbf{S}_2$, and so on. The latter consists of computing first **P** and then adding a left corner matrix **S'** of $\mathbf{S}_1$ such that (1) can be applied. The enlargement is repeated until **a** is obtained.

Connection between the theorem of combination and SCHUR's *relation* (1). Taking in (1) a matrix **P** of one row, so that **P** is an element a_1 of the matrix, we obtain

$$\mathbf{S}_1 = \mathbf{S} - a_1^{-1}\mathbf{R}\mathbf{Q}, \text{ where}$$
$$\mathbf{R}' = (a_2\, a_3 \ldots a_n) \text{ and } \mathbf{Q} = (b_1\, c_1\, d_1 \ldots n_1).$$

Then $\mathbf{S}_1$ is the determinant of order $n - 1$ remaining after the application of the combination-theorem. Thus SCHUR's relation (2) for $p = 1$ is a special case of the combination-theorem. Conversely this theorem is a special case of SCHUR's relation for $p = 1$. Let us confine ourselves to a determinant of three rows. Then by the theorem of combination:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \rightarrow \begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 - q_1 a_2 & b_3 - q_1 a_3 \\ 0 & c_2 - q_2 a_2 & c_3 - q_2 a_3 \end{vmatrix}, \text{ where } q_1 = b_1/a_1,\ q_2 = c_1/a_1.$$

The matrix on the left is nothing but the product of

$$\mathbf{A} = \begin{vmatrix} 1 & 0 & 0 \\ q_1 & 1 & 0 \\ q_2 & 0 & 1 \end{vmatrix}$$

and the matrix on the right hand side. Consequently the proposition of combination decomposes the original matrix into a product, identical to (1) if $P = a_1$, $Q = (a_2 a_3)$, ... *Hence relation* (1) *may be considered as a generalisation of the combination-theorem.* So we have to investigate whether this generalisation yields an advantage in computation. We shall see that it is not the case.

Methods of Computing the inverse matrix.

Method I. Since $\mathbf{a}^{-1} = (A_{ki}/a) = \mathbf{A}/a$, where $\mathbf{A}$ is the matrix of the subdeterminants, we have first to compute all minors A_{ki} of a_{ki}. The determinant a itself will then be given by the linear combination

$$a = a_{1i} A_{1i} + a_{2i} A_{2i} + \ldots + a_{ni} A_{ni}.$$

By choosing various values for i we obtain many controls for the computation.

Method II. Inverting (1) we obtain

$$\left.\begin{aligned} &\begin{pmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{R} & \mathbf{S} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{E} & -\mathbf{P}^{-1}\mathbf{Q}\mathbf{S}_1^{-1} \\ \mathbf{O} & \mathbf{S}_1^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{P}^{-1} & \mathbf{O} \\ -\mathbf{R}\mathbf{P}^{-1} & \mathbf{E} \end{pmatrix} \text{ or} \\ &\begin{pmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{R} & \mathbf{S} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{P}^{-1} + \mathbf{T}\mathbf{U} & -\mathbf{T} \\ -\mathbf{S}_1^{-1}\mathbf{U} & \mathbf{S}_1^{-1} \end{pmatrix}, \text{ where } \mathbf{T} = \mathbf{P}^{-1}\mathbf{Q}\mathbf{S}_1^{-1},\ \mathbf{U} = \mathbf{R}\mathbf{P}^{-1}. \end{aligned}\right\} \quad (3)$$

This formula is not established by SCHUR explicitly, but, as it is a simple consequence of (1), the formulae (1—3) may be denoted as SCHUR's *relations.*

Also formula (3) has been rediscovered by BANACHIEWICZ independently (lit. 3, p. 54), and also here he described a method of condensation and a method of enlargement.

But also this paper of BANACHIEWICZ remained unknown, and so FRAZER, DUNCAN and COLLAR published relation (3) in 1938 anew (lit. 4). The same was the case with HOTELLING (lit. 5 and 6), DUNCAN (lit. 4a), WAUGH (lit. 7) and finally with the last paper of L. GUTTMAN (lit. 8), who even says (l.c., p. 336) that "the basis for such routines has also been noticed before, but does not seem to have attracted the attention it merits". Then he recognises a partial priority of FRAZER, DUNCAN, COLLAR and WAUGH and adds that "a possible reason for this lack of attention may be the belief that the methods apply only to a restricted class of matrices". Thus the full and general considerations of BANACHIEWICZ have escaped his attention.

The number of multiplications and additions.

The number of multiplications and additions necessary for the calculation of a determinant of the order n will be denoted by M_n' and A_n' if method I is used (WHITTAKER's "CHIÒ's rule of pivotal elements") and by M_n'' and A_n'' if method II is used.

A. Calculation of a determinant.

Method I. Let a_{11} be the pivotal element. Then the elements of the first row are multiplied by a_{i1}/a_{11} in turn. This requires n multiplications (divisions) and $n-1$ additions in every row (the first one excepted), thus $n(n-1)$ multiplications and $(n-1)^2$ additions for all rows. Finally we have the multiplication by a_{11}. So

$$M'_n = n^2 - n + 1 + M'_{n-1} \text{ and } A'_n = (n-1)^2 + A'_{n-1}.$$

The new determinant is dealt with in the same manner and so on, until a determinant of order 2 results. Adding we obtain

$$\left.\begin{aligned} M'_n &= \sum_3^n (n^2 - n + 1) + 2 = \tfrac{1}{3} n (n^2 + 2) - 2 \\ A'_n &= \sum_3^n (n-1)^2 + 1 = \tfrac{1}{6} n (n-1)(2n-1). \end{aligned}\right\} \quad \ldots \quad (4)$$

Method II. Application of (2) requires the multiplication of several matrices. The multiplication of a *pm*- with a *mq*-matrix requires *pmq* multiplications and $pq(m-1)$ additions, for every element of the product requires m multiplications and $m-1$ additions. Consequently (2) requires successively the calculation of

1) matrix $\mathbf{P}^{-1}$ the order of which is assumed to be greater than 1. Let the number of multiplications and additions necessary for this calculation be denoted by $M(p)$ and $A(p)$.

2) $\mathbf{P}^{-1}\mathbf{Q}$. But $M(\mathbf{P}^{-1}\mathbf{Q}) = p^2(n-p)$, $A(\mathbf{P}^{-1}\mathbf{Q}) = p(p-1)(n-p)$

3) $\mathbf{S}_1 = \mathbf{S} - \mathbf{R}\mathbf{P}^{-1}\mathbf{Q}$, but

$$M(\mathbf{S}_1) = M(\mathbf{R}\mathbf{P}^{-1}\mathbf{Q}) = p(n-p)^2, \quad A(\mathbf{S}_1) = (n-p)^2 + A(\mathbf{R}\mathbf{P}^{-1}\mathbf{Q}) = p(n-p)^2$$

4) $\det S_1$, thus M''_{n-p} and A''_{n-p}

5) $\det P \det S_1$, thus one multiplication.

In total we have therefore

$$M''_n = M(p) + M''_{n-p} + pn(n-p) + 1, \quad A''_n = A(p) + A''_{n-p} + p(n-1)(n-p).$$

The computation of S_1 goes in the same manner, that is by separating a left corner-determinant of the order p, and so on. Now put $n = mp$. Then applying the above formulae repeatedly, we obtain

$$M''_n = p^3 \sum_{i=2}^{m} i(i-1) + m - 1 + (m-1) \cdot M(p) + M''_p =$$

$$= \tfrac{1}{3} p^3 (m^3 - m) + m - 1 + (m-1) \cdot M(p) + M''_p$$

$$A''_n = p^3 \sum_{i=2}^{m} i^2 - p^2 \sum_{i=2}^{m} i - p^3 \sum_{i=2}^{m} i + p^2(m-1) + (m-1) \cdot A(p) + A''_p =$$

$$= \tfrac{1}{3} p^3 (m^3 - m) - \tfrac{1}{2} p^2 (m^2 - m) + (m-1) \cdot A(p) + A''_p.$$

Here $M(p)$ and $A(p)$ are unknown, but anticipating the result of

p. 56, (11) we assume that the inverse matrices will also be computed by SCHUR's relation (3) so that

$$M(p)=p^3,\ A(p)=p^3-2p^2+2p,$$

hence

$$\left.\begin{aligned} M_n'' &= \tfrac{1}{3}n^3+\tfrac{3}{3}p^2n-p^3+M_p''+m-1 \\ A_n'' &= \tfrac{1}{3}n^3+\tfrac{3}{3}p^2n-p^3-\tfrac{1}{2}n^2-\tfrac{3}{2}pn+2p^2+2n-2p+A_p''. \end{aligned}\right\} \quad (5)$$

These considerations are not valid when $p=1$. But in this case we saw that SCHUR's relation is identical to the application of the combination-theorem, so that formulae (4) hold.

For $p>1$, we have $M_p''=M_p'$ and $A_p''=A_p'$ in (5). Then by (4) we obtain

$$\left.\begin{aligned} M_n'' &= \tfrac{1}{3}n^3+\tfrac{3}{3}p^2n-\tfrac{3}{3}p^3+\tfrac{3}{3}p-3m \\ A_n'' &= \tfrac{1}{3}n^3+\tfrac{3}{3}p^2n-\tfrac{1}{2}n^2-\tfrac{3}{2}pn+2n-\tfrac{3}{3}p^3+\tfrac{3}{2}p^2-\tfrac{11}{6}p. \end{aligned}\right\} \quad (6)$$

Comparison between I and II.

$$M_n''-M_n'>\tfrac{3}{3}(p^2-1)(n-p)>0,\ \text{if } p>1$$
$$A_n''-A_n'=\tfrac{1}{6}(4p^2-9p+11)(n-p)>0,\ \text{if } p>1.$$

Consequently:

For $p=1$ the two methods are identical. For $p>1$ the combination-theorem is much more preferable to the application of SCHUR's relation. In other words, the best method for solving linear equations is the scholastic one which eliminates successively the unknowns.

Hence the remarks of BANACHIEWICZ (lit. 3, p. 54—6, p. 61) that his (viz. SCHUR's) relation considerably saves work is refuted.

B. Calculation of the inverse matrix.

Method I. We have to compute first the n^2 minors of order $n-1$. Now by (4) each minor requires

$\tfrac{1}{3}(n-1)(n^2-2n+3)-2$ multipl. and $\tfrac{1}{6}(n-1)(n-2)(2n-3)$ additions.

So all minors together will require n^2 times as much so that

$$M_n'\approx\tfrac{1}{3}n^5,\quad A_n''\approx\tfrac{1}{3}n^5 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

But these numbers may be replaced by much smaller ones as is it not necessary to compute all these minors. For we have the following proposition, the proof of which is immediate.

Theorem. The minor A_{ij} of each element a_{ij} that is not cancelled by the reduction (viz. each minor A_{ij} with $i\neq r,\ j\neq s$ where a_{rs} is the pivotal element) is equal to the minor in the reduced determinant of that element into which a_{ij} transforms by the reduction.

For let the pivotal element be a_{11}. Then every row R_k with the elements a_{kl} is transformed into the row with the elements a'_{kl}:

$$a'_{kl} = a_{kl} - \frac{a_{k1}}{a_{11}} a_{1l}.$$

Now f.i. the minor A_{22} which contains the element a_{11}, may be treated in the same manner as was the original determinant. But then the *k-th* row of A_{22} contains the elements a'_{kl} again. Q.e.d.

Thus the first application of the combination-theorem yields the determinant of order $n-1$, the $(n-1)^2$ minors of which equal the corresponding minors of a itself, apart from the pivotal element as factor. There are still missing the $2n-1$ minors of the row and column which are crossing in the pivotal element. To find them the process is to be repeated with a pivotal element which is not standing in the same row or column as the first element. So we obtain $2n-3$ further minors. To find finally the two missing minors the process will be repeated with a third element which does not stand in the same row or in the same column as either of the former pivotal elements.

By the aid of the combination-theorem we have therefore reduced the determinant D_n of order n to three determinants of order $n-1$, or symbolically:

$$D_n = D_{n-1} + \bar{D}_{n-1} + \bar{\bar{D}}_{n-1}.$$

Here all minors of D_{n-1} have to be computed, but only $2n-3$ minors of $\bar{D}_{n-1}$ (the minors of a row and a column with the exception of two) and 2 minors of $\bar{\bar{D}}_{n-1}$.

But to pass from D_n to a determinant of order $n-1$:

$$t_n = n^2 - n$$

multiplications are necessary (p. 52). Now let be denoted by

m_n: the number of multipl. necessary for computing all minors of D_n

$\bar{m}_n$: the number of multipl. necessary for computing the minors of a row and a column of D_n

$\bar{\bar{m}}_n$: the number of multipl. necessary for computing 2 minors of D_n.

Then

$$\begin{aligned} m_n &= (t_n + m_{n-1}) + (t_n + \bar{m}_{n-1}) + (t_n + \bar{\bar{m}}_{n-1}) \\ \bar{m}_n &= \qquad\qquad (t_n + \bar{m}_{n-1}) + (t_n + \bar{\bar{m}}_{n-1}) \\ \bar{\bar{m}}_n &= \qquad\qquad\qquad\qquad (t_n + \bar{\bar{m}}_{n-1}). \end{aligned}$$

Thus f.i.

$$m_n = 3t_n + 6t_{n-1} + 10t_{n-2} + 15t_{n-3} + 21t_{n-4} + m_{n-5} + 5\bar{m}_{n-5} + 15\bar{\bar{m}}_{n-5},$$

where the coefficients 3, 6, 10, 15, ... form an arithmetical series of order 2, while the coefficients 1, 5, 15 form a series of order 4. Proceeding generally to the term t_{n-i}, the coefficients of the t_k form a series of order

2 (the second sum of 1, 1, 1, ...), whereas the coefficients of m_{n-i-1}, $\overline{m}_{n-i-1}$, $\overline{\overline{m}}_{n-i-1}$ form a series of order i (the i-*th* sum of the series 1, 1, 1, ...).

Now by choosing $i = n-3$ only minors of second order determinants have to be calculated. But for this no further labor is required. Yet the minors of the second order determinants must be multiplied by all the pivotal elements used until now. These were $n-2$ elements. Now we have three kinds of second order determinants: Those of which all 4 minors must be computed, those of which 3 minors and those of which 2 minors must be computed. The numbers of them will be furnished by the first three terms of the $n-3$th sum of 1, 1, 1, ..., that is by $1, n-2, (n-1)(n-2)/2$. So the totality of the second order determinants will require

$$(n-2)[4 + 3(n-2) + (n-1)(n-2)] = n^3-2n^2 \text{ multiplications.}$$

Consequently we obtain

$$m_n = 3t_n + 6t_{n-1} + \ldots + \tfrac{1}{2}n(n-1)t_3 + n^3-2n^2,$$

To sum m_n we have to form the second sum of $t_3, t_4, \ldots, t_n$. Now the sum of $t_3, t_4, \ldots, t_m$ is

$s_{0,m} = t_3 + t_4 + \ldots + t_m = \frac{1}{6}(m^3-m) - 2$ thus

$s_{1,m} = s_{0,3} + \ldots + s_{0,m} = \frac{1}{12}(m^4 + 2m^3-m^2-26m) + 2$ and

$s_{2,m} = s_{1,3} + \ldots + s_{1,m} = \frac{1}{60}(m^5 + 5m^4 + 5m^3-65m^2 + 54m)$. Now

$$m_n = s_{2,n+1} - t_{n+1} + n^3 - 2n^2 = \tfrac{1}{60}(m^5 + 10n^4 + 95n^3 - 190n^2 - 96n) \quad . \quad (8)$$

Similarly we obtain that the number a_n of additions necessary for computing all minors of D_n is

$$a_n = 3u_n + 6u_{n-1} + \ldots + \tfrac{1}{2}n(n-1)u_3, \text{ where } u_k = (k-1)^2.$$

$$a_n = \tfrac{1}{120}(2n^5 + 15n^3 + 40n^3-135n^2-42n) \quad . \quad . \quad . \quad . \quad (8')$$

The corresponding numbers M'_n and A'_n for the inverse matrix are

$$M'_n = m_n + n^2, \quad A'_n = m_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

We add the following table for the first values of n.

n :	3	4	5	6	7	8	9	10
M'_n :	36	120	292	600	1106	1888	3042	4684
A'_n :	12	51	142	321	637	1154	1953	3134.

Method II. Formula (3) requires successively the computation of

$\mathbf{P}^{-1}$: M''_p multiplications and A''_p additions

$\mathbf{U} = \mathbf{RP}^{-1}$: $p^2(n-p)$ multiplications and $p(p-1)(n-p)$ additions

$\mathbf{P}^{-1}\mathbf{Q}$: $p^2(n-p)$ multiplic. and $p(p-1)(n-p)$ additions

$\mathbf{RP}^{-1}\mathbf{Q}$: $p(n-p)^2$ multipl. and $(p-1)(n-p)^2$ additions

$\mathbf{S}_1$: 0 multipl. and $(n-p)^2$ additions

$\mathbf{S}_1^{-1}$: M''_{n-p} multipl. and A''_{n-p} additions

$\mathbf{S}_1^{-1}\mathbf{U}$: $p(n-p)^2$ multipl. and $p(n-p)(n-p-1)$ additions
$\mathbf{T} = \mathbf{P}^{-1}\mathbf{Q}\mathbf{S}_1^{-1}$: $p(n-p)^2$ multipl. and $p(n-p)(n-p-1)$ additions
$\mathbf{TU}$: $p^2(n-p)$ multipl. and $p^2(n-p-1)$ additions
$\mathbf{P}^{-1} = \mathbf{TU}$: 0 multipl. and p^2 additions.

In total we have

$$M_n'' = M_p'' + M_{n-p}'' + 3pn(n-p), \quad A_n'' = A_p'' + A_{n-p}'' + p(n-p)(3n-4)$$

or, by putting $n = mp$:

$$M_n'' = M_p'' + M_{(m-1)p}'' + 3p^3(m^2-m),$$
$$A_n'' = A_p'' + A_{(m-1)p}'' + p^2(3m^2p - 3mp - 4m + 4).$$

The new determinant is to be treated similarly and so on, until a determinant of order $2p$ arises, which will split into two determinants of order p. Adding all equations we have

$$\left.\begin{aligned} M_n'' &= m \cdot M_p'' + p^3(m^3-m) = m \cdot M_p'' + n^3 - p^2 n \\ A_n'' &= m \cdot A_p'' + p^3(m^3-m) - 2p^2(m^2-m) = m \cdot A_p'' + n^3 - p^2 n - 2n^2 + 2pn, \end{aligned}\right\} \quad (10)$$

especially

for $p = 1$, then $M_n'' = n^3$ and $A_n'' = n^3 - 2n^2 + 2n$

for $p = 2$, then (because of $M_2'' = 6$ and $A_2'' = 1$)

$$M_n'' = n^3 \text{ and } A_n'' = n^3 - 2n^2 + \frac{n}{2}$$

for $p = 3$, then (because of $M_3'' = 27$ and $A_3'' = 15$)

$$M_n'' = n^3 \text{ and } A_n'' = n^3 - 2n^2 + 2n.$$

Generally, if the values of $M_p'' = p^3$ and $A_p'' = p^3 - 2p^2 + 2p$ following from the formula of above are substituted into (10), we obtain for every p:

$$M_n'' = n^3 \text{ and } A_n'' = n^3 - 2n^2 + 2n \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Consequently, *under these conditions the number of multiplications and additions is independent of* p. The most simple is therefore to take $p = 1$. Yet taking $p = 2$ the number may be slightly diminished, as we saw above.

Comparison between I and II. Comparing (9) and (11) we see that *method II requires much less work of computation than method I.* Only the small values $n = 3$ or 4 must be examined separately. But the table below shews that also for them method II is preferable.

n :	3	4	5	6	7	8	9	10
M_n'' :	27	64	125	216	343	512	729	1000
A_n'' :	15	40	85	156	259	400	585	820.

Summary.

1. Formulae (1—3) are discovered by SCHUR 30 years ago and rediscovered by BANACHIEWICZ in 1937.

2. For the computation of a determinant the method of pivotal elements is preferable to the method founded on SCHUR's relation.

3. For the computation of the inverse matrix the converse holds.

3a. The most convenient method is to use (3) for $p = 1$.

REFERENCES.

1. SCHUR, J., Crelle's Journal f. reine u. angewandte Mathematik, **147**, 217 (1917).
2. BANACHIEWICZ, T., Calcul des déterminants par la méthode des cracoviens. Bulletin internat. Académie Polonaise Sciences, A, 1937, 109—120 (1937).
3. BANACHIEWICZ, T., Zur Berechnung der Determinanten wie auch der Inversen und zur darauf basierten Auflösung der Systeme linearer Gleichungen. Acta Astronomica, sér. c. **3**, 41—67 (1937).
4. FRAZER, R. A., W. J. DUNCAN and R. A. COLLAR, Elementary Matrices, Cambridge University Press, (1938).

4a. DUNCAN, W. J., Some devices for the solution of large sets of simultaneous linear equations, Philos. Magaz., **35**, 660—670 (1944).

5. HOTELLING, H., Some new methods in matrix calculation. Annals of Mathematical Statistics, **14**, 1—34 (1943).
6. HOTELLING, H., Further points on matrix calculation and simultaneous equations. Annals of Mathematical Statistics, **14**, 440—1 (1943).
7. WAUGH, F. V., A note concerning Hotelling's method of inverting a partitioned matrix. Annals Mathemat. Statistics, **16**, 216—217 (1945).
8. GUTTMAN, L., Enlargement methods for computing the inverse matrix. Annals Mathemat. Statistics, **17**, 336—345 (1946).

I am greatly indebted to professor VAN DER CORPUT for having corrected the english text.

Mathematics. — *On the dissection of rectangles into squares.* (Second communication.) By C. J. BOUWKAMP. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of December 21, 1946.)

5. *Summary of the possible planar networks without wires in parallel or series connection ($T \leq 14$). Complexity of a network.*

Four different types of networks can be distinguished. In the first place there is a number of networks like that of fig. 3*a*; they all show one or more zero-currents after a source is placed in any of their wires (the remaining wires have equal resistances). The second type, for which fig. 3*b* may serve as an example, shows equal currents in a trivial manner after a source is placed in some of its wires. There are networks belonging to either of these two types. These networks, as far as they are not "compound", are tabulated in fig. 5; double arrows indicate pairs of dual networks. Networks without arrows are self-dual.

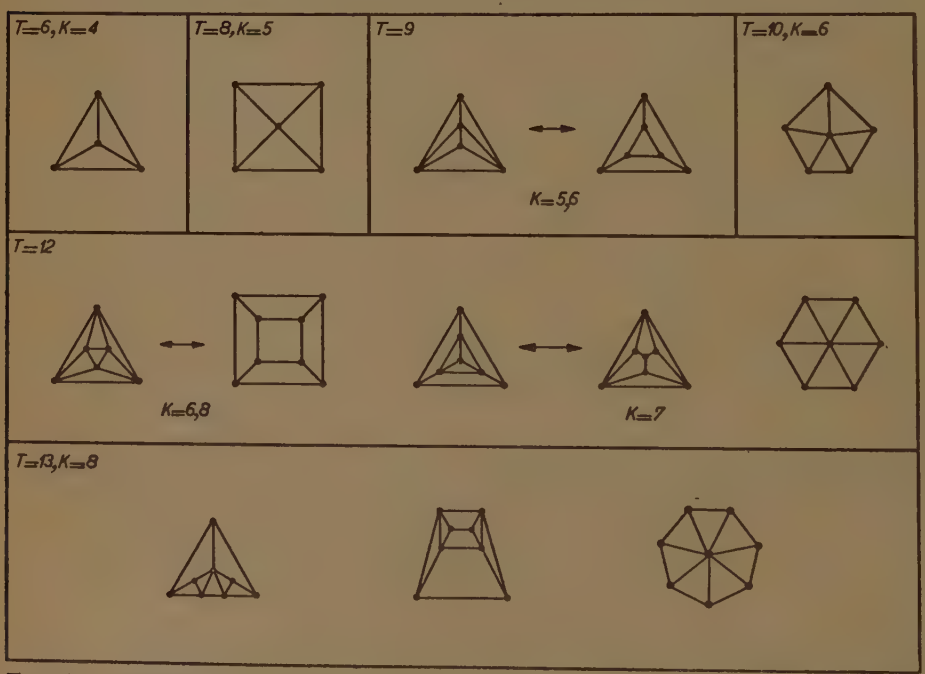

Fig. 5. Networks (not compound) giving either zero elements or trivially imperfect squarings.

The third type of networks is compound, having two or more smaller networks in series or parallel connection. They are gathered in fig. 6. Of equivalent networks only one is given.

None of the networks of figs 5, 6 gives rise to perfect or non-trivially imperfect squarings, except those in fig. 6 marked with α, β, γ. Each of the marked networks generates one of the non-trivially compound squarings mentioned in section 1; one of them is perfect (γ), two are non-trivially imperfect (α, β). The three exceptional squarings consist of 4 squares and 1 rectangle, the latter being dissected into nine squares according to one of the three possibilities of fig. 4. Their codes are

α: (14, 13) (1, 12) (6, 4, 5) (3, 1) (6) (5, 1) (4)
β: (67, 65) (2, 63) (36, 33) (5, 28) (25, 9, 2) (7) (16)
γ: (131, 130) (1, 129) (72, 60) (28, 32) (56, 16) (40, 4) (36),

of which only the last corresponds to a perfect squaring.

Networks of the fourth type, which give rise to at least one squaring of the required character, are drawn in figs 7, 8. Their total number is 87, of which 17 are self-dual. It will be obvious that the sieving process must be carried out rather carefully in order to get them all [1]).

The networks in figs 7, 8 are arranged according to increasing "complexity". The *complexity* C of a network is the total number of "complete trees" (german: "volständiger Baum") that can be drawn in the network. A *complete tree* is a sub-network that (*i*) is connected, (*ii*) contains all the vertices and some (or all) of the wires of the original network, and (*iii*) does not contain any closed circuit.

The numerical value of the complexity of a given network can be found in the following way. Enumerate the K vertices arbitrarily: $P_1, P_2 \ldots P_K$. Let further a square matrix of K rows be formed by elements a_{ik}, such that for diagonal elements $a_{ii} =$ total number of wires at the vertex P_i, and for non-diagonal elements ($i \neq k$) $a_{ik} = -1$ or 0, according as to whether or not P_i and P_k are connected by a wire. The determinant of this symmetrical matrix obviously vanishes. Furthermore its first sub-determinants have a common absolute value. This common value is the complexity of the network under consideration [2]).

Dual networks have the same complexity; the inverse is not true, as can be seen from the networks $C = 1015$, 1088 in fig. 8, or from the networks α, β, γ, in fig. 6, which all have $C = 1040$ (there is in fig. 8 also a network of $C = 1040$).

This concept of complexity is very important with regard to the problem of squared rectangles. It was shown in paper A that *all squared rectangles derived from a network of complexity* C *can be drawn in such*

---

[1]) We are indebted to Prof. DE BRUIJN for constructing independently all the networks containing 14 wires. This diminishes to a large extent the probability of having overlooked one of the required graphs. With regard to the graphs involving 13 wires, it may be noticed here that our number of 67 perfect squared rectangles of order 12 is in full agreement with the result of paper A. Furthermore, in TELLEGEN's papers the possible networks are drawn for $T \leqq 12$, none of which is missing here.

[2]) For a proof of this theorem, see paper A. As was already remarked there, the theorem is quite trivial from known results in early Kirchhoff network theory.

a manner that their semi-perimeters are equal to C, whilst at the same time all the elements of all those squarings have integer sides.

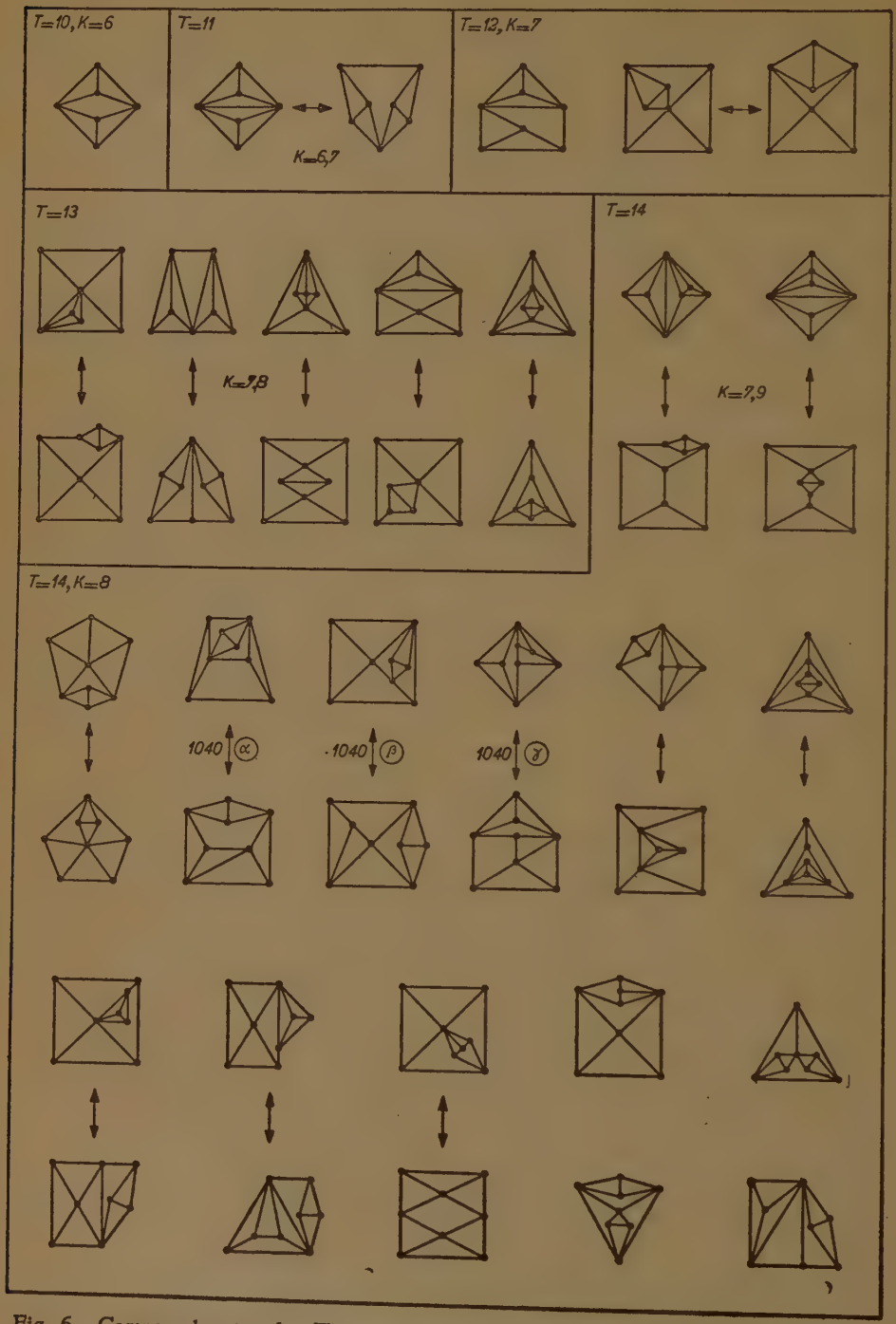

Fig. 6. Compound networks. The only networks that lead to non-trivially compound squarings are marked with α, β or γ.

The last part of this theorem shows that any squared rectangle has commensurable sides (and elements). The converse is also true. In paper A

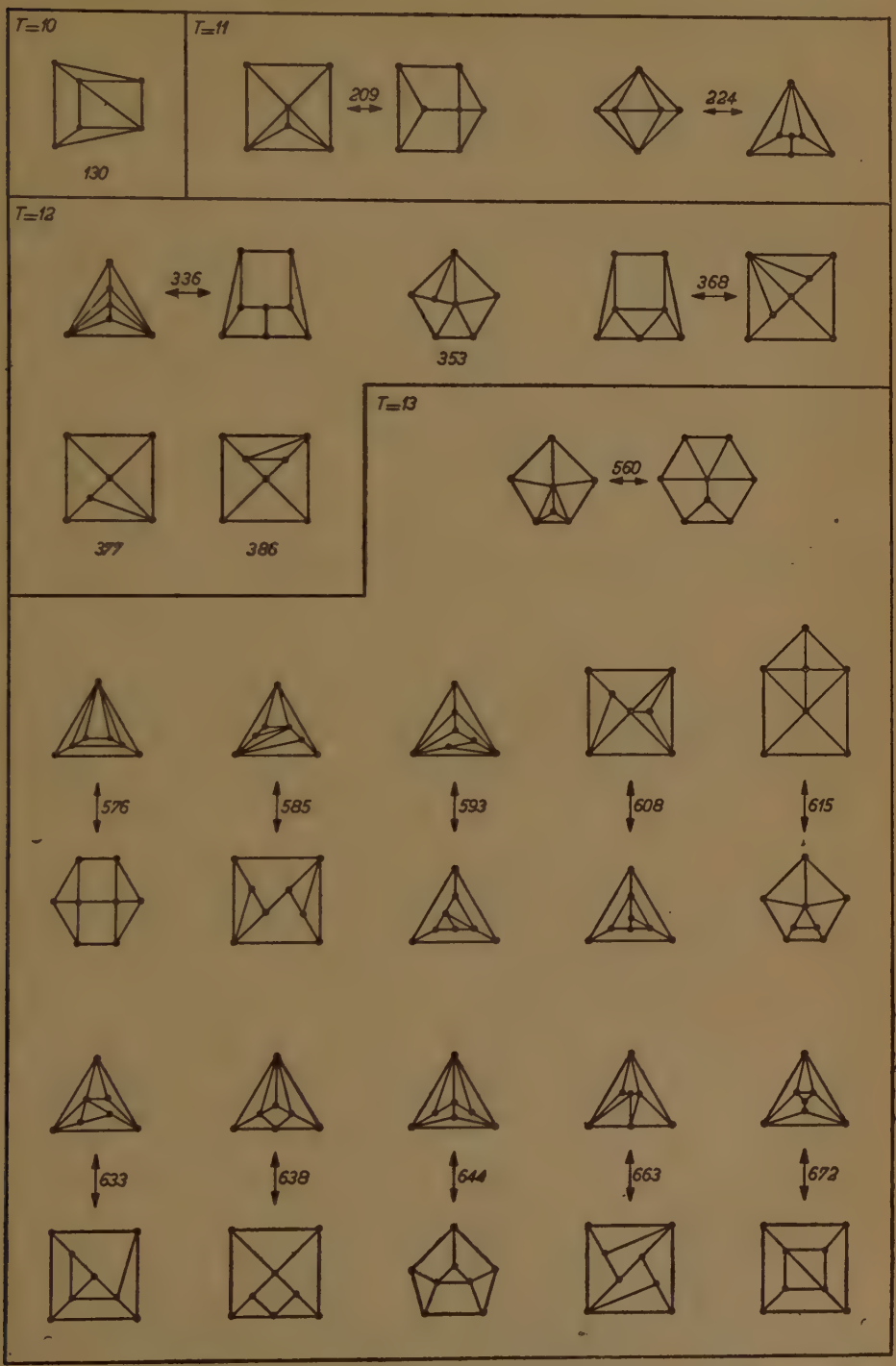

Fig. 7. Networks giving at least one simple squaring of order up to 12. The numbers denote corresponding values of complexity.

it is shown that any rectangle of commensurable sides can be dissected into unequal squares, even in an infinite number of essentially different ways. It is not known, however, whether every such rectangle is perfectible and *simple*.

The integer sides referred to above are called the *full* elements of the squaring. It often happens that the full elements of a particular squaring have a common factor. After division by the highest common factor one

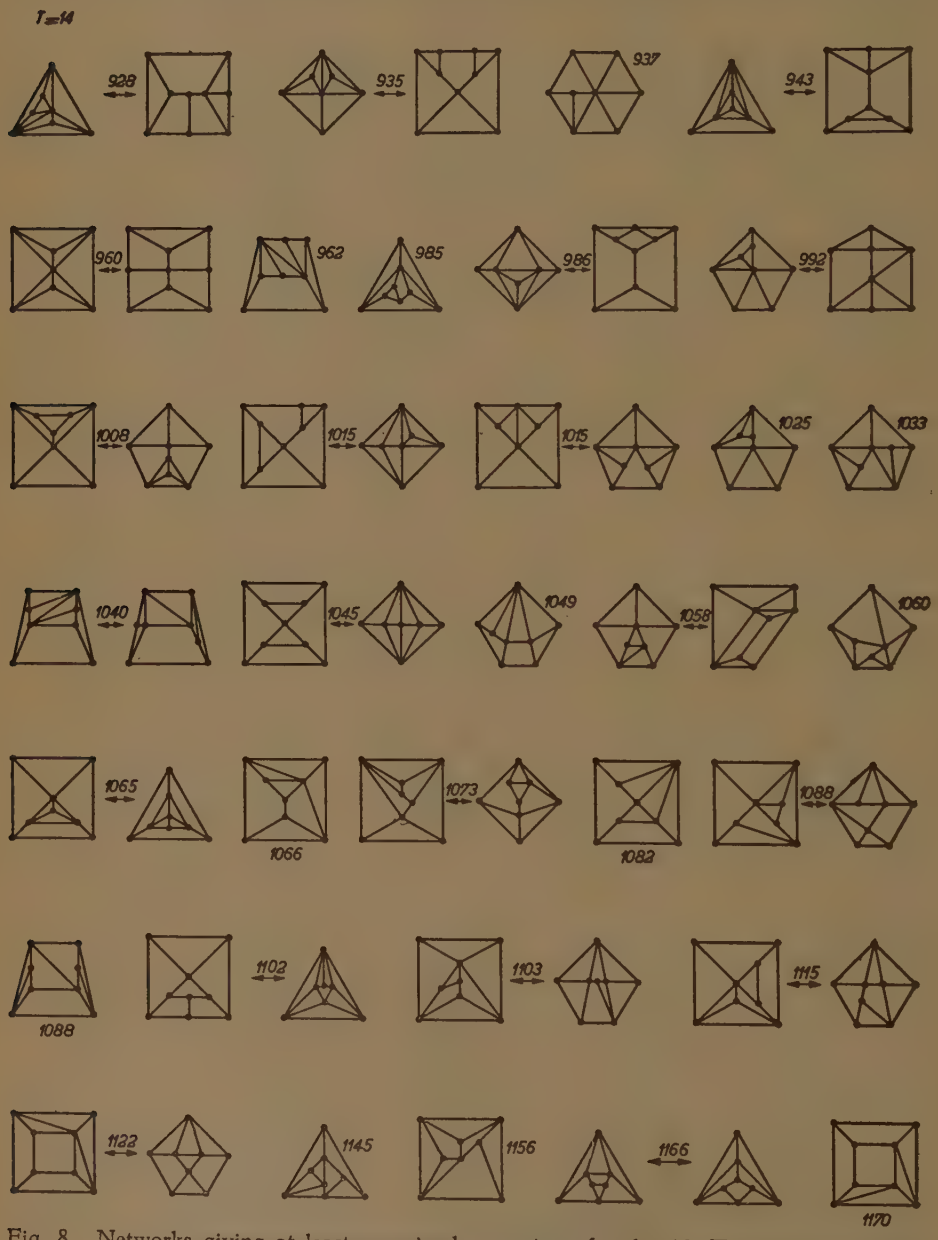

Fig. 8. Networks giving at least one simple squaring of order 13. The numbers denote corresponding values of complexity.

gets the *reduced* elements; this factor is called *reduction factor*. In paper A many theorems on reduction were given; for example, all non-trivially imperfect squarings are reducible. The inverse of this is not true. Further, if C is prime, all the squarings are irreducible. For details and proofs we may refer the reader to the fundamental paper A.

6. *Summary and classification of all simple squared rectangles of order less than* 14.

Even if the required networks have been constructed, it is still a matter of patience to derive from them the possible squared rectangles, though this is by no means difficult. It is not necessary to apply the solution of the Kirchhoff equations in the well-known form of a quotient of two determinants. It is much more simple to assume a certain number of unknown currents to be equal to $x, y, z, \ldots$, and then calculate — by use of the current- and voltage-equations — the remaining currents, which are integral linear combinations of the $x, y, z, \ldots$. Usually, not more than three or four unknown currents are already sufficient. Finally, a system of independent equations is obtained, homogeneous in $x, y, z, \ldots$. The number of equations is one less than the total number of assumed quantities $x, y, z, \ldots$. This diophantine system can be easily solved in smallest numbers. By repeating the process with these numerical values of $x, y, z, \ldots$, the reduced elements of the squaring are easily found.

The squarings will now be *classified* as follows. Undoubtedly, the most important feature is the order of the squaring. The second important property is the least [3]) possible value of the complexity of all networks that will generate it. Therefore the squarings are conveniently arranged in increasing order, and further in classes according to increasing complexity. The order will be denoted by *roman*, the complexity by *arabic*, figures. Squarings in the same class are arranged in such a way that the difference of full horizontal and vertical sides decreases; so the rectangles tend more and more to a square. The elements in a class are distinguished by letters $a, b, c, \ldots$, a prime denoting (non-trivial) imperfection. It may be noticed that in our coding system only the reduced elements occur.

With this in mind, we have got the following *classification of all simple squared rectangles of order less than* 14:

Order IX.

$C = 130$	a'	(6, 4, 5) (3, 1) (6) (5, 1) (4)
	b	(36, 33) (5, 28) (25, 9, 2) (7) (16)
	c	(18, 15) (7, 8) (14, 4) (10, 1) (9)

a' has reduction factor 5; that of c is 2.

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[3]) As follows from the reduction of the order in the case of zero-currents (cf. the end of section 2), there may exist "higher" networks that generate the squared rectangle in question. Cf. the concept of "normal" polar networks in paper A.

Order X.

$C = 209$ a (45, 44, 41) (3, 38) (12, 35) (34, 11) (23)
b (60, 55) (16, 39) (34, 15, 11) (4, 23) (19)
c (57, 54) (3, 7, 44) (41, 15, 4) (11) (26)
d (60, 45) (19, 26) (44, 16) (12, 7) (33) (28)

$C = 224$ a (25, 17, 23) (11, 6) (5, 24) (22, 3) (19)
b (30, 27) (3, 11, 13) (25, 8) (17, 2) (15)

Both squarings have reduction factor 2.

Order XI.

$C = 336$ a (72, 71, 66) (5, 61) (1, 19, 56) (55, 18) (37)
b (95, 90) (5, 24, 61) (56, 25, 19) (6, 37) (31)

$C = 353$ a (85, 57, 67) (47, 10) (77) (59, 26) (7, 40) (33)
b (100, 94) (29, 65) (59, 25, 16) (9, 7) (36) (34)
c (97, 94) (26, 28) (65, 32) (9, 17) (33, 8) (25)
d (99, 88) (10, 78) (1, 9) (67, 25, 8) (17) (42)
e (100, 85) (43, 42) (68, 32) (1, 41) (4, 40) (36)
f (99, 78) (21, 57) (77, 43) (16, 41) (34, 9) (25)

$C = 368$ a (89, 49, 71) (27, 22) (5, 88) (32) (70, 19) (51)
b (105, 94) (19, 75) (64, 33, 8) (27) (31, 2) (29)
c (51, 47) (8, 39) (35, 11, 5) (1, 7) (6) (24)
d (102, 89) (40, 49) (75, 27) (48, 19) (10, 39) (29)
e (105, 80) (33, 47) (78, 27) (19, 14) (5, 56) (51)

c is of reduction 2.

$C = 377$ a (92, 64, 53) (11, 42) (44, 31) (76, 16) (73) (60)
b (102, 97) (16, 81) (76, 15, 11) (4, 23) (19) (42)
c (102, 92) (31, 23, 38) (81, 21) (8, 15) (60) (53)

$C = 386$ a (43, 29, 40) (19, 10) (9, 1) (41) (38, 5) (33)
b (96, 56, 57) (55, 1) (58) (81, 15) (66, 4) (62)
c (105, 100) (6, 13, 81) (76, 28, 1) (7) (20) (48)
d (50, 48) (7, 19, 22) (45, 5) (12) (28, 3) (25)
e (105, 90) (15, 31, 44) (86, 34) (18, 13) (57) (52)
f (56, 41) (17, 24) (40, 14, 2) (12, 7) (31) (26)

a, d and f are of reduction 2.

Order XII.

$C = 560$ a (124, 120, 109) (11, 98) (44, 87) (83, 41) (1, 43) (42)
b (164, 149) (51, 98) (83, 41, 40) (4, 47) (1, 43) (42)
c (154, 149) (41, 108) (103, 51) (14, 27) (1, 13) (52, 40)
d (154, 143) (11, 12, 120) (109, 41, 14, 1) (13) (27) (68)
e' (41, 31) (14, 17) (27, 14) (11, 3) (20) (13, 1) (12)

e' has reduction 4.

$C = 576$ a' (15, 16, 15) (11, 4) (4, 11) (3, 10, 3) (7) (7)
b' (21, 19) (6, 13) (11, 6, 4) (3, 7) (5, 1) (4)

Both of reduction 8.

$C = 585$ a' (9, 8, 5, 7) (3, 2) (9) (3, 8) (7, 2) (5)
b (132, 128, 93) (35, 58) (36, 104, 23) (100, 32) (81) (68)
c (171, 152) (42, 110) (91, 37, 20, 23) (17, 3) (68) (54)
d' (33, 31) (9, 22) (20, 6, 7) (5, 1) (4, 13) (9)
e (155, 152) (26, 126) (123, 32) (9, 17) (33, 8) (25) (58)
f (55, 44) (12, 11, 21) (41, 14) (1, 10) (13) (31) (27)

Reductions of a', d' and f are 13, 5 and 3, respectively.

$C = 593$

a	(113, 76, 96, 101) (56, 20) (111, 5) (106) (94, 19) (75)
b	(121, 129, 118) (11, 107) (104, 17) (9, 35, 96) (26) (61)
c	(123, 129, 101) (28, 73) (117, 6) (111, 152) (7, 66) (59)
d	(143, 73, 120) (43, 30) (13, 17) (56) (137) (114, 29) (85)
e	(166, 160) (6, 47, 107) (101, 43, 28) (15, 13) (60) (58)
f	(175, 149) (29, 120) (94, 43, 35, 3) (32) (8, 59) (51)
g	(166, 155) (11, 27, 117) (106, 55, 16) (43) (51, 4) (47)
h	(160, 155) (5, 8, 19, 123) (118, 44, 3) (11) (30) (74)
i	(165, 149) (19, 130) (114, 44, 7) (4, 15) (11) (26) (70)
j	(176, 137) (67, 70) (104, 44, 28) (16, 76, 3) (73) (60)
k	(165, 143) (67, 76) (120, 45) (75, 28, 9) (19, 66) (47)
l	(175, 130) (45, 85) (113, 55, 52) (12, 73) (3, 61) (58)
m	(176, 121) (47, 74) (20, 27) (120, 44, 12) (32) (101) (76)

$C = 608$

a	(136, 123, 118) (5, 113) (20, 108) (95, 34, 7) (27) (61)
b'	(31, 33, 28) (13, 15) (29, 2) (27, 8) (19, 2) (17)
c	(129, 132, 92) (33, 59) (7, 26) (126, 3) (123, 19) (104)
d'	(37, 24, 23) (6, 17) (19, 5) (11) (31, 6) (25) (28)
e	(166, 163) (3, 44, 116) (113, 56) (15, 29) (57, 14) (43)
f	(83, 79) (4, 12, 61) (59, 19, 9) (1, 11) (10) (40)
g	(85, 77) (12, 65) (57, 17, 7, 4) (3, 13) (10) (40)
h	(163, 158) (5, 24, 129) (124, 25, 19) (6, 37) (31) (68)
i'	(19, 10, 11) (9, 1) (12) (17, 7, 4) (3, 13) (10)
j	(165, 154) (19, 135) (124, 33, 8) (27) (31, 2) (29) (60)
k	(165, 148) (52, 37, 59) (130, 35) (15, 22) (95, 7) (88)
l	(170, 135) (35, 100) (133, 72) (7, 25, 68) (61, 18) (43)

f, *g* have reduction 2; *b'*, *d'* have 4 and *i'* has 8.

$C = 615$

a	(144, 97, 127) (67, 30) (37, 120) (103, 41) (21, 83) (62)
b	(142, 84, 127) (41, 43) (39, 2) (37, 135) (120, 22) (98)
c	(174, 164) (51, 113) (103, 43, 28) (15, 13) (2, 62) (60)
d	(58, 48) (20, 28) (41, 17) (7, 13) (24) (5, 23) (18)
e	(165, 143) (22, 67, 54) (142, 45) (13, 41) (97, 28) (69)

d has reduction 3.

$C = 633$

a	(138, 116, 132) (51, 49, 16) (33, 115) (109, 29) (82) (80)
b	(141, 113, 123) (28, 75, 10) (133) (115, 54) (7, 68) (61)
c	(139, 93, 136) (59, 34) (25, 9) (16, 129) (126, 13) (113)
d	(48, 28, 44) (12, 16) (8, 4) (17, 47) (43, 13) (30)
e	(144, 116, 93) (23, 70) (28, 64, 47) (136, 36) (117) (100)
f	(180, 164) (39, 125) (109, 48, 23) (25, 37) (61, 12) (49)
g	(57, 56) (2, 12, 42) (41, 11, 4, 1) (3) (7) (30)
h	(170, 168) (47, 59, 62) (125, 45) (80, 12) (68, 3) (65)
i	(56, 27, 29) (25, 2) (31) (17, 8) (43, 13) (39) (30)
j	(170, 164) (29, 135) (129, 25, 16) (9, 7) (36) (34) (70)
k	(171, 158) (13, 64, 81) (133, 51) (82, 33) (16, 65) (49)
l	(168, 158) (10, 49, 37, 62) (139, 39) (12, 25) (100) (87)
m	(60, 46) (18, 28) (45, 15) (11, 7) (4, 3) (31) (30)

d, *g*, *i*, *m* of reduction 3.

$C = 638$

a	(71, 60, 62) (27, 31, 2) (64) (55, 16) (39, 4) (35)
b	(150, 92, 111) (73, 19) (54, 76) (135, 15) (120, 22) (98)
c	(175, 164) (10, 19, 135) (1, 9) (124, 44, 8) (36) (80)
d	(175, 160) (15, 73, 72) (128, 62) (1, 71) (4, 70) (66)
e	(82, 80) (2, 11, 31, 36) (75, 9) (20) (46, 5) (41)
f	(88, 72) (16, 18, 38) (71, 31, 2) (20) (9, 49) (40)

a, *e*, *f* of reduction 2.

$C = 644$ a (72, 52, 69) (35, 17) (26, 60) (57, 15) (42, 8) (34)
b (89, 88) (8, 23, 57) (56, 26, 7) (15) (4, 34) (30)
c (88, 77) (13, 28, 36) (69, 17, 2) (15) (52, 8) (44)
d (79, 74) (15, 23, 36) (70, 26, 8) (18, 13) (49) (44)
e (44, 37) (7, 9, 21) (36, 13, 2) (11) (23, 1) (22)

c of reduction 4; the rest of reduction 2.

$C = 663$ a (148, 108, 130) (59, 27, 22) (5, 147) (32) (129, 19) (110)
b' (12, 7, 10) (4, 3) (1, 12) (1, 4) (10, 3) (7)
c (160, 113, 95) (18, 77) (72, 59) (135, 25) (13, 123) (110)
d (52, 36, 32) (4, 28) (16, 24) (49, 19) (11, 41) (30)
e (60, 58) (6, 7, 45) (43, 13, 4) (9, 1) (8) (30)
f (180, 155) (25, 53, 77) (148, 57) (29, 24) (5, 96) (91)
g (174, 160) (33, 53, 74) (155, 19) (52) (32, 21) (95) (84)

d, e of reduction 3; b' of 13.

$C = 672$ a (76, 47, 70) (24, 23) (20, 73) (5, 19) (67, 14) (53)
b (97, 72) (29, 43) (70, 23, 2) (19, 14) (5, 52) (49)

Both of reduction 2.

Order XIII.

$C = 928$ a (168, 164, 152, 149) (3, 146) (12, 143) (45, 131) (127, 41) (86)
b (196, 204, 193) (51, 142) (139, 49, 8) (41, 131, 40) (91) (90)
c (204, 200, 181) (19, 162) (4, 72, 143) (139, 69) (1, 71) (70)
d' (24, 25, 23) (2, 21) (20, 3, 1) (2, 7, 19) (5) (12)
e' (52, 51, 37) (14, 23) (1, 14, 41, 9) (40, 13) (32) (27)
f' (52, 31, 49) (13, 18) (8, 5) (3, 18, 51) (48, 15) (33)
g (280, 241) (75, 166) (127, 61, 56, 36) (20, 91) (5, 71) (66)
h (135, 123) (40, 83) (71, 36, 28) (8, 17, 43) (35, 9) (26)
i (133, 125) (8, 36, 81) (73, 40, 28) (19, 45) (33, 7) (26)
j (266, 247) (19, 60, 168) (149, 61, 34, 41) (27, 7) (108) (88)
k (250, 247) (3, 8, 52, 184) (181, 49, 18, 5) (13) (31) (132)
l (246, 241) (41, 200) (195, 51) (14, 27) (1, 13) (52) (40) (92)
m' (35, 25) (10, 15) (21, 10, 9, 5) (4, 16) (1, 12) (11)
n (270, 195) (75, 120) (193, 69, 38, 45) (31, 7) (24, 148) (124)

Reductions 2(h, i), 4(e', f') and 8(d', m').

$C = 935$ a (199, 210, 184) (26, 158) (143, 56) (45, 59, 132) (87, 14) (73)
b (39, 42, 31) (11, 20) (36, 3) (33, 14, 9) (5, 24) (19)
c (269, 254) (15, 81, 158) (143, 71, 70) (4, 77) (1, 73) (72)
d (269, 243) (26, 37, 180) (154, 71, 59, 11) (48) (12, 95) (83)
e (254, 243) (11, 12, 25, 195) (184, 66, 14, 1) (13) (52) (118)
f (264, 208) (56, 52, 100) (4, 48) (199, 81, 44) (37, 155) (118)

Reduction of b is 5.

$C = 937$ a (229, 154, 177) (131, 23) (200) (148, 81) (39, 92) (67, 14) (53)
b (268, 253) (90, 163) (148, 67, 53) (14, 39) (81) (17, 73) (56)
c (259, 256) (90, 166) (163, 96) (14, 76) (67, 24, 5) (19) (43)
d (259, 253) (81, 172) (166, 93) (38, 43) (73, 20) (53, 5) (48)
e (268, 229) (119, 110) (172, 96) (9, 101) (36, 92) (76, 20) (56)
f (264, 232) (23, 209) (9, 14) (177, 67, 24, 5) (19) (43) (110)
g (264, 209) (55, 154) (200, 119) (53, 101) (81, 38) (5, 48) (43)

$C = 943$ a (188, 164, 105, 151) (59, 46) (13, 184) (65, 171) (147, 41) (106)
b (207, 194, 184) (10, 174) (13, 27, 164) (151, 55, 14) (41) (96)

	c	(275, 246) (70, 176) (147, 55, 32, 41) (23, 9) (14, 106) (92)
	d	(263, 253) (10, 69, 174) (164, 50, 59) (41, 9) (32, 105) (73)
	e	(275, 205) (70, 135) (188, 92, 65) (27, 50, 123) (96, 23) (73)
$C = 960$	*a'*	(28, 25, 23) (2, 21) (8, 19) (16, 7, 5) (2, 11) (9)

Reduction 8.

$C = 962$	*a'*	(17, 9, 7, 12) (2, 5) (11) (17) (12, 5) (2, 9) (7)
	b	(138, 127) (38, 89) (78, 31, 15, 14) (1, 13) (16) (51) (47)
	c	(265, 254) (65, 189) (178, 87) (33, 32) (1, 31) (91, 30) (61)
	d	(276, 221) (127, 94) (189, 87) (33, 61) (15, 117, 28) (102) (89)

Reductions 2(*b*), 13(*a'*).

$C = 985$	*a*	(209, 205, 194) (11, 183) (44, 172) (168, 41) (1, 43) (42) (85)
	b	(204, 220, 169) (71, 98) (188, 16) (172, 64) (44, 27) (125) (108)
	c	(43, 44, 25) (11, 14) (8, 3) (17) (42, 1) (41, 12) (29)
	d	(269, 264) (5, 71, 188) (183, 91) (24, 47) (1, 23) (92) (70)
	e	(269, 253) (16, 27, 210) (194, 76, 15) (4, 23) (19) (42) (118)
	f	(264, 253) (11, 27, 215) (204, 55, 16) (43) (51, 4) (47) (98)
	g	(304, 209) (91, 118) (64, 27) (168, 76, 60) (145) (16, 108) (92)

Reduction of *c* is 5.

$C = 986$	*a*	(91, 62, 84, 85) (40, 22) (18, 87, 1) (86) (80, 11) (69)
	b	(219, 158, 216) (61, 97) (39, 177) (174, 70, 36) (34, 138) (104)
	c	(281, 264) (26, 61, 177) (160, 68, 44, 9) (35) (24, 116) (92)
	d	(136, 135) (1, 12, 35, 87) (86, 40, 11) (23) (6, 52) (46)
	e	(281, 244) (39, 97, 106) (182, 80, 58) (22, 124, 9) (115) (102)
	f	(132, 121) (11, 17, 40, 53) (108, 29, 6) (23) (79, 13) (66)

Reduction 2 (*a*, *d*, *f*).

$C = 992$	*a*	(224, 180, 211) (104, 76) (45, 166) (153, 71) (121) (11, 93) (82)
	b'	(31, 17, 24) (10, 7) (3, 28) (13) (21, 10) (1, 12) (11)
	c'	(61, 31, 48) (14, 17) (11, 3) (8, 60) (19) (47, 14) (33)
	d	(288, 263) (85, 178) (153, 83, 52) (44, 8) (93) (70, 13) (57)
	e	(279, 266) (85, 181) (168, 68, 43) (25, 18) (7, 11) (100) (96)
	f	(137, 133) (40, 93) (89, 48) (12, 28) (41, 19) (3, 25) (22)
	g	(143, 127) (28, 99) (83, 48, 12) (40) (35, 13) (9, 31) (22)
	h	(274, 263) (71, 192) (181, 93) (26, 45) (7, 19) (88, 12) (76)
	i	(275, 254) (45, 209) (188, 56, 31) (13, 32) (25, 6) (19) (132)
	j'	(60, 33, 39) (27, 6) (45) (56, 31) (13, 32) (25, 6) (19)
	k	(275, 244) (108, 136) (198, 77) (121, 64) (43, 93) (57, 7) (50)
	l	(286, 211) (83, 128) (209, 77) (38, 45) (31, 7) (24, 156) (132)

Reductions 2 (*f*, *g*); 4 (*c'*, *j'*) and 8 (*b'*).

$C = 1008$	*a'*	(57, 56, 39) (14, 25) (3, 11) (15, 44) (43, 14) (36) (29)
	b	(71, 72, 52) (19, 33) (70, 1) (5, 14) (69, 4) (9) (56)
	c	(285, 266) (75, 191) (172, 57, 56) (15, 116) (43, 14) (29) (72)
	d	(266, 263) (3, 44, 216) (213, 56) (15, 29) (57, 14) (43) (100)
	e	(95, 76) (24, 19, 33) (70, 25) (5, 14) (20, 9) (56) (45)

Reductions 3 (*b*, *e*); 4 (*a'*).

$C = 1015$	*a*	(180, 143, 167, 173) (37, 82, 24) (185, 6) (179) (172, 45) (127)
	b	(219, 215, 199) (16, 183) (17, 47, 167) (163, 43, 13) (30) (120)
	c'	(41, 43, 39) (4, 16, 19) (39, 2) (37, 12) (25, 3) (22)
	d	(244, 157, 207) (107, 50) (57, 200) (163, 61, 20) (41, 143) (102)
	e	(235, 178, 195) (57, 104, 17) (212) (172, 73, 47) (26, 125) (99)

f' (47, 30, 43) (17, 13) (16, 40) (36, 16, 12) (4, 24) (20)
g (222, 164, 207) (80, 41, 43) (39, 2) (37, 215) (200, 22) (178)
h (247, 154, 192) (116, 38) (230) (175, 72) (49, 67) (103, 18) (85)
i (284, 274) (10, 81, 183) (173, 73, 48) (25, 23) (2, 102) (100)
j (244, 164, 144) (20, 124) (80, 104) (219, 81, 24) (57, 195) (138)
k (284, 268) (16, 57, 195) (179, 80, 41) (39, 59) (99, 20) (79)
l (294, 253) (38, 215) (3, 35) (174, 91, 32) (67) (83, 8) (75)
m' (55, 54) (4, 10, 40) (39, 9, 4, 3) (1, 6) (5) (30)
n (274, 268) (6, 26, 31, 205) (199, 61, 20) (41, 5) (36) (138)
o (285, 253) (29, 224) (8, 21) (192, 70, 18, 5) (13) (52) (122)
p (275, 253) (22, 107, 124) (212, 85) (127, 48, 17) (31, 110) (79)
q (280, 247) (125, 122) (208, 72) (52, 20) (3, 119) (32, 116) (84)
r (270, 253) (17, 82, 59, 95) (222, 65) (23, 36) (157, 13) (144)
s' (57, 46) (25, 21) (43, 14) (4, 17) (29, 14) (1, 16) (15)

Reduction 5 (c', f', m', s').

$C = 1025$ a' (52, 36, 29) (7, 22) (28, 15) (37) (36, 16) (4, 24) (20)
b' (57, 55) (36, 10, 11) (17, 38) (9, 1) (8, 4) (21) (17)
c (51, 36, 25) (11, 14) (24, 20, 3) (17) (42, 9) (37) (33)
d (56, 55) (16, 39) (38, 18) (3, 4, 9) (20, 1) (5) (14)
e (57, 52) (20, 13, 19) (39, 18) (7, 6) (25) (3, 24) (21)
f' (55, 53) (9, 44) (42, 6, 7) (5, 1) (4, 13) (9) (22)
g (55, 51) (15, 8, 9, 19) (44, 11) (7, 1) (10) (33) (29)

All have reductions 5.

$C = 1033$ a (241, 169, 198) (140, 29) (227) (184, 57) (46, 11) (35, 116) (81)
b (236, 128, 116, 113) (3, 110) (12, 107) (140) (217) (204, 32) (172)
c (288, 259) (32, 227) (198, 76, 11, 3) (8, 27) (19) (46) (122)
d (304, 241) (119, 122) (184, 76, 44) (32, 12) (128, 3) (125) (108)
e (285, 259) (29, 230) (204, 43, 35, 3) (32) (8, 59) (51) (110)
f (285, 236) (107, 59, 70) (227, 58) (48, 11) (81) (169, 44) (125)
g (288, 230) (58, 172) (227, 119) (16, 43, 113) (108, 11) (27) (70)

$C = 1040$ a' (16, 12, 10, 13) (7, 3) (7, 5) (16) (13, 3) (12) (10)
b (252, 144, 197) (91, 53) (250) (27, 64) (195, 47, 10) (37) (148)
c (304, 263) (53, 210) (169, 73, 50, 12) (65) (23, 27) (96) (92)
d (290, 263) (39, 224) (197, 65, 28) (16, 23) (37, 7) (30) (132)
e' (29, 25) (12, 13) (21, 8) (13, 5, 2) (1, 12) (3) (8)
f' (65, 37, 30) (7, 23) (28, 16) (39) (63, 30) (3, 36) (33)

Reductions 4 (f'); 10 (e') and 13 (a').

$C = 1045$ a (225, 203, 205) (22, 82, 97, 2) (207) (187, 60) (127, 15) (112)
b (280, 278) (2, 77, 97, 102) (207, 75) (132, 20) (112, 5) (107)

$C = 1049$ a (236, 168, 229) (65, 103) (42, 187) (3, 62) (180, 59) (145) (121)
b (261, 150, 197) (103, 47) (244) (8, 95) (180, 89) (2, 93) (91)
c (298, 266) (47, 219) (187, 96, 15) (62) (19, 43) (91, 24) (67)
d (293, 266) (42, 224) (197, 59, 24, 13) (11, 2) (44) (35) (138)
e (261, 138, 153) (123, 15) (168) (236, 96, 52) (44, 8) (176) (140)
f (293, 244) (123, 121) (219, 74) (54, 67) (145, 52) (93, 13) (80)
g (298, 229) (89, 140) (224, 74) (54, 35) (19, 16) (3, 153) (150)

$C = 1058$ a (248, 169, 216) (115, 54) (7, 209) (61) (177, 71) (35, 141) (106)
b (233, 157, 225) (99, 58) (41, 17) (7, 218) (24) (210, 23) (187)
c (273, 161, 159) (2, 157) (163) (192, 81) (8, 149) (30, 141) (111)
d (253, 123, 83, 126) (40, 43) (163) (169) (220, 33) (187, 9) (178)
e (300, 281) (85, 196) (177, 80, 43) (20, 23) (17, 3) (111) (97)

	f	(145, 143) (45, 98) (96, 26, 23) (3, 20) (29) (12, 53) (41)
	g	(286, 281) (71, 210) (205, 81) (15, 56) (55, 41) (14, 83) (69)
	h	(290, 273) (67, 99, 107) (205, 85) (35, 32) (123, 8) (120) (115)
	i	(143, 138) (4, 7, 17, 110) (1, 3) (105, 39) (10) (27) (66)
	j	(150, 124) (56, 68) (105, 45) (15, 29, 12) (17, 63) (60) (46)
	k	(286, 253) (33, 58, 55, 107) (233, 86) (3, 52) (61) (159) (147)
	l	(312, 225) (78, 147) (9, 69) (209, 112) (67, 149) (97, 15) (82)
	m	(156, 109) (43, 66) (4, 16, 23) (108, 40, 12) (28) (89) (68)
	n'	(12, 11) (1, 3, 7) (11, 2) (5) (2, 5) (4, 1) (3)

Reductions 2 (*f*, *i*, *j*, *m*) and 23 (*n'*); *n'* is a *square*.

$C = 1060$	*a*	(122, 95, 102) (37, 51, 7) (109) (89, 33) (23, 14) (65) (56)
	b	(152, 137) (33, 104) (89, 38, 25) (7, 26) (13, 19) (51) (45)
	c	(147, 134) (8, 11, 115) (5, 3) (14) (102, 38, 12) (26) (64)
	d'	(27, 15, 14) (1, 13) (16) (23, 4) (4, 9) (19, 5) (14)
	e	(76, 61) (29, 32) (52, 24) (10, 16, 3) (35) (28, 6) (22)
	f	(147, 127) (20, 48, 59) (109, 58) (37, 11) (51, 7) (70) (44)
	g	(135, 134) (7, 127) (59, 51, 19, 6) (13) (32) (8, 75) (67)

Reductions 2, except *d'* (10) and *e* (4).

$C = 1065$	*a*	(73, 75, 63) (20, 43) (71, 2) (69, 8) (5, 38) (33)
	b	(247, 129, 232) (71, 58) (13, 45) (84) (52, 225) (210, 37) (173)
	c	(240, 188, 165) (47, 118) (52, 112, 24) (71) (232, 60) (189, 172)
	d	(284, 278) (6, 47, 225) (219, 43, 28) (15, 13) (60) (58) (118)
	e	(285, 248) (37, 112, 99) (247, 75) (28, 71) (172, 15) (43) (114)

Reduction of *a* is 3.

$C = 1066$	*a*	(121, 86, 115) (38, 48) (19, 96) (90, 31) (28, 10) (77) (59)
	b	(264, 162, 167) (157, 5) (172) (209, 55) (154, 58) (38, 134) (96)
	c	(153, 137) (31, 106) (90, 44, 19) (4, 27) (23) (2, 48) (46)
	d	(297, 280) (22, 49, 209) (192, 100, 5) (27) (76) (92, 8) (84)
	e	(153, 121) (50, 71) (106, 47) (29, 21) (25, 67) (59, 17) (42)
	f	(140, 132) (25, 48, 59) (121, 19) (2, 23) (21) (81, 11) (70)
	g	(297, 242) (55, 69, 118) (230, 88, 34) (20, 49) (54) (167) (142)

Reductions 2 (*a*, *c*, *e*, *f*).

$C = 1073$	*a*	(244, 192, 197) (100, 92) (87, 110) (196, 48) (156, 23) (148) (133)
	b	(252, 156, 89, 111) (67, 22) (133) (135, 88) (221) (213, 39) (174)
	c	(259, 133, 216) (79, 54) (25, 29) (100, 4) (249) (206, 53) (153)
	d	(261, 168, 164) (4, 41, 119) (135, 37) (78) (219, 42) (197) (177)
	e	(272, 168, 145) (23, 122) (92, 99) (12, 80) (221) (216, 68) (148)
	f	(301, 275) (53, 222) (196, 78, 27) (80) (49, 29) (10, 89) (69)
	g	(296, 275) (48, 227) (206, 68, 22) (17, 5) (12, 41) (29) (138)
	h	(293, 274) (42, 232) (213, 37, 20, 23) (17, 3) (68) (54) (122)
	i	(290, 274) (39, 235) (219, 48, 23) (25, 37) (61, 12) (49) (110)
	j	(272, 116, 164) (68, 48) (20, 192) (88) (249, 111) (27, 165) (138)
	k	(301, 244) (64, 61, 119) (227, 74) (3, 58) (67) (12, 165) (153)
	l	(293, 252) (99, 79, 74) (235, 58) (5, 69) (20, 64) (177) (133)

$C = 1082$	*a*	(126, 82, 114) (50, 32) (41, 105) (93, 33) (27, 23) (64) (60)
	b	(239, 162, 232) (77, 85) (15, 217) (210, 106) (100) (104, 2) (102)
	c	(241, 146, 228) (64, 82) (46, 18) (89, 239) (226, 15) (61) (150)
	d	(154, 140) (33, 107) (93, 41, 20) (1, 32) (21) (52, 10) (42)
	e	(154, 126) (53, 73) (107, 47) (22, 31) (11, 62) (60, 9) (51)
	f	(312, 241) (77, 164) (217, 89, 6) (83) (128, 44) (40, 124) (84)
	g	(156, 116) (41, 75) (113, 32, 11) (10, 31) (21) (81, 3) (78)

Reductions 2 (*a*, *d*, *e*, *g*).

$C = 1088$

a	(242, 172, 219) (99, 43, 30) (13, 17) (56) (236) (213, 29) (184)
b'	(33, 24, 19) (6, 13) (15, 8, 1) (7) (28) (27, 6) (21)
c	(280, 168, 145) (23, 122) (92, 99) (85, 7) (228) (215, 65) (150)
d	(310, 281) (65, 216) (187, 87, 36) (38, 63) (13, 25) (100) (88)
e	(296, 281) (51, 230) (215, 41, 40) (4, 47) (1, 43) (42) (132)
f	(298, 279) (23, 24, 232) (213, 81, 4) (26, 1) (25) (51) (132)
g'	(35, 21, 16) (5, 11) (20, 6) (17) (29, 6) (23, 3) (20)
h'	(37, 35) (12, 10, 13) (27, 10) (7, 3) (17, 5) (16) (12)
i'	(37, 35) (6, 29) (27, 6, 4) (3, 7) (5, 1) (4) (16)
j'	(7, 6, 5) (1, 4) (4, 3) (4, 3) (7) (1, 6) (5)
k'	(37, 33) (12, 8, 13) (29, 8) (3, 5) (1, 2) (21) (20)
l'	(57, 41, 42) (40, 1) (43) (33, 24) (9, 51, 4) (47) (42)
m'	(81, 59) (26, 33) (51, 26, 4) (23, 7) (40) (25, 1) (24)
n	(280, 279) (29, 250) (104, 63, 85, 28) (57) (41, 22) (164) (145)
o	(149, 125) (24, 40, 61) (121, 52) (19, 21) (17, 2) (84) (69)
p	(155, 119) (52, 67) (115, 40) (24, 13, 15) (11, 2) (84) (75)
q	(324, 221) (87, 134) (40, 47) (219, 81, 24) (57, 7) (188) (138)

Reductions 2 (o, p); 4 (l', m'); 8 (b', g', h', i', k') and 32 (j').

$C = 1102$

a	(120, 101, 115) (44, 43, 14) (29, 100) (95, 25) (1, 71) (70)
b	(239, 192, 232) (47, 105, 40) (65, 207) (200, 86) (28, 142) (114)
c	(119, 90, 113) (24, 43, 23) (20, 116) (5, 19) (110, 14) (96)
d	(249, 154, 230) (78, 76) (2, 65, 239) (17, 63) (220, 46) (174)
e	(249, 202, 164) (38, 126) (47, 105, 88) (238, 58) (17, 197) (180)
f	(152, 127) (25, 39, 63) (120, 43, 14) (29, 24) (5, 82) (77)

Reductions 2 (a, c, f).

$C = 1103$

a	(258, 204, 176) (28, 148) (105, 127) (207, 51) (156) (7, 141) (134)
b	(254, 148, 231) (73, 75) (71, 2) (69, 8) (239) (216, 38) (178)
c	(248, 204, 163) (41, 122) (52, 112, 81) (240, 8) (60) (203) (172)
d	(285, 137, 171) (103, 34) (69, 136) (105, 67) (225, 60) (203) (165)
e	(305, 286) (60, 226) (207, 57, 41) (16, 85) (73) (4, 81) (77)
f	(301, 285) (19, 34, 232) (216, 78, 7) (4, 15) (11) (60) (138)
g	(286, 128, 171) (85, 43) (42, 172) (127) (232, 54) (178, 3) (175)
h	(296, 286) (51, 235) (225, 43, 28) (15, 13) (2, 62) (60) (122)
i	(296, 285) (81, 76, 128) (226, 70) (5, 71) (156) (19, 109) (90)
j	(286, 285) (38, 247) (137, 112, 37) (75) (53, 134) (109, 28) (81)
k	(305, 258) (64, 75, 119) (235, 70) (53, 11) (42, 44) (165) (163)
l	(312, 248) (67, 62, 119) (5, 57) (231, 78, 3) (75) (176) (153)
m	(301, 254) (101, 153) (247, 54) (103, 52) (64, 141) (90, 13) (77)
n	(312, 240) (44, 60, 136) (28, 16) (76) (239, 101) (37, 175) (138)

$C = 1115$

a	(239, 205, 219) (60, 131, 14) (233) (213, 26) (86) (15, 116) (101)
b	(262, 154, 222) (86, 68) (35, 255) (69, 17) (52) (215, 47) (168)
c	(260, 145, 228) (88, 57) (31, 26) (254) (27, 92) (222, 65) (157)
d'	(53, 34, 36) (25, 9) (7, 29) (16) (47, 6) (41, 6) (35)
e'	(52, 51) (21, 30) (32, 20) (12, 20, 9) (39) (36, 8) (28)
f'	(61, 58) (5, 7, 46) (43, 13, 5) (3, 2) (9) (8) (30)
g	(292, 144, 157) (131, 13) (170) (79, 52) (230, 62) (27, 195) (168)
h	(304, 288) (13, 40, 235) (3, 10) (219, 81, 7) (17) (57) (138)
i	(292, 290) (47, 243) (122, 125, 45) (92) (119, 3) (116, 12) (104)
j	(304, 274) (30, 100, 144) (233, 101) (31, 69) (132) (25, 119) (94)
k	(324, 254) (100, 154) (213, 81, 30) (51, 79) (132) (25, 129) (104)
l	(305, 262) (105, 157) (243, 62) (87, 80) (28, 129) (7, 101) (94)
m	(324, 239) (101, 138) (228, 68, 28) (12, 52, 37) (40) (175) (160)
n	(288, 274) (14, 51, 87, 122) (265, 37) (88) (52, 35) (157) (140)

Reductions 5 (d', e', f').

$C = 1122$ *a* (42, 31, 39) (20, 11) (3, 36) (33, 9) (14) (24, 5) (19)
b (124, 80, 118) (42, 38) (35, 121) (11, 31) (115, 9) (20) (86)
c (287, 142, 164) (120, 22) (186) (25, 95) (242, 70) (7, 179) (172)
d (329, 248) (88, 160) (216, 106, 7) (95) (23, 137) (4, 114) (110)
e (329, 234) (88, 146) (30, 58) (230, 76, 23) (53) (25, 179) (154)
f (154, 127) (27, 29, 71) (126, 53, 2) (31) (20, 11) (82) (73)

Reductions 2 (*b*, *f*); 6 (*a*).

$C = 1145$ *a* (257, 169, 237) (120, 49) (30, 19) (11, 245) (41) (225, 32) (193)
b (277, 180, 176) (45, 131) (139, 41) (86) (235, 42) (12, 205) (193)
c (62, 60) (4, 9, 47) (45, 15, 2) (6) (1, 8) (7) (30)
d (300, 149, 159) (139, 10) (169) (12, 87, 40) (237, 75) (209) (162)
e (60, 28, 32) (24, 4) (36) (8, 16) (49, 19) (11, 41) (30)
f (310, 268) (42, 86, 140) (257, 95) (51, 35) (16, 19) (162) (159)
g (300, 277) (55, 91, 131) (268, 32) (87) (51, 40) (11, 160) (149)

Reductions 5 (*c*, *e*).

$C = 1156$ *a* (68, 46, 47) (30, 16) (15, 32) (14, 17) (60, 8) (52) (49)
b (138, 84, 97) (71, 13) (42, 68) (121, 17) (16, 26) (104) (94)
c (153, 152) (8, 23, 121) (120, 26, 7) (15) (4, 34) (30) (64)
d (141, 78, 85) (71, 7) (92) (133, 8) (51, 28) (23, 97) (74)
e (152, 141) (13, 60, 68) (133, 17, 2) (15) (32) (84, 8) (76)
f (76, 69) (15, 17, 37) (68, 8) (21, 2) (19) (39, 1) (38)

All have reductions 2, except *a*, *f* having 4.

$C = 1166$ *a* (264, 152, 247) (66, 86) (46, 20) (97, 9) (256) (239, 71) (168)
b (140, 96, 86) (10, 76) (63, 43) (20, 23) (121, 19) (102) (99)
c (271, 192, 170) (22, 148) (88, 126) (262, 9) (97) (59, 215) (156)
d (324, 271) (59, 84, 128) (247, 71, 6) (65) (40, 44) (176) (172)
e (154, 140) (33, 43, 64) (135, 19) (42, 10) (32, 21) (85) (74)
f (162, 131) (21, 32, 78) (10, 11) (128, 44) (43) (84, 3) (81)

Reductions 2 (*b*, *e*, *f*).

$C = 1170$ *a* (44, 27, 41) (14, 13) (12, 1) (3, 11) (42) (39, 8) (31)
b (91, 62, 58) (4, 54) (29, 37) (88, 32) (24, 13) (67) (56)
c (56, 42) (16, 26) (41, 13, 2) (11, 7) (4, 3) (29) (28)

Reductions 3 (*b*) and 6 (*a*, *c*).

$C = 1176$ *a* (47, 38) (7, 12, 19) (2, 5) (36, 13) (10, 7) (26) (23)

Reduction 7.

Mathematics. — *On the dissection of rectangles into squares.* (Third communication.) By C. J. BOUWKAMP. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of December 21, 1946.)

7. *Some special squared rectangles.*

Two differently squared rectangles are called *congruent* if their reduced sides are two by two equal. They are called *conformal* if their reduced sides are proportional. From the list in section 6 we can easily obtain all cases of conformal or congruent rectangles of order less than 14.

The simplest example of two conformal rectangles is provided by IX, 130, *c* and XII, 585, *f*, in such a sense that they have together 21 elements, 21 being the least possible number of elements totally involved. This example was already given in A (in that paper one should read $cf = 55$, instead of 15, p. 327). It may be noticed that the 12-th order squaring above shows two horizontal line segments at a same level, corresponding to a pair of equipotential vertices in the network from which it was derived. When this pair of conformal rectangles is made equal in size, they will show two elements in common. The imperfect squaring XIII, 1040, f' is also conformal with either of the squarings above.

Next there are two pairs of conformal rectangles containing 10 and 13 elements. The first pair is provided by X, 224, *a*, and XIII, 1008, *b*, whilst the second pair is formed by X, 224, *b* and XIII, 1008, *e*. Either of the 13-th order squarings contains two horizontal line segments at a same level. Upon transformation on the same size, four common elements are found in both cases.

Furthermore, the 13-th order squarings XIII, 1060, *e* and *f* are conformal; they show five common elements after transformation on equal size.

The following cases of conformal rectangles all include imperfections: XIII, 1088, k', l', m', two of which are even congruent, and XIII, 1088, g', h', i', j', three of which are congruent.

The remaining conformal rectangles are all of the congruent type. Two congruent rectangles of different full sides are XII, 615, *d* and XIII, 1025, *g*; they have one element in common. Also congruent and of different full sides are XIII, 935, *b* and XIII, 1122, *a*. It is interesting to note that they contain the same set of elements. *It is the simplest example of building up a rectangle with the same set of elements in two different ways,* as is seen by inspection of the list in section 6. This example is not new, as it was already derived in paper A, from certain general considerations, starting from networks with a pair of equipotential vertices.

All remaining cases of congruence, as far as the list of section 6 is

concerned, are such that they give pairs of squarings of the same order and the same full sides. The simplest example is provided by XII, 608, *f* and *g*. They have four elements in common [1]).

A pair of congruent simple squarings of order 13, one of which is imperfect, is formed by XIII, 1025, *b'* and *c*.

Five common elements occur in each of the following pairs of congruent perfect squarings:

XIII, 928, *h* and *i*; XIII, 992, *f* and *g*; XIII, 1088, *o* and *p*.

Four common elements in

XIII, 1015, *j* and *k*; XIII, 1088, *e* and *f*; XIII, 1115, *j* and *k*.

Three common elements in

XIII, 1073, *k* and *l*.

One common element in

XIII, 1015, *d* and *e*; XIII, 1073, *b* and *c*.

Last but not least, there is in our list a pair of congruent squarings with no common elements, namely XIII, 1015, *g* and *h*. *In fact it is the simplest example of a rectangle of given order that can be dissected into two*

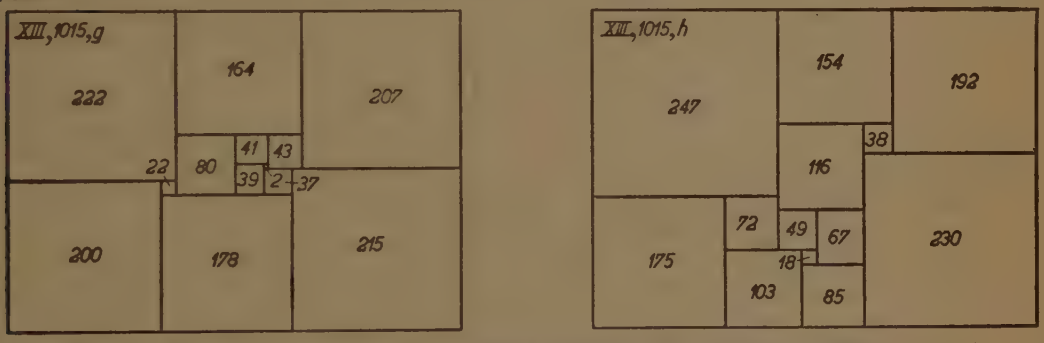

Fig. 9. The simplest example of congruent squarings with completely different sets of elements.

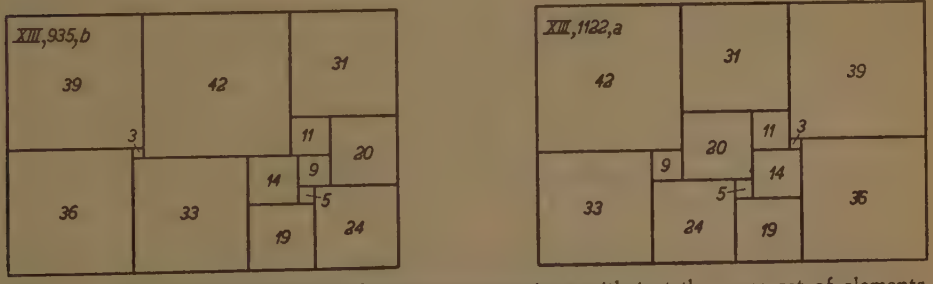

Fig. 10. The simplest example of congruent squarings with just the same set of elements differently arranged.

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[1]) See also paper A, p. 330; the concept of "equivalence" in A is completely different from that defined in section 3. In the sense of paper A, equivalent squarings are conformal and have the same full sides.

completely different sets of squares. It may be remarked that they originate from essentially different networks, though their complexity is the same.

Figs 9, 10 show the simplest examples of congruent rectangles that contain either a completely different set of elements, or just the same elements differently arranged.

8. *Perfect squares.*

We have already given the simplest example of a simple squared square; cf. fig. 1. It is not at all perfect, however, as it contains double elements. One naturally asks whether a simple *perfect* square can be constructed. The authors of paper A have given a device how this may be done. They pretend to have constructed "a simple 'uncrossed' perfect square of order 55, which, when drawn out, disguises its symmetrical origin very skilfully" (A, p. 334). In this section, however, we shall prove that actually the suggested construction *fails*, and so we are led to the conclusion that, unfortunately, there is no simple perfect squared square known at present.

We first show how *compound* perfect squares can be obtained. Three different methods were developed in paper A.

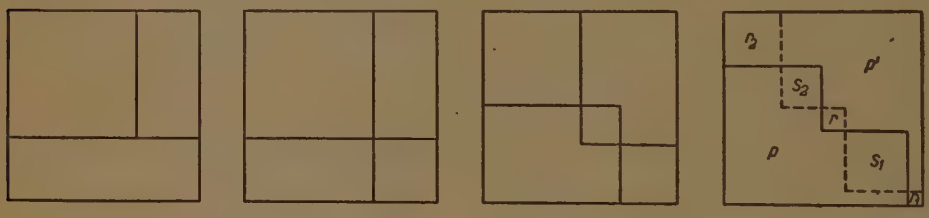

Fig. 11. Several methods for the construction of perfect squared squares.

1. Fig. 11, α shows a compound square, consisting of one square and two rectangles which in their turn are supposed to be dissected into squares.

From careful examination of the list given in section 6 it appears that only *three* different solutions exist if both rectangles are subject to the condition that their orders do not exceed 13. *Two* solutions are non-trivially imperfect, as for each of them both rectangles are non-trivially imperfect themselves. One solution contains a square of side 16, together with the squared rectangles XIII, 962, a' and XII, 585, a'. The other consists of a square 22, together with the squarings XII, 663, b' and XIII, 1040, a'.

Only the third solution happens to be perfect. It contains a square of side 231, together with the perfect squarings XII, 608, a and XIII, 985, a. Thus we have obtained a perfect square of order 26. Presumably *it is the simplest* (that means of minimum order) *perfect squared square known at present.* It was already given in paper A; its code has been given in section 1 of the present paper.

2. Fig. 11, β shows a compound square, consisting of two squares and two rectangles of equal size which in their turn may be dissected into squares.

Obviously, it is not difficult to construct non-trivially imperfect squares of this character: we merely take twice any rectangle of our list in section 6. On the other hand, there is in our list only one couple of squarings leading to a perfect square. This square is of the order 28, containing two squares of sides 422, 583, respectively, together with the pair of totally different congruent squarings of fig. 9.

3. Cf. fig. 11, γ.

For this purpose two perfect squarings of equal size are required, having only one element in common, subject to the condition that this common element is corner element in either of the two rectangles. In paper A a device was given how to construct a pair of such special rectangles. One of the simplest examples of perfect squares, constructed in this way, is of the order 39. Its code reads as follows:

(900, 393, 520) (263, 130) (3, 224, 293) (133) (244, 152) (155, 69) (61, 91) (362) (31, 30) (276) (476, 205, 219, 275) (191, 14) (177, 56) (113, 218, 638) (8, 105) (80, 199, 97) (437, 119) (420) (318).

All squares so far obtained are compound. We now investigate in detail the method that (according to the authors of A) would lead to simple perfect squared squares.

Consider the "rotor" network of fig. 12, with terminals A_1, A_2, A_3. Let its wires have unit conductance, and let currents $87a$, $87b$, leave the network at A_2, A_3, respectively. The current entering at A_1 must then be $87\ (a + b)$. The complete set of currents is uniquely determined, and is shown in fig. 12. The currents are integral linear combination of a and b. Without lack of generality, we may suppose a and b to be integers, subject to $0 < b < a$.

This network is a generalization of the "polar" networks treated before, in so far that now more than two terminals are present. It corresponds to a *squared polygon* of angles $\pi/2$ and $3\pi/2$. For example, the rotor network of fig. 12, in action, corresponds to a squared polygon P the dimensions of which are shown in fig. 13.

The typical corner elements C_1, C_2 (shaded in fig. 13) have sides $27a - 8b$, $8a + 35b$, respectively. It must be noted that the situation of fig. 13 is possible only if $27a - 8b < 49\ (a - b)$ and $8a + 35b < 87b$; thus $41/22 < a/b < 13/2$. Otherwise at least one of the corner elements is too large. If the inequality above is not fulfilled, it is impossible to draw in fig. 13 the rectangle R which is important in the further construction.

The vertical left side of the polygon may be considered as the terminal A_1, and the remaining vertical boundaries at the right correspond to A_3, A_2. The current flows horizontally from left to right. The ingoing current $87(a + b)$ is equal to the left vertical side, the two outgoing currents $87a$, $87b$, are equal to the other vertical boundaries.

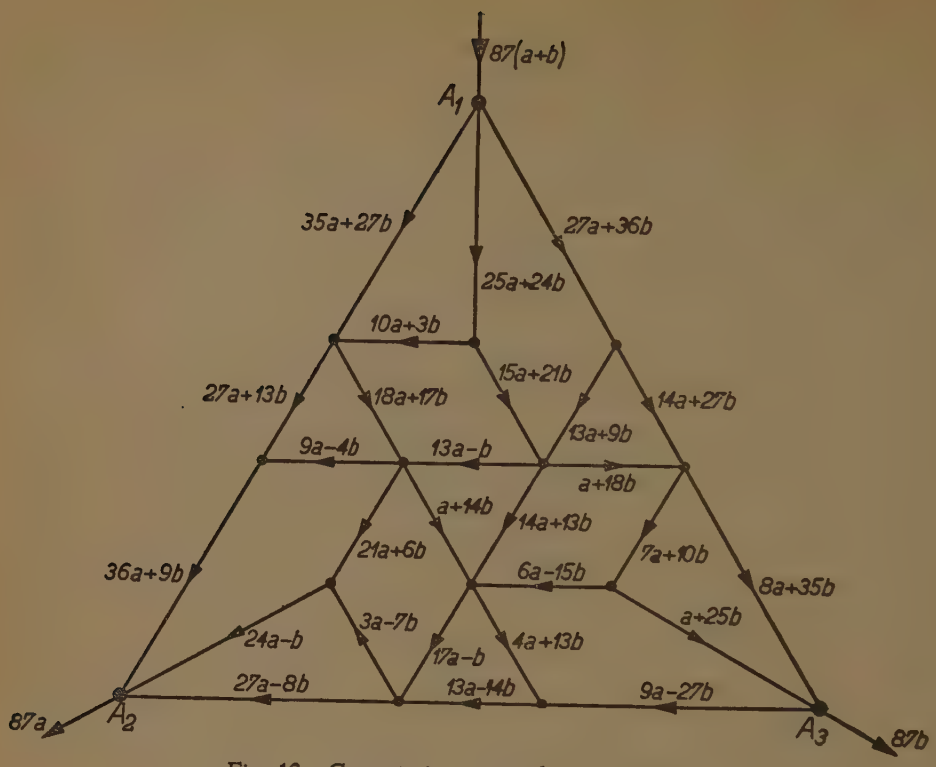

Fig. 12. Currents in a typical rotor network.

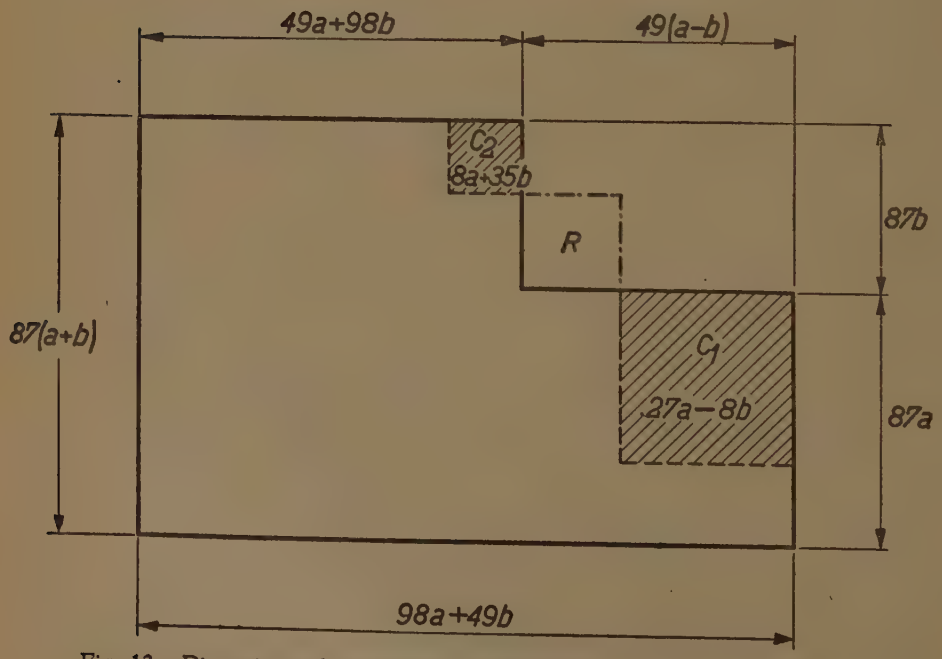

Fig. 13. Dimensions of the polygon corresponding to the rotor of fig. 12.

If the skew-symmetrical rotor network of fig. 12 is replaced by its reflection (leaving the currents at A_1, A_2, A_3, invariant), the new squared polygon P' will have the same shape as the old one P; this follows from the triad symmetry of the rotor. The set of currents in the reflected rotor are easily found from those in fig. 12. We also could have interchanged a and b, without reflecting the rotor; we prefer, however, the former method, in order to have always $a > b$ in the following.

The corner elements C'_1, C'_2 of P' have sides $27a - 9b$, $9a + 36b$, respectively. In order to be able to draw the analogous rectangle R', another condition has to be fulfilled, namely $40/22 < a/b < 17/3$. If both P and P' have existing rectangles R, R', we thus have the condition

$$41/22 < a/b < 17/3.$$

Let us now consider fig. 11, δ. It contains two congruent polygons; one of them (p) has full-drawn boundaries, the other (p') is partly dashed. Suppose (i) either of the two polygons is perfectly squared, (ii) the polygons p, p' have no common elements, except the pairs of typical corner elements $c_1 = c'_1 = S_1$, and $c_2 = c'_2 = S_2$, overlapping two by two. Then it clearly follows: r, r_1, r_2 are squares. Furthermore, if these squares are mutually unequal, and none of them is equal to some element of p, p', then fig. 11, δ obviously leads to a *simple perfect squared square*. The elements not drawn out in fig. 11, δ belong to p, p'.

This is in fact the construction proposed in paper A, in order to obtain simple perfect squares. For an actual example we only need two suitably chosen polygons p, p'. The authors of A suggest that the rotor network of fig. 12 leads to such a pair of polygons. This, however, is not true, as will be seen below.

It is important to note that the shape of P, P' can be varied by varying a and b. Obviously, there is only *one* degree of freedom, as only the ratio a/b is significant. It must be taken in mind that the elements of P, P' alter correspondingly, by variation of a/b. Now, P and P' can be used as p, p' respectively, if, and only if, $C_1 = C'_1$; $C_2 = C'_2$. That means

$$27a - 8b = 27a - 9b,$$
$$8a + 35b = 9a + 36b.$$

These independent conditions admit only $a = b = 0$. Hence *the construction of fig* 11, δ *fails for the rotor network of fig.* 12.

This can also be seen in a somewhat differing manner. If P is to be used as p, then the rectangle R in fig. 13 has to be a square, corresponding to r in fig. 11, δ. From this condition it follows $87b - (8a + 35b) = 49(a - b) - (27a - 8b)$; thus $10a = 31b$. For the special values $a = 31$, $b = 10$ (the inequality above is fulfilled!), one can only *hope* that the now uniquely determined polygon P' can be used as the counter part p' of $p = P$. This, however, is not so. R' is even not a square, let

alone it is equal to R, as is required. In fact, R' is a square, if, and only if, $a = 91$, $b = 31$.

Thus the construction fails. There is, consequently, no simple perfect squared square known at present. Moreover, it will be very unlikely, that for some other rotor network the above construction is effective.

July, 1946.

Natuurkundig Laboratorium der N.V. Philips' Gloeilampenfabrieken, Eindhoven, the Netherlands.

Mathematics. — *On the elastic stability of sandwich plates.* I. By P. P. BIJLAARD. (Communicated by Prof. F. K. TH. VAN ITERSON.)

(Communicated at the meeting of December 21, 1946.)

As we remarked at the end of our preceding paper [1]), our method of subdividing the elastic behaviour of a system may be applied also to composite plates, as used in aeroplane construction, with arbitrary boundary conditions.

Let us consider a sandwich plate (fig. 1), simply supported at the edges $x = 0$, $y = 0$, $x = a$ and $y = b$ and subjected to compressive stresses ϱ_1 and ϱ_2 in X- and Y-directions respectively. We firstly assume

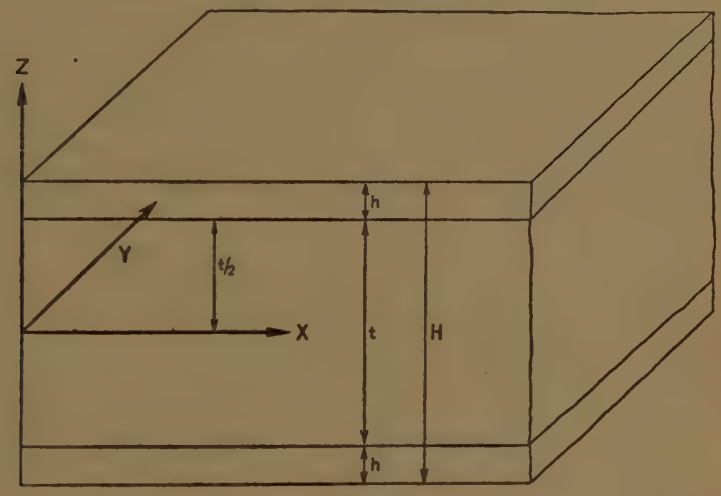

Fig. 1

the modulus of rigidity G of the weak inner layer to be infinite, as far as its resistance to shearing stresses τ_{zx} and τ_{zy} is concerned. According to the usual theory of thin plates, the rigidity of the outer plates with respect to these shearing stresses, is anyhow assumed to be infinite. For this *first case* the partial differential equation for the buckling of isotropic plates applies, being [2])

$$D\left\{\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right\} + P_{xm}\frac{\partial^2 w}{\partial x^2} + P_{ym}\frac{\partial^2 w}{\partial y^2} = 0. \quad . \quad (1)$$

Value w is in our case the deflection of the composite plate, D is its flexural rigidity and P_{xm} and P_{ym} are the compressive forces $2h\,\varrho_1$ and

1) BIJLAARD. On the elastic stability of thin plates, supported by a continuous medium. Proceedings Royal Netherlands Academy of Sciences. 1946. Nr. 10.

2) TIMOSHENKO. Theory of elastic stability, p. 324.

$2h\,\varrho_2$ per unit breadth, assumed to be taken by the outer plates alone. In computing D we leave the proper moments of inertia $h^3/12$ of the outer plates provisionally out of account, as we did also in our preceding paper, because here too these outer plates cannot be assumed as rigid in the second case. In this way we get

$$D = \tfrac{1}{2} E_1' h (t+h)^2 + \tfrac{1}{12} E' t^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which $E'_1 = E_1/(1-\nu_1^2)$ and $E' = E/(1-\nu^2)$, E_1 and E being the moduli of elasticity of outer plates and layer respectively. Assuming

$$w = w_0 \sin(m\pi/a)\, x \sin(n\pi/b)\, y \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which satisfies the boundary conditions, whilst $P_{ym} = \beta P_{xm}$ and $\beta < 1$, eq. (1) yields, after simple ranging, the critical thrust

$$P_1 = P_{xm} = \frac{\pi^2 (n^2 a^2 + m^2 b^2)^2}{a^2 b^2 (\beta n^2 a^2 + m^2 b^2)} D. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In the *second case* we should now assume only the modulus of rigidity G of the layer with respect to shearing stresses τ_{zx} and τ_{zy} to have a finite rigidity, but, as with the buckling of the strips, as considered in our preceding paper, both cases cannot be entirely separated. In fig. 2 the

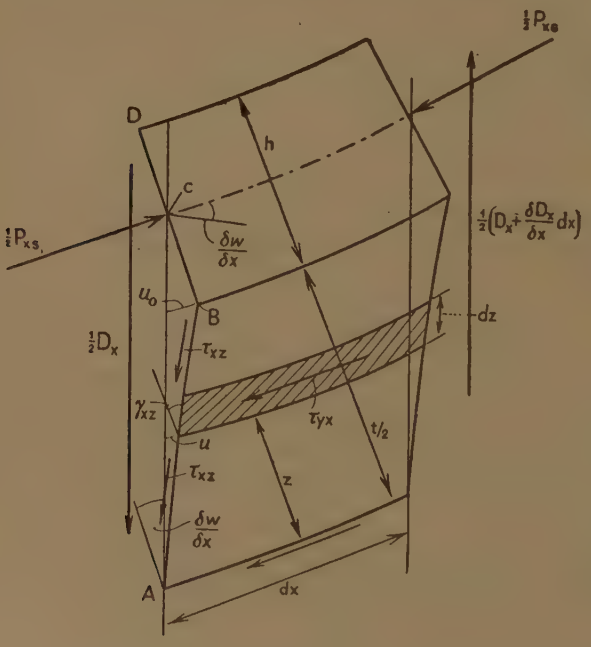

Fig. 2

deformation of an element $\frac{1}{2} H dx dy$ of the upper half of the composite plate is shown, looking in the direction of the positive Y-axis, the outer plates being assumed here too to have no flexural rigidity. As in the second

case in our preceding paper, we neglect here the strains in the layer in X- and Y-directions. Their influence is indeed immaterial, as even the total effect of the stresses σ_x and σ_y in the layer is of no practical value. In eq. (35) of our preceding paper the effect of the stresses σ_x in the layer is given by the last terms of numerator and denominator in the last fraction, which, in the example given there, had only an influence of less than 0.1 %. Moreover this formula (35), derived in the indicated way, coincides practically with our formula (28), found by expansion of sinh and cosh in our exact solution in series and gave in our example the same results.

Furthermore, with the deformation shown in fig. 2, the displacement u in X-direction varies also in Y-direction, by which torsional stresses τ_{yx} will originate in the layer. The total effect of these stresses τ_{yx} is of the same order of magnitude as that of the stresses σ_x and σ_y in the layer, the total effect of all these stresses being represented in the first case by the term $E' t^3/12$ of the flexural rigidity D in eq. (2). Moreover the effect of these torsional stresses in the second case is very small, as they only cause an increase of the angle γ_{xz} in fig. 2 with increasing z, so that the line AB will be curved a little, whilst without these stresses it would be straight. With a given deflection, however, the angle between the straight line AB and the vertical AC, being $(h/t)\,\partial w/\partial x$, is fixed, so that the torsional stresses do neither influence the average shearing strain γ_{xz} nor the average shearing stress τ_{xz} between A and B, nor the total shearing force D_x along AB. Only in B, at the bottom of the outer plate, γ_{xz} and thus also the shearing stress $\tau_{zx} = \tau_{xz}$ at the bottom of the plate, is increased by the torsional stresses τ_{yx}, so that the vertical shearing force along BC will be somewhat increased by them. We will neglect these torsional stresses too, whilst moreover we will show at the end, that their influence is indeed negligible in this case.

The line AB being now straight, we find that between A and B (fig. 2)

$$\tau_{xz} = G\gamma_{xz} = G\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = G\,\frac{t+h}{t}\,\frac{\partial w}{\partial x}. \quad . \quad . \quad . \quad (5)$$

The shearing force, conveyed to the outer plate by the shearing stresses τ_{zx} at its bottom, will be equally distributed over the total height h of the plate, so that from B to D in fig. 2 the vertical shearing stress τ_{xz} will decrease linearly to zero. Thus the total shearing force D_x acting on an element $Hdxdy$ of the composite plate will be

$$D_x = (t+h)\,\tau_{xz} = \frac{(t+h)^2}{t}\,G\,\frac{\partial w}{\partial x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

whilst in the same way we obtain

$$D_y = \frac{(t+h)^2}{t}\,G\,\frac{\partial w}{\partial y}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The equilibrium in Z-direction of the element $Hdxdy$ requires

$$\frac{\partial D_x}{\partial x}\,dx\,dy+\frac{\partial D_y}{\partial y}\,dx\,dy-P_{xs}\frac{\partial^2 w}{\partial x^2}\,dx\,dy-P_{ys}\frac{\partial^2 w}{\partial y^2}\,dx\,dy=0$$

or, using eqs, (6) and (7)

$$\frac{(t+h)^2}{t}\,G\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)-P_{xs}\frac{\partial^2 w}{\partial x^2}-P_{ys}\frac{\partial^2 w}{\partial y^2}=0. \quad . \quad . \quad (8)$$

A deflection according to eq. (3) satisfies also this differential equation. Substitution yields, with $P_{ys}/P_{xs}=\beta$

$$P_2=P_{xs}=\frac{n^2a^2+m^2b^2}{\beta n^2a^2+m^2b^2}\,\frac{(t+h)^2}{t}\,G. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Furthermore the rigidity of the outer plates, left out of account till now, will cause an excess critical thrust P_{xo}. As for this case eqs. (1) and (3) apply, this excess critical thrust will be given too by eq. (4), if D is equaled to the total flexural rigidity $E_1' h^3/6$ of the two outer plates.

The fact that eq. (3) satisfies also eq. (8) and thus all cases of buckling, causes of course that our method will yield very accurate results here, as it did also in the case of the strips in our preceding publication, where buckling in a half sine wave satisfied also all cases. In general the results will be less accurate the more the buckling forms for the cases in which the system is divided differ from each other.

So the critical thrust of the composite plate is now, according to our preceding publication

$$P_x=P_{x0}+(P_{xm}^{-1}+P_{xs}^{-1})^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

which yields, after using eqs. (4) and (9) and some transformation

$$\left.\begin{aligned}P_x=2h\varrho_1=\frac{\pi^2(n^2a^2+m^2b^2)^2E_1'h^3}{6(\beta n^2a^2+m^2b^2)a^2b^2}+\\+\frac{\pi^2(n^2a^2+m^2b^2)^2(t+h)^2DG}{(\beta n^2a^2+m^2b^2)a^2b^2(t+h)^2G+\pi^2(n^2a^2+m^2b^2)(\beta n^2a^2+m^2b^2)tD}\end{aligned}\right\} \quad (11)$$

in which D is given by eq. (2), β is P_y/P_x and the numbers of half waves m and n in X- and Y-direction respectively have to be chosen in such a way that P_x is as small as possible. With compression in one direction only, so with $P_y=2h\varrho_2=0$ and $\beta=0$, in eqs. (4) and (9) value n appears only in the numerators, so that P_{xo}, P_{xm} and P_{xs}, and thus also P_x, will be as small as possible with $n=1$. Denoting furthermore the half wave length a/m by L and substituting D according to eq. (2), we obtain as the critical thrust with compression in X-direction

$$\left.\begin{aligned}P_x=2h\varrho_1=\frac{\pi^2}{6}\left(\frac{b^2+L^2}{b^2L}\right)^2E_1'h^3+\\+\frac{\pi^2E_1'(b^2+L^2)^2(t+h)^2h+(\pi^2\,6)E'(b^2+L^2)^2t^3}{2\pi^2(1+\nu)E_1'(b^2+L^2)b^2th+2b^4L^2E+(\pi^2/3)(1+\nu)E'(b^2+L^2)b^2t^4/(t+h)^2}E\end{aligned}\right\} \quad (12)$$

in which the half wave length L, that has to be an integer part of the total length a, but that with a larger than about $4b$ may sufficiently accurately be chosen arbitrarily, has to be determined such that P_x is as small as possible, respectively a minimum. As according to eqs. (4) and (9) and with $\beta = 0$ and $n = 1$, P_1 and P_2 will be minimum if $L = a/m$ is equal to b or to zero respectively, the real half wave length L will have to be smaller than b here.

The amount of work, saved by our method, may be appreciated by comparing it with the computations executed by VAN DER NEUT in order to obtain the critical thrust of sandwich plates with the same boundary conditions [3]). Moreover his method is already a simplification with respect to the exact one and, as he assumes h to be infinitely small, his case is simpler, so that in our second case the torsional stresses would be zero and the proof of their insignificance, as given under here, would be superfluous. He obtains, transformed in our notations and taking into account that in his equations we have, according to our notations (fig. 1), to replace t by $t + h$

$$P_x = \frac{\pi^2}{6}\left(\frac{b^2+L^2}{b^2 L}\right)^2 E_1' h^3 + \left. \vphantom{\frac{a}{b}} \right\} \quad (N_3)$$

$$+ \frac{\pi^2 E_1' (b^2+L^2)^2 (t+h)^2 h + (\pi^2/6) E' (b^2+L^2)^2 (t+h)^3}{2\pi^2 (1+\nu) E_1' (b^2+L^2) b^2 (t+h) h + 2 b^4 L^2 E} E \left. \vphantom{\frac{a}{b}} \right\}$$

the difference with our result being principally caused by the fact, that he assumes the outer plates and the intermediate layer to be connected to one another in the middle plane of the outer plates, whilst we took the real connection at the bottom of the plates into account. The error of his formula is of the same order of magnitude as that of his formula (N_2) for strips, quoted in our preceding paper.

Finally we will show, that with the deformation according to our second case (fig. 2), the torsional stresses have indeed a negligible influence. Expressing them in the displacements u and v in X- and Y-direction respectively, we have

$$\tau_{yx} = G\gamma_{yx} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right). \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The force acting on an element $dxdydz$, indicated by cross hatching in fig. 2, is

$$\left\{\tau_{yx} - \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\, dy\right)\right\} dx\, dz = -\frac{\partial \tau_{yx}}{\partial y}\, dx\, dy\, dz$$

working in the positive direction of τ_{yx}, as shown in fig. 2. By that τ_{zx} increases with increasing distance z, as

$$\frac{\partial \tau_{zx}}{\partial z}\, dz\, dx\, dy = -\frac{\partial \tau_{yx}}{\partial y}\, dx\, dy\, dz$$

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[3]) VAN DER NEUT. Die Stabilität geschichteter Platten. Nationaal Luchtvaart Laboratorium, Bericht S. 286, 1943.

from which, after substitution of $\tau_{zx} = \tau_{xz}$ and τ_{yx} according to eqs. (5) and (13) respectively, we obtain the following partial differential equation

$$-\frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x\, \partial y} \quad \ldots\ldots\ldots \quad (14a)$$

and after interchanging x and y as well as u and v

$$-\frac{\partial^2 v}{\partial z^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x\, \partial y} \quad \ldots\ldots\ldots \quad (14b)$$

According to fig. 2 and with the same interchanging we get

$$\left.\begin{aligned} u = u_0 Z_x = Z_x \frac{h}{2} \frac{\partial w}{\partial x} \\ v = v_0 Z_y = Z_y \frac{h}{2} \frac{\partial w}{\partial y} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (15)$$

in which Z_x and Z_y are functions of z only. Substitution of these values in eqs. (14a) and (14b) and using again eq. (3) yields

$$\left.\begin{aligned} Z_x'' = \frac{n^2 \pi^2}{b^2} (Z_x + Z_y) \\ Z_y'' = \frac{m^2 \pi^2}{a^2} (Z_x + Z_y) \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (16)$$

an accent indicating a differentiation with respect to z. Addition of both equations (16) yields

$$(Z_x + Z_y)'' = \lambda^2 (Z_x + Z_y) \quad \ldots\ldots\ldots \quad (17)$$

in which

$$\lambda^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) \quad \ldots\ldots\ldots \quad (18)$$

whilst the general solution of eq. (17) is

$$Z_x + Z_y = C_1 \cosh \lambda z + C_2 \sinh \lambda z. \quad \ldots\ldots\ldots \quad (19)$$

At $z = 0$ the displacements u and v are zero, so that, according to eq. (15), also $Z_x + Z_y$ has to equal zero there, yielding $C_1 = 0$. At $z = t/2$ the values u and v equal u_0 and v_0, so that $Z_x + Z_y = 2$, yielding $C_2 = 2/\sinh \varphi$, in which $\varphi = \lambda t/2$. Thus eq. (19) transforms in

$$Z_x + Z_y = 2 \sinh \lambda z / \sinh \varphi. \quad \ldots\ldots\ldots \quad (20)$$

Substitution of this equation in eq. (16) and integrating two times yields

$$Z_x = \frac{2 n^2 \pi^2}{b^2 \lambda^2} \frac{\sinh \lambda z}{\sinh \varphi} + D_1 z + D_2. \quad \ldots\ldots\ldots \quad (21)$$

After introduction of the boundary conditions at $z = 0$ and $z = t/2$,

being $Z_x = 0$ and $Z_x = 1$ respectively, using eq. (18) and interchanging of a/m and b/n to find Z_y, we obtain

$$\left.\begin{aligned} Z_x &= \frac{2\,n^2\,a^2}{n^2\,a^2+m^2\,b^2}\,\frac{\sinh \lambda z}{\sinh \varphi} - \frac{2\,(n^2\,a^2-m^2\,b^2)}{t\,(n^2\,a^2+m^2\,b^2)}\,z \\ Z_y &= \frac{2\,m^2\,b^2}{n^2\,a^2+m^2\,b^2}\,\frac{\sinh \lambda z}{\sinh \varphi} + \frac{2\,(n^2\,a^2-m^2\,b^2)}{t\,(n^2\,a^2+m^2\,b^2)}\,z \end{aligned}\right\} \quad . \quad . \quad (22)$$

so that, using also eqs. (5) and (15), we find

$$\tau_{xz} = G\,\frac{\partial w}{\partial x}\left[1+\frac{h}{2}\left\{\frac{2\,n^2\,a^2\,\lambda}{n^2\,a^2+m^2\,b^2}\,\frac{\cosh \lambda z}{\sinh \varphi} - \frac{2\,(n^2\,a^2-m^2\,b^2)}{t\,(n^2\,a^2+m^2\,b^2)}\right\}\right] \quad (23)$$

and

$$\left.\begin{aligned} D_x &= 2\int_0^{t/2} \tau_{xz}\,dz + h\,(\tau_{xz})_{z=t/2} \\ &= G\,\frac{\partial w}{\partial x}\left[(t+h)+h+h^2\,\frac{(2\,\varphi\coth\varphi-1)\,n^2\,a^2+m^2\,b^2}{(n^2\,a^2+m^2\,b^2)\,t}\right] \end{aligned}\right\} \quad . \quad (24)$$

The term $(t + h)$ relates to the integral, so that comparison with eq. (5) demonstrates, that the shearing force along the layer is indeed the same as with neglection of the torsional stresses. Furthermore, as in all computations in this paper we neglected the influence of stresses σ_z perpendicular to the middle plane of the plates, as is done also in the theory of thin plates, we may state that the equations derived here will only be accurate if the ratios a/mt and b/nt are more than about 8 or 10. So according to eq. (18) value $\varphi = \lambda t/2$ is smaller than 0,28 and φ *coth* φ is less than 3 % more than unity. Thus the last term between the large brackets in eq. (24), that is relatively very small, may be equated to h^2/t, by which eq. (24) transforms in eq. (6), so that, also if the torsional stresses are taken into account, the same value P_{xs} as in eq. (9) is obtained, q.e.d.

Also with other boundary conditions our method may be applied. We will consider for example the buckling of a uniformly compressed rectangular plate, simply supported along the loaded sides, $x = 0$ and $x = a$, and having various boundary conditions along the other two sides (fig. 3). Dividing the system in the same cases as before, for the first case

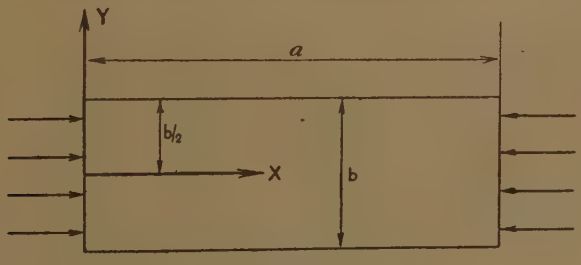

Fig. 3

(P_{xm}) as well as for the outer plates (P_{xo}) differential equation (1), with $P_{ym} = 0$, applies, whilst for the second case differential equation (8) is valid. Assuming

$$w = Y \sin (m\pi/a)\, x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

in which Y is a function of y only, substitution in eqs. (1) and (8) yields respectively

$$Y_1'''' - 2\frac{m^2\pi^2}{a^2} Y_1'' - \frac{m^2\pi^2}{a^2}\left[\frac{P_{xm}}{D} - \frac{m^2\pi^2}{a^2}\right] Y_1 = 0 \quad . \quad . \quad (26)$$

$$Y_2'' + \frac{m^2\pi^2}{a^2}\left[\frac{t}{(t+h)^2}\frac{P_{xs}}{G} - 1\right] Y_2 = 0 \quad . \quad . \quad . \quad . \quad (27)$$

Owing to the restraint at the unloaded sides, $P_{xm}/D > m^2\pi^2/a^2$ and $P_{xs}/G > (t+h)^2/t$, so that the terms in the large brackets in eqs. (26) and (27) are positive and the general solutions of these equations are represented respectively by

$$Y_1 = C_1 \cosh \alpha y + C_2 \sinh \alpha y + C_3 \cos \beta y + C_4 \sin \beta y \; . \quad . \quad (28)$$

$$Y_2 = C_5 \cos \psi y + C_6 \sin \psi y \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

in which

$$\alpha = \sqrt{\frac{m^2\pi^2}{a^2} + \sqrt{\frac{m^2\pi^2}{a^2}\frac{P_{xm}}{D}}}, \quad \beta = \sqrt{-\frac{m^2\pi^2}{a^2} + \sqrt{\frac{m^2\pi^2}{a^2}\frac{P_{xm}}{D}}} \; . \quad (30)$$

$$\psi = \frac{m\pi}{a}\sqrt{\frac{t}{(t+h)^2}\frac{P_{xs}}{G} - 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

The constants C_1—C_6 depend on the boundary conditions along the edges $y = \pm b/2$. If, for example, both sides are built in, the required symmetry demands C_2, C_4 and C_6 to be zero. Further the boundary conditions $w = 0$ and $\partial w/\partial y = 0$ at $y = b/2$ lead here for eq. (28) to two homogeneous linear equations in C_1 and C_3. Equation of the denominator determinant of these equations to zero yields, in the same way as with the stability of isotropic plates, to the buckling condition

$$\alpha \tanh (\alpha b/2) = -\beta \tan (\beta b/2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

With eq. (29) the condition $w = 0$, and so $Y = 0$, at $y = b/2$, leads for the smallest critical thrust to the condition $\psi b/2 = \pi/2$, yielding according to eq. (31)

$$P_{xs} = \frac{a^2 + m^2 b^2}{m^2 b^2}\frac{(t+h)^2}{t} G \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

and being of course identical to eq. (9), with $\beta = 0$ and $n = 1$.

In order to find an approximate value of the critical thrust P_x we may solve eqs. (32) and (33) for various values m and substitute the results in

eq. (10). Computing P_{xm} we have to equal D in eqs. (30) to the value given in eq. (2), whilst P_{xo} is found by replacing P_{xm} and D in eqs. (30) by P_{xo} and $E'_1 h^3/6$ respectively. Value m has to be chosen such as to make P_x in eq. (10) as small as possible.

In this case the middle plane of the plate has a different form in cases 1 and 2, as in the first case eq. (28) holds, with C_1 and C_3 differing from zero, whilst in the second case eq. (29), with $C_6 = 0$, is valid. As we equal the values a/m, the middle planes coincide where $w = 0$. In reality both planes will have to coincide everywhere, giving an excess restraint and so increasing P_x, Thus the approximate value, which we obtain in the above indicated way, deviates to the safe side, P_x being in point of fact somewhat more [4]).

With other boundary conditions the same method may be followed.

As we stated already in our preceding paper [1]), page 1195, we considered till now sandwich plates with both plates buckling in the same direction. Symmetrical buckling will be considered in a following paper.

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[4]) A somewhat similar case occurs in our theory of plastic stability, see e.g. BIJLAARD. A theory of plastic stability and its application to thin plates of structural steel. Proceedings Royal Netherlands Academy of Sciences, 1938, Nr. 7, where we equaled the real buckling stress σ_B to $(\sigma_E + 3\,\sigma_P)/4$, σ_E and σ_P being the buckling stresses if the plate deforms entirely elastically or entirely plastically respectively. Considering buckling just beneath the yield stress, with a plate compressed in X-direction and being simply supported at the edges $y = \pm b/2$, whilst in X-direction it is very long, so that practically any half wave length may establish itself in that direction, we find $h\sigma_E = 4{,}40\pi^2 EI/b^2$ (with $L = b$), whilst $h\sigma_P = 3{,}40\ \pi^2 EI/b^2$ (with $L = 0{,}82\,b$), yielding $h\sigma_B = 3{,}65\ \pi^2 EI/b^2$. In this way we calculate sufficiently accurately and safely, the real half wave length being between $0{,}82\,b$ and b, so that our ideal elastic and plastic plates are prevented from choosing their optimum wave lengths, by which the real buckling stress will be somewhat higher than that, computed in the above mentioned way. If necessary, we may find the real wave length by computing directly with values A, B, D and F being $\frac{1}{4}$ times these values with elastic plus $\frac{3}{4}$ times these values with plastic deformation. In this way we find $a = 0{,}88\,b$ and $h\sigma_B = 3{,}67\ \pi^2 EI/b^2$, the difference in buckling stress being negligible, although the wave lengths diverge rather much, the reason being the small variation of the buckling stress in the neighbourhood of its minimum. The same method may, however, not be followed in the present paper, as the optimum wave lengths for the two cases diverge here too much.

Cytology. — *An electron-microscopical study of bull sperm I.* By L. H. BRETSCHNEIDER and WOUTERA VAN ITERSON. (From the Zoological Laboratory at Utrecht and the Netherlands Institute for Electron Microscopy at Delft.) (Communicated by Prof. CHR. P. RAVEN.)

(Communicated at the meeting of November 30, 1946.)

1. *Introduction.*

The Research Group for Endocrinology at Utrecht, under the auspices of the organization for Applied Scientific Research (T.N.O.), wanted for their studies on artificial insemination a thorough investigation of the development and structure of bull sperm.

With the light microscope we can distinguish as main features of the spermatozoon a flat head with a tender cap, a neck, the body and a long tail. In these, finer structures are hardly visible, as they lie beyond the possibilities of resolution of the instrument. To reveal such details it seemed necessary to use the 100-fold improvement in resolution given by the electron microscope. A review of the recent American literature taught us that some submicroscopical investigations of spermatozoa had already been undertaken by various workers (2, 6, 8). The present communication bears only a preliminary character.

Apart from the aim with which this study was undertaken, it may gvie a first impression of the possibilities of the Dutch electron microscope with a view to the solution of problems of morphology and genesis of the healthy and abnormal sperm. This microscope, which will be described elsewhere, was designed and built by Mr. J. B. LE POOLE in the Physical Institute of the Technical University at Delft during the later years of the war. It is characterized amongst others by a three-lens system which, without changing the pole pieces and without image distortion, enables a continuous variation of the magnification from 1000 to 60.000 times. The accelerating voltage can be varied from 50—120 kv, opening the possibility to work with higher tensions, if the density of the object makes this desirable.

The image formation with the electron microscope depends on the electron scattering power of the structure of the specimen. The magnitude of this scattering is dependent on the thickness and density of the object. In general an object thickness of 0.1 μ is considered suitable for electron microscope research. The relatively large bull sperm (thickness $\pm$ 1μ—400mμ) hardly answers this requirement, but owing to the heterogenity of its structure it still proved possible to detect some interesting details. An electron accelerating voltage of 90 kv proved to give a suitable contrast. Moreover, we succeeded in increasing this heterogenity with the aid of the so-called shadow-casting method of WILLIAMS and WYCKOFF (10).

2. *Technique.*

The sperm used was taken from a normal bull. Colloidal material, when present in an electron microscope preparation, will dry up against objects and spoil their image. For this reason the sperm was cleaned by repeated centrifugation. After this procedure a part of the sperm was mounted fresh in a drop of distilled water on the film of the specimen holder, another part was fixed in chloramine. The chloramine treatment appeared to be favourable for the contrast in the tender structures which will be due to a shrinkage of the protein structures. In particular the so-called axial filament showed better in the fixed preparations.

As object holders for the apparatus at Delft no screens are used, but small silver cones with a very tiny hole of 0.3 mm diameter in the top to allow for the passage of the electron beam. The film is prepared by bringing with a glass capillary one drop of a solution of "Geisselthallack" on the surface of distilled water in a 10 cm Petri dish. After the film has settled, which takes some minutes, the specimen holder mounted on a metal rod is brought into touch with the film, either in the center or in any suitable area. Then the holder is removed by turning the rod, so as to roll the surplus of the film around the conical sides. On top of this film the objects in distilled water are now mounted with the aid of a platinum loop. After the drop has dried in the air, the specimens are locked in the vacuum of the electron microscope or subjected to the procedure of shadow-casting.

The procedure of drying fresh sperm in the air on the object film for electron microscope examination is comparable to the light-microscopical technique in which air-dried smears are used. The drying has the advantage of flattening the sperm, thus increasing its electron-permeability. The width of the head increases about 20%.

The influence of the vacuum on the sperm is unknown to us; heat damage as a consequence of electron absorption was not observed.

To increase the contrast of tender structures two methods are in use. The influence of selective staining will be considered in a subsequent report. In these experiments a part of the preparations were submitted to a second technique, a shadow-casting with gold.

WILLIAMS and WYCKOFF (10) developed a method of increasing contrast for electron microscopy by evaporating particles of a piece of metal (gold or chromium for instance) heated in vacuum. When the object holders are placed in an oblique position to this metal piece in an auxiliary vacuum chamber, the evaporated metal particles will condense all over the specimen, but as in the case of a snow storm, in the shelter of a higher structure no particles will settle, thus causing a blank "shadow" in proportion to the height of the structure. This procedure of coating objects with a few atoms of metal has the double advantage of increasing the density of hardly perceptible structures and making it possible to estimate their relative heights (compare the micrographs 1 and 2, or 3 and 7).

When considering the photographs one should realize that in favour of the effect the prints of shadow-castings are made in negative. In our case

the shadowing with gold sometimes showed the disadvantage of a rough coagulation of this metal when the object was too intensely "illuminated" by the electron beam which spoiled the details of finer structures (Cf. Plates, fig. 10). But in general the micrographs obtained in this way, in contrast to what holds for the ordinary pictures, greatly facilitate the mental reconstruction of the three-dimensional structure.

3. *Spermiogenesis.*

For a better understanding of the disclosed facts we give a brief account of the growing process of the spermatozoon.

A. The spermatid (Text fig. 1a). From this primitive stage the so complicately built spermatozoon develops by differentiation. In spermiogenesis we notice first a globular cell body in which we distinguish: in the center a round nucleus, the diplosomes, the acroblast originating from a Golgi-body, numerous mitochondria and some separate Golgi-bodies.

B. The meta-spermatozoon. After this first development has come to an end, the cell stretches longitudinally. The nucleus moves to the apical cell pole and the acroblast assumes a caplike position by covering the apical part of the nucleus (Text fig. 1b). The diplosomes move antapically towards the axis of the cell. An axis fibril, the axial filament, grows from the peripheral centrosome; it can be seen to continue outside the cell body in the direction of the longitudinal axis. The Golgi-bodies and mitochondria arrange themselves in a vortex around this axial filament.

C. *The immature spermatozoon stage.* A sheathlike lamella, the collar, differentiates at the equator of the nuclear membrane and grows in a caudal direction round the cell-axis, until a length of 165 mμ is reached. It encloses the axial filament, the diplosomes, Golgi-bodies, mitochondria and the central protoplasm, the so-called involucrum. Where this collar ends, the axial filament surrounded by the involucrum continues and reaches a length of 70μ. The cell plasm not used for construction contracts to a drop and is separated as residual protoplasm. By dehydration and loss of protoplasm all structures are still more condensed, by which process the nucleus flattens.

As a consequence of this, the structures of the mature sperm are packed into a space of less than 1 μ thickness, making a further analysis with the light microscope impossible, thus necessitating the electron-microscopical investigations.

4. *The structure of the mature bull sperm.*

The electron-microscopical image shows:

A. The structure of the sperm as also visible with the ordinary microscope.

B. Submicroscopical structures leading down to the macromolecular protein components.

Spermiogenesis

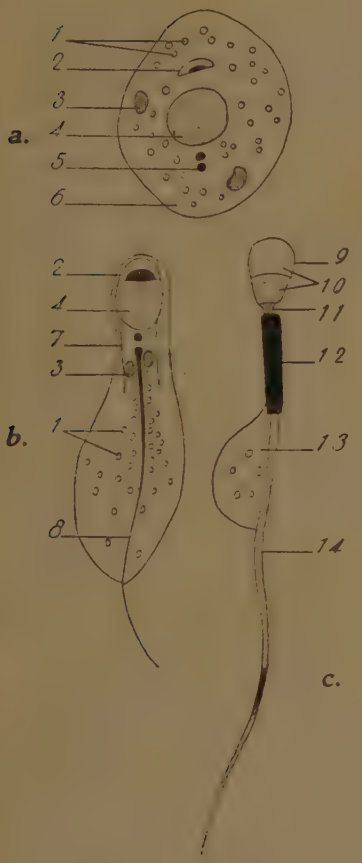

Fig. 1.

1. mitochondria.
2. acroblast.
3. Golgi-body.
4. nucleus.
5. diplosomes.
6. protoplasm.
7. collar.
8. axial filament.
9. protoplasmic-cap.
10. head.
11. neck.
12. middle piece.
13. residual plasm.
14. tail.

Schematized resumption of the electron-microscopical observations.

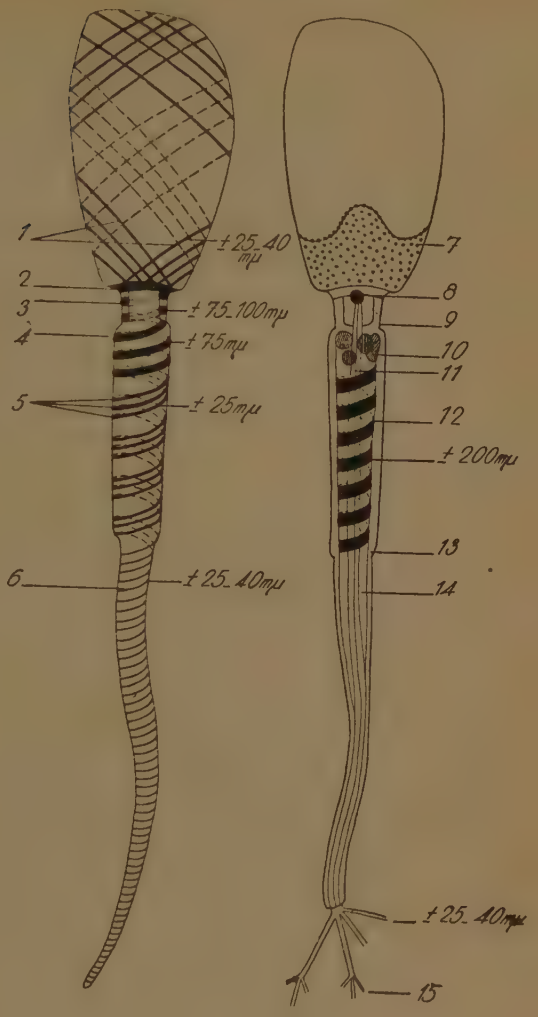

Fig. 2.

1. head fibrils.
2. collar.
3. articular strands.
4. cortical helix (middle piece).
5. splitting up into thinner threads.
6. cortical helix (tail).
7. chromosome shadow.
8. centrosome.
9. articular strands.
10. Golgi-bodys.
11. axial filament.
12. broad helix.
13. cortical plasm.
14. involucrum.
15. protofibrils (axial filament).

A. Description of the more conspicuous structural elements.

a. *The head* (Cf. Plates, fig. 1, 2, 3). Its shape is slightly asymmetrical, both at the concave base and at the top. On one side the top is a little rounder than on the other side. The apex has shifted over to one side; however, the deviation in the axis is so small that it is not detectable with the light microscope.

The apical part is covered by a tender cap, called protoplasmic cap by BAYLOR, NALBANDOV and CLARK 1934 (2), which shows in the photographs as a lighter zone surrounding the outside. Evidently the cap is lifted a little from the head as indicated by the broader zone at the top. (Cf. Plates, fig. 3.)

The distal part of the cap is marked by a line or a notch in the profile. Here in the equator of the head a double dark shadow band becomes visible. This apparent doubling is caused by different extension of the cap at front and back side, both margins covering each other incompletely. These shadows are often lined by a lighter zone, the "sub-equatorial zone".

Both zones can be made visible for the light microscope by staining with iron haematoxylin. They can be understood as differences in thickness of the nucleus or nuclear structures in this area.

In spermiogenesis the chromosomes are completely condensed in the nucleus- or head-base. In our electron microscope pictures we can see them indicated as a vague "chromosome shadow" (Cf. Plates, fig. 3, 4).

b. *The neck* (Cf. Plates, fig. 4, 5). The tail articulates with the broad (1,5—2 μ) base of the head by 2 lateral columns, the "articular strands". These strands are connected with the collar, a ring-shaped membrane fitting closely to the head-base (Cf. Plates, fig. 4). As an artifact the collar sometimes detaches itself, by which the articulation points are displayed (Cf. Plates, fig. 5). In extreme cases protoplasmic threads are seen to stretch between headbase and collar. The proximal centrosome is situated in the center of the neck, against the base of the nucleus. It is the origin of the axial filament, and is often covered by the collar. The remaining space is found to be empty both on examination in the light and in the electron microscope, and in vivo is probably filled with a liquid (Cf. Plates, fig. 4, 5).

c. *The middle piece* (Cf. Plates, fig. 6, 7). The neck passes into the cylindrical middle piece (length 16—18μ, width 750 mμ—1μ). Cytogenetically it is distinguished by containing several cell structures which for a large part envelop each other. As a consequence their images overlap in electron microscope pictures, thus complicating their interpretation.

In the proximal part the axial filament runs through the middle piece. This is seen as a lighter medial zone in ordinary pictures (Cf. Plates, fig. 4).

In the micrographs of this part, immediately behind the neck a more dense structure of irregular shape can be distinguished, and in some cases more-

over vacuoles and larger granules. In cytogenesis the mitochondria are deposited here. These irregular structures are transformed into a broad helix. The regular coils of this helix run obliquely backwards, and encircle the axial filament until the end of the middle piece. As a proof of this anticlockwise spiral nature, both the upper and lower parts of the coil can be seen as a diagonally crossing system in shadow-cast pictures (Cf. Plates, fig. 6). The thickness of the spiral filament is 200 mμ and the number of coils in the middle piece amounts to $\pm$ 25.

This spiral originates from mitochondria linked together and after insemination of the ovulum it disperses again into mitochondria. Excess mitochondria are often found in the separated residual protoplasm (Cf. Plates, fig. 9). Perhaps they are identical with the granules measuring also 200 mμ which we can distinguish vaguely on some of the electron micrographs. Axial filament and helix are embedded in the endoplasm, the so-called involucrum, while this is surrounded by the ectoplasm being richer in vacuoles.

d. *The tail* (Cf. Plates, fig. 4, 7, 8, 10). The middle piece is continued into the long tail (45 μ). Its width diminishes from beginning to end, from 800 mμ to 400 mμ. As the large helix is lacking here one can often notice that the axial filament is composed of a number of subfibrils. Especially in the chloramine-fixed preparations this is evident. In the upper part of the tail the axial filament is still enclosed by the involucrum. In the proximal part of the tail the involucrum diminishes quickly in width, so that at the end the axial filament is only wrapped in ectoplasm. In such a construction the tail of the sperm resembles a flagellum or cilium of animal cells. It is known that both of these consist of a fibrillar axis enveloped by ectoplasm.

It seems evident that in vivo also the terminal point of the tail is surrounded by ectoplasm. When the specimen is washed and dried for electron-microscopical examination, the protoplasmic sheath probably breaks up so as to free the axial filaments, which now split up into a brush of 9 subfibrils (Cf. Plates, fig. 10). Each of the fibrils has a width of $\pm$ 30 mμ, their length varying with the location of the ectoplasm rupture. They can have a length of several μ.

B. The more delicate electron-microscopical structures.

As is the case in every animal cell, also the sperm is bordered by a thin exterior limiting layer, the so-called cortical plasm. In accordance with the great mechanical requirements of the motile sperm, there are to be found in this outer layer appropriately arranged mechanical fibrils, the so-called tender helix.

On examining the head in shadow-cast pictures (Cf. Plates, fig. 1, 2), one is struck by a regular structure of crossing fibrils on its surface. The same is revealed in ordinary pictures (Cf. Plates, fig. 3, 4), where — although with some difficulty — a mosaiclike structure is unmistakably observed. In this connection we want to emphasize the difference noticed in chloramine and non-chloramine fixed preparations. Noticeably the latter pictures

of the head region show (Cf. Plates, fig. 5) on a rather dark background still darker marbled indications of the spiral systems, while the chloramine-fixed heads are much more penetrable to the electron beam, but show this structure less clear (Cf. Plates, fig. 3, 4). They, however, have the advantage of giving a better "chromosome shadow".

In the neck we observe as a continuation of the head fibril structures 3—4 knots in the articular strands, corresponding on both sides, which might represent transverse bands. From the collar to the end of the tail an anti-clockwise spiral is wound, enwrapping first with wide coils the involucrum, the broad helix and the axial filament, but more backwards (with smaller coils) it seems only to hold the fibrils of the axial filament together in a string-like way (Cf. Plates, fig. 10).

It seems reasonable to consider all of this spiral structure as a whole. This would mean that we are dealing with one continuous fibril system from head to tail, with some kind of transition in the articular strands. In the base of the head the fibrils cross at an angle of $\pm$ 90°, near the top this angle is enlarged to $\pm$ 120°. The significance of this value should not, however, be overrated without knowledge of the real shape of the head, as the angle only represents some projection in the plane of the photograph. About 120 fibrils in each direction cover the whole head.

The diameter of the fibrils in the head is $\pm$ 25 — 40 mμ. The diameter of the first part of the cortical helix is $\pm$ 75 mμ, more caudally it is diminished to $\pm$ 25 — 40 mμ, but greater magnifications vaguely indicate that this probably results from a splitting up into 2—3 thinner threads.

At the broken end the subfibrils of the axial filament of $\pm$ 25—40 mμ show a sub-division into a tuft of finer strands, the protofibrils. As we are suffering from lack of contrast in our micrographs of these very tender structures, it is difficult to count these strands, but there may be at least some 5 of these subunits. Their diameter will fall in some tens of Å units. This means that the electron microscope reveals here structures of only a small number of protein molecules, united to larger fibrillar units, these being bundled again to form the axial filament.

When considering the pictures we are tempted to describe a banded structure both in the axial and in the helical fibrils of the cortex. However, we give this observation in this preliminary report only with great reserve, as more evidence is needed to exclude influences of underlying structure of the supporting film, while in shadow-cast pictures a disturbance by the coarseness of the gold grains has to be eliminated.

As an unexpected fact we observed on the head somewhat below the equatorial zone a porus, shaped like a crater, *i.e.*, consisting of a wall and a pit (Cf. Plates, fig. 1). We are unable to give any interpretation of this unknown formation.

5. *Discussion.* (Cf. Textfig. 2.)

The preceding shows clearly that electron-microscopical investigation

greatly adds to our knowledge of the structure of the bull sperm. After these first attempts many questions arise. We expect that an increase in contrast by applying selective stains will be of value.

In agreement with HARVEY and ANDERSON'S observations (6) on the sperm of *Arbacia*, and with those of BAYLOR, NALBANDOV and CLARK (2) on the bull sperm, we also found the axial filament to consist of ± 10 subfibrils. Our analysis, however, went further, since we were able to detect in our micrographs (not printed here) a secondary splitting up of these subfibrils into protofibrils, having the dimension of only a relatively small number of protein molecules.

Surpassing the observations of BAYLOR, NALBANDOV and CLARK (2) we were able to demonstrate in the middle piece the broad helix, formed out of mitochondria, the centrosome, and the axial filament originating from the latter. We also stressed the existence of a cortical structure, consisting probably of 2 crossing fibril systems in the head region, and one anti-clockwise helix in the middle piece and tail, whereas both are connected in a rather uncertain way in the neck region.

A photograph of a part of an assumedly comparable helix from an ultrasonically fragmented human sperm tail is given by F. O. SCHMITT (8)..

HARVEY and ANDERSON (6) noticed regularly cross striations in the tail of their *Arbacia* sperm. They hesitated, however, to compare these with the phenomenon reported for collagen fibers by SCHMITT (see below), and feared that it might be produced in the washing and drying process. We suppose, however, that also in the case of *Arbacia* this cross striation may be attributed to the presence of such a helical fibril in the sheath of the tail. The process of drying for electron-microscopical purposes is as a dehydration one of the least radical preparation techniques. Already B. M. KLEIN (7) noticed in 1928 the favourable influence of drying as compared to fixation with chemical agents on the structure of the protoplasm.

Even if the banded structure we noticed in the axial as well as in the helical fibrils should have to be considered as an artifact due to drying, it still would reveal something of a structural regularity. However, when we succeed in excluding disturbing influences from the molecular structure of the supporting film and the gold coagulate, we expect some cross striation to remain as a demonstration of some underlying fundamental law in nature. In this respect we want to refer to the recently published electron micrographs of tendon (F. O. SCHMITT e.a. (8)) and muscle (RICHARD ANDERSON and HANCE 1942 (5)) and most striking of all, to ASTBURY'S crystalline protein fibre. From the fact that in these cases the value of the regular spacing between the successive bands is in fair agreement with x-ray diffraction measurements, it is inferred that the cross striations should be designed as subperiods in the polypeptide chains. We hope to bring in a future publication more evidence that the same holds for the protein fibrils observed by us.

The most striking result of our investigation seems to be the continuous fibril system enwrapping head and tail of the sperm cell. Spiral structures in cell walls are by no means an unknown feature in nature. We want to remind of this phenomenon in the pellicula of the *Ciliata (Ophryoscolecidæ)* (3, 4). An inspection of our pictures of the head region strongly recalls in mind the spiral structures in the cell walls of the green algae *Valonia* and *Cladophora* as described amongst others by ASTBURY (9 and 1). One can see an analogy between the continuation of the spiral structures from the head over the tail in the sperm cell and the situation occurring in the branching of *Cladophora* cells. In the latter case the spiral striations indicating the direction of the cellulose chains, clearly pass over on the wall of the newly formed cell. In our case the protein chains from the tender helix might split up at the base of the head into two spiral systems running in opposite directions. When considering the pictures we should realize that we have on the head at least two or three layers on the upper and lower side covering each other. Apart from the lower side — which in shadow-cast photographs does not so much effect the whole picture as in ordinary micrographs — we have nuclear membrane and the cortical sheath and in the upper part moreover the cap as three structures complicating each other.

With regard to the motility of the sperm we expect that there will be an antagonism between axial filament and cortical helix. Which of the two will play the active and which the passive part, we do not venture to decide.

Summary.

The underlying study is a first application of the Delft electron microscope in the investigation of sperm structure.

Normal bull sperm, fresh and after chloramine fixation, in some cases submitted to a procedure of shadow-casting with gold, has been studied and photographed at an electron emission voltage of 90 kv. It is concluded that the electron microscope can be of great use in elucidating the morphology of the sperm cell. We were able to observe:

A. As structures also visible with the ordinary microscope:

1. The asymmetrical head, covered by the cap and containing the chromosomes showing as a vague shadow.
2. The neck, where the tail is seen to articulate by the articular strands with the collar, a ring-shaped membrane. In the neck the centrosome is situated from which the axial filament originates.
3. The middle piece with broad helix, involucrum and several other elements.
4. The tail with cortical sheath and axial filament consisting of 9 subfibrils.

Indicates 1 μ in all pictures.

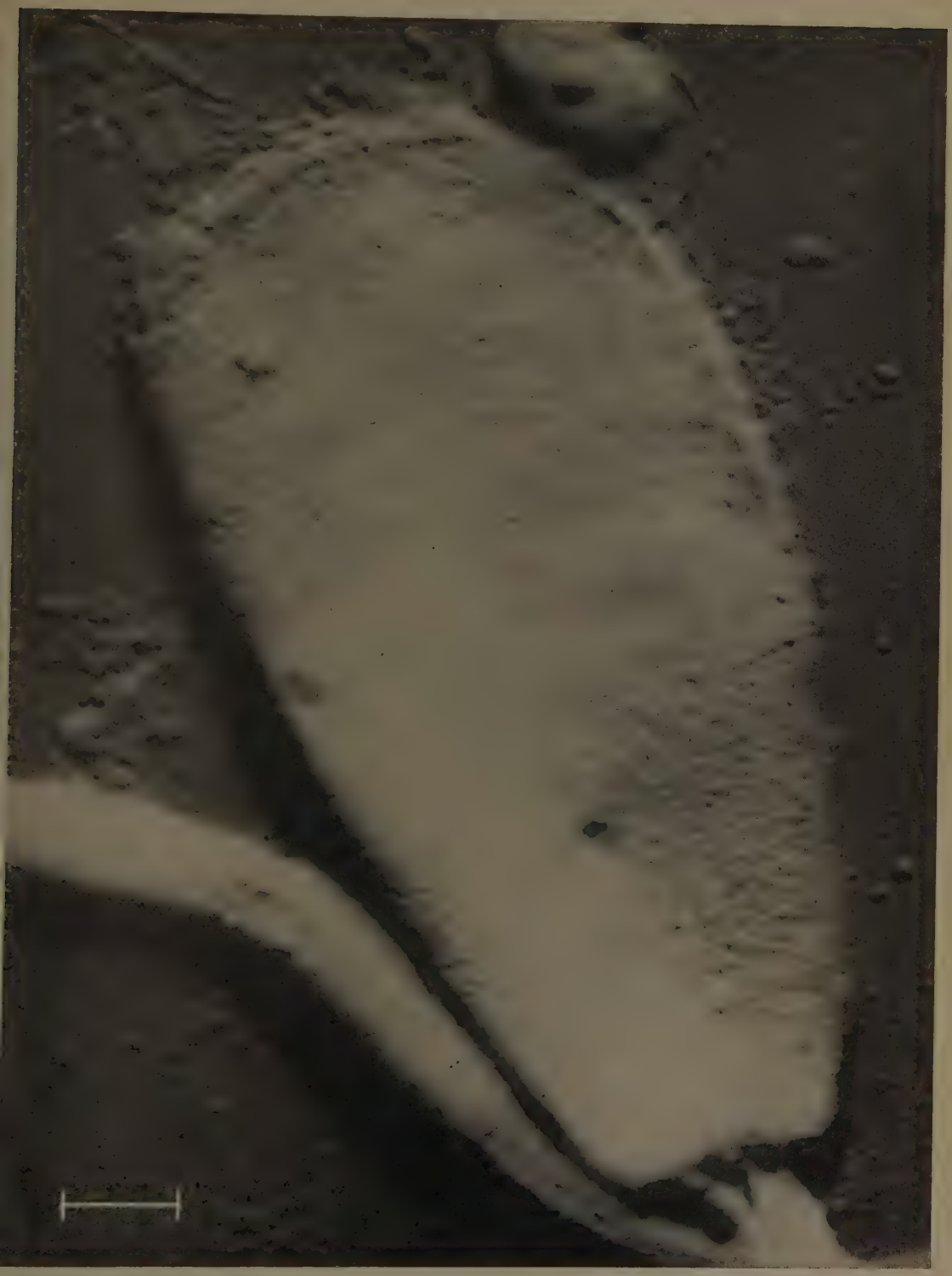

Fig. 1. Head of bull sperm. Shadow-cast 18.000 × 90 kv. Showing: crossing fibril system; in the equator the edge of the cap from front and back side; below these a porus.

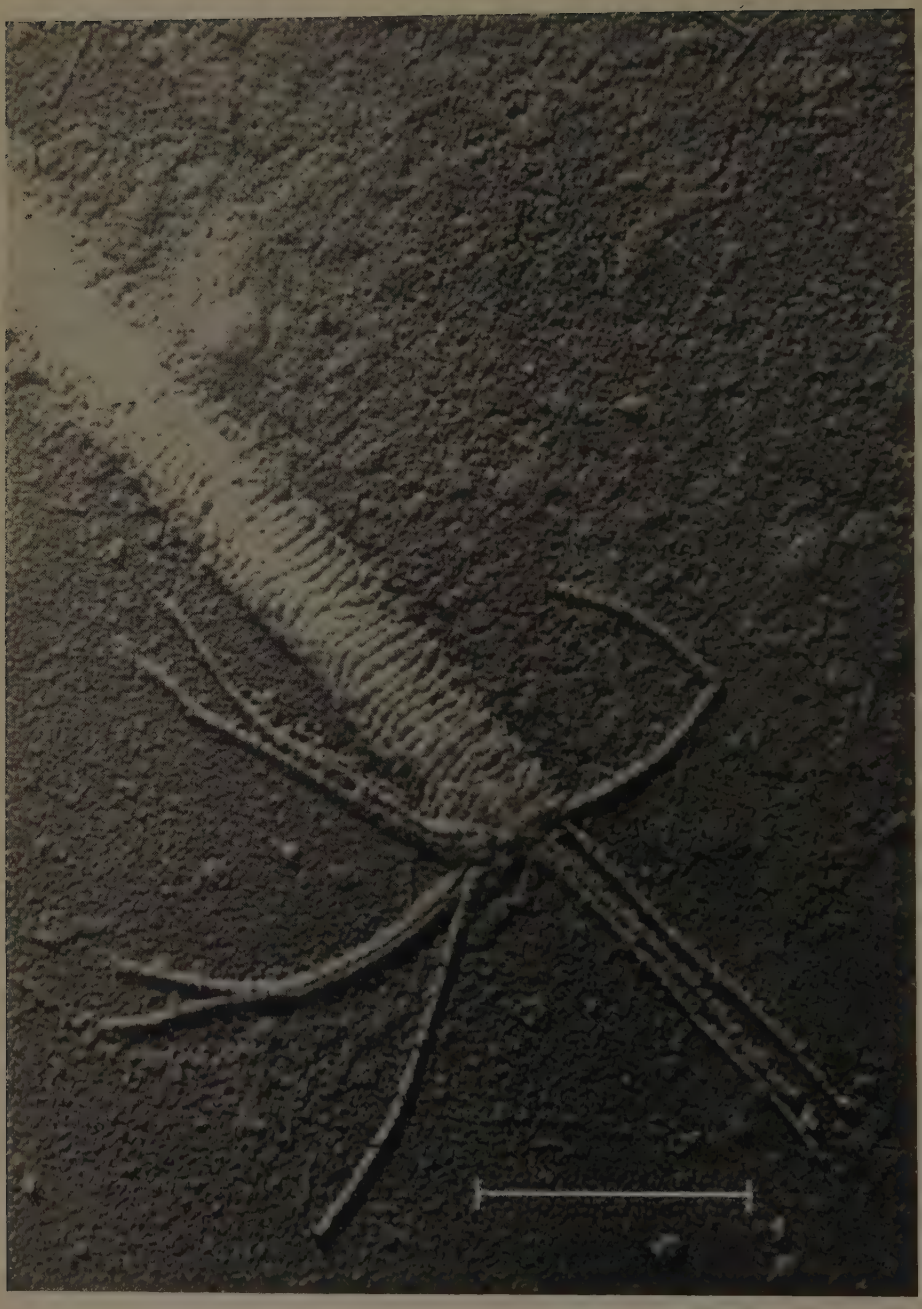

Fig. 10. Broken end of tail. Chloramine-fixed. Shadow-cast 36.000 × 90 kv.
Showing: tuft of subfibrils.

Fig. 5. Displayed articular strands. 42.000 × 90 kv. Notice: detached centrosome.

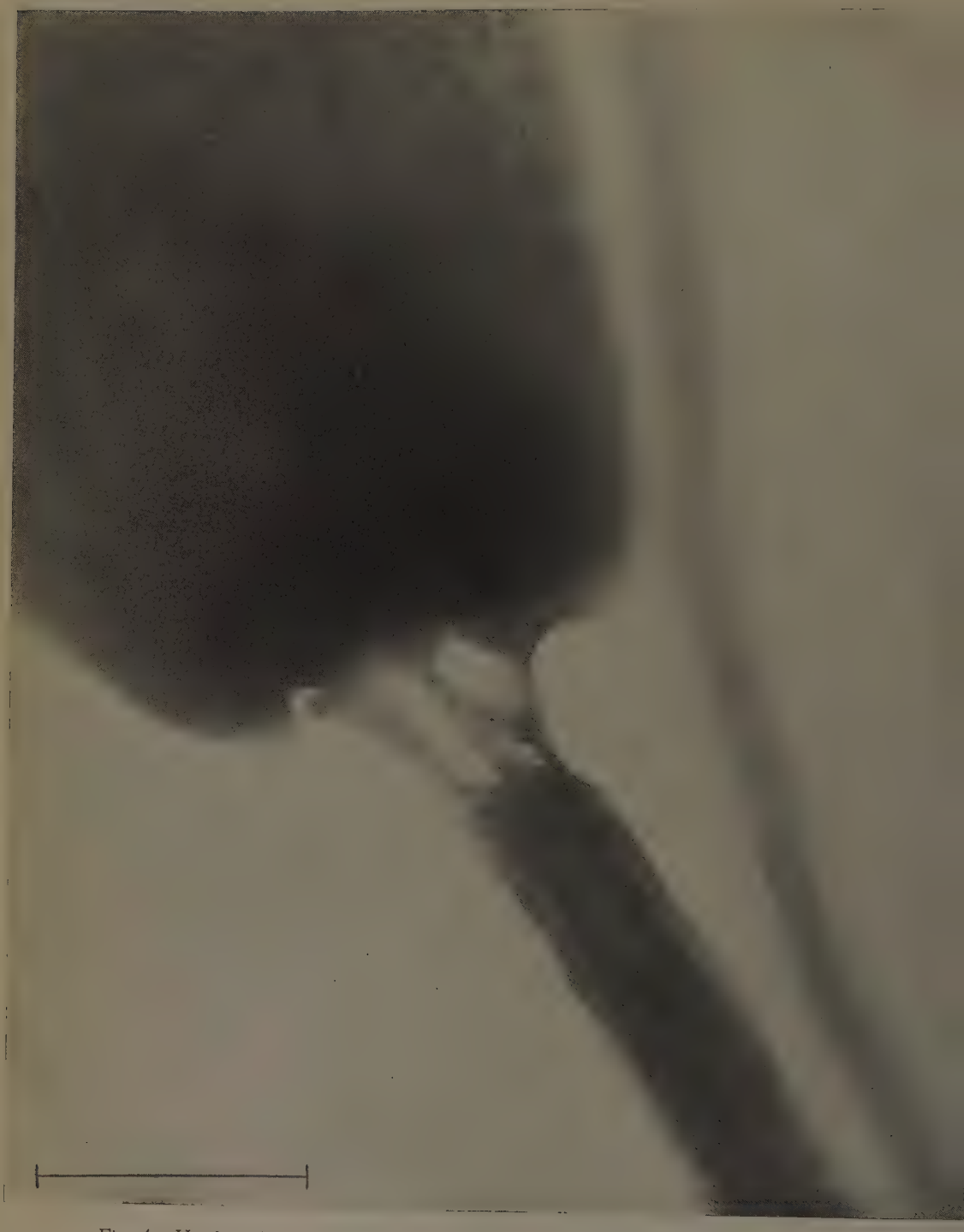

Fig. 4. Head, neck and proximal part of middle piece. Chloramine-fixed 42.000 × 90 kv. Showing: chromosome shadow; articular strands with knots, attached to collar; axial filament attached to centrosome. To the right: part of tail with tender helix and axial filament.

Fig. 6. Middle piece. Shadow-cast 20.000 × 90 kv. Showing: tender helix enwrapping broad helix.

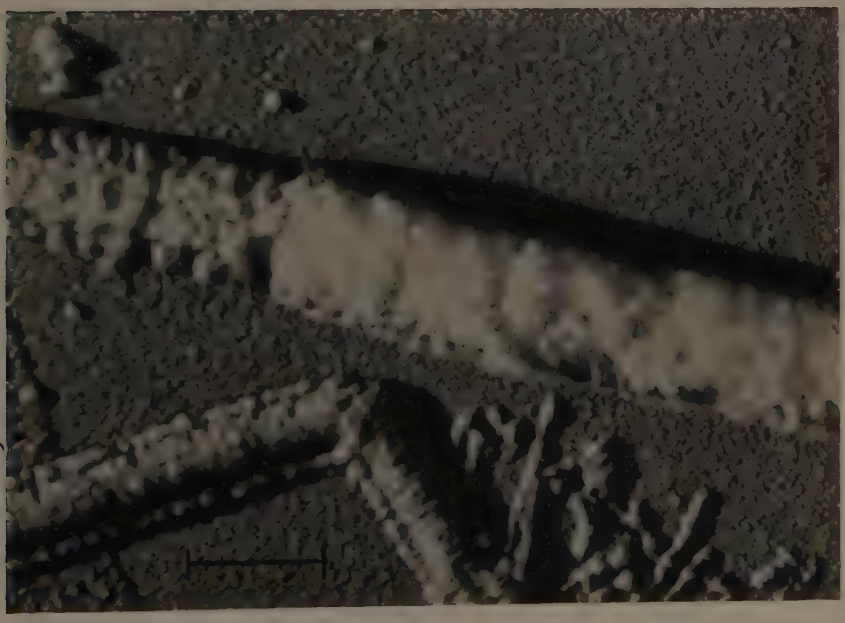

Fig. 7. Transition of middle piece to tail. Shadow-cast 20.000 × 90 kv.

Fig. 3. Head of bull sperm. Chloramine-fixed 20.000 × 90 kv. Notice: at the outside lighter zone of the cap; its edge as a line in the equator; less dense the sub-equatorial zone; chromosome shadow; porus very indistinct.

Fig. 8. Fragments of tail. Chloramine-fixed. Shadow-cast 36.000 × 90 kv. Showing: tender helix and axial filament (at the left: part of head and neck).

Fig. 9. Fragment with residual protoplasm. Chloramine-fixed. Shadow-cast 20.000 × 90 kv. Notice: mitochondria.

Fig. 2. Head of bull sperm. Shadow-cast 18.000 × 90 kv. Showing: crossing fibril system; edge of cap and porus vaguely in the equator; surrounding top: escaped protoplasmic contents.

B. Submicroscopical structures:

1. The cortical plasm exists of a crossing fibril system in the head region which continues after an uncertain transition in the neck into an anti-clockwise winding spiral, the "tender helix" enveloping middle piece and tail as the sheath.
2. The subfibrils of the axial filament showed to consist of protofibrils with a diameter of only some tens of Å units.
3. Tentatively we have described in helical and axial filaments a cross striation structure, but further evidence is needed in order to exclude disturbing influences of object film and gold coagulate.
4. As a new structural element a porus was observed near the equatorial zone of the head.

With regard to sperm motility an antagonism between axial filament and cortical helix is deemed probable.

1. ASTBURY, W. T. and R. D. PRESTON, Proc. Roy. Soc. B. **129**, 54 (1940).
2. BAYLOR, M. R. B., A. NALBANDOV and G. L. CLARK, Proc. Soc. Expt. Biol. Med. **54**, 229 (1943).
3. BRETSCHNEIDER, L. H., Verhandl. D. Zool. Ges. 324 (1931).
4. BRETSCHNEIDER, L. H., Arch. f. Protistenk. **82**, 298 (1934).
5. BURTON, E. F. and W. H. KOHL, The Electron Microscope, 268 (1946).
6. HARVEY, E. B. and TH. F. ANDERSON, Biol. Bull. **85**, 151 (1943).
7. KLEIN, B. M., Arch. f. Protistenk. **62**, 177 (1928).
8. SCHMITT, F. O., Advances in Protein Chemistry, Vol. I, 37 (1946).
9. PRESTON, R. D. and W. T. ASTBURY, Proc. Roy. Soc. B. **122**, 76 (1937).
10. WILLIAMS, R. C. and R. W. G. WYCKOFF, J. Appl. Phys. **17**, 23 (1946).

Mathematics. — *On irreducible convex domains.* By K. MAHLER (Manchester). (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of December 21, 1946.)

Let K be a bounded symmetrical star domain, or short, *a star domain*, in the (x_1, x_2)-plane, i.e. a bounded closed point set of the following kind:

1) The origin $O = (0, 0)$ is an *inner* point of K.
2) When $X = (x_1, x_2)$ belongs to K, then so does the symmetrical point $-X = (-x_1, -x_2)$.
3) The boundary C of K is a JORDAN curve which meets every radius vector from O in just *one* point.

The star domain K is *a convex domain*, if it contains with any two points X and Y also all the points [1])

$$(1-t)X + tY, \qquad 0 \leqslant t \leqslant 1,$$

of the line segment XY joining these two points.

A lattice Λ of basis

$$X_1 = (x_{11}, x_{12}), \qquad X_2 = (x_{21}, x_{22})$$

and of determinant

$$d(\Lambda) = |x_{11}\, x_{22} - x_{12}\, x_{21}| > 0$$

consists of all points

$$P = u_1 X_1 + u_2 X_2 \qquad (u_1, u_2 = 0, \mp 1, \mp 2, \ldots).$$

Λ is called *K-admissible* if no lattice point except O is an *inner* point of K. The lower bound

$$\Delta(K) = \text{l. b. } d(\Lambda)$$

of $d(\Lambda)$ extended over all K-admissible lattices Λ is a finite positive number. There exists at least one *critical lattice of K*, i.e. a K-admissible lattice Λ such that

$$d(\Lambda) = \Delta(K).$$

It is easily seen that

$$\Delta(K) \geqslant \Delta(H)$$

if K contains H. We say that K *is irreducible* if the stronger inequality

$$\Delta(H) < \Delta(K)$$

holds for all star domains H contained in, but different from, K. I do not

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[1]) We use the notation of sums and scalar products of points or vectors as usual in linear algebra or vector analysis.

know whether every star domain K contains an irreducible star domain K' (not necessarily different from K) such that

$$\Delta(K') = \Delta(K).$$

In this note, I show that this is true at least for convex domains, and, in fact, prove the slightly stronger result:

Theorem 1: *Every convex domain K contains an irreducible convex domain K'* (not necessarily different from K) *such that*

$$\Delta(K') = \Delta(K).$$

§ 1. The parallelogram.

To prove Theorem 1, a number of simple lemmas are required. We begin with an example of an irreducible domain.

Lemma 1: *Every parallelogram with centre at O is irreducible.*

Proof: By affine invariance, it suffices to prove the assertion for the unit square,

$$K_0: \qquad |x_1| \leqq 1, \ |x_2| \leqq 1,$$

for which, by MINKOWSKI's theorem on linear forms,

$$\Delta(K_0) = 1.$$

Let H be any star domain contained in, but different from, K_0. Then at least one point P_0 on the boundary C_0 of K_0 lies outside H; without loss of generality, this point P_0 belongs to the line segment

$$P_0 = (\xi, 1), \qquad 0 < \xi < 1,$$

of C_0. Denote by ϑ the number satisfying

$$0 < \vartheta < 1$$

for which ϑP_0 lies on the boundary of H, and by Λ_0 the lattice of basis

$$P_1 = \vartheta P_0 = (\vartheta\xi, \vartheta), \qquad P_2 = (1, \vartheta - 1).$$

This lattice is of determinant

$$d(\Lambda_0) = \begin{vmatrix} 1 & \vartheta - 1 \\ \vartheta\xi & \vartheta \end{vmatrix} = 1 - (1-\vartheta)(1-\vartheta\xi) < 1 = \Delta(K_0).$$

It is H-admissible since $\pm \vartheta P_0$ are its only points which are inner points of K_0. Hence

$$\Delta(H) \leqq d(\Lambda_0) < \Delta(K_0),$$

as asserted.

Corollary: *Theorem* 1 *holds for all parallelograms with centre at O.*

§ 2. The parallelograms with three vertices on C.

Let K be a convex domain, and let P_1 be any point on its boundary. The line L_1 through O and P_1 divides the (x_1, x_2)-plane into two semiplanes, P_+ and P_-, say. Denote by C_+ the half of C in P_+, by $C + P_1$, and

$C_+ + P_1$, the sets of all points $P + P_1$ where P runs over C, and C_+, respectively. The two points $-P_1 + P_1 = O$ and $P_1 + P_1 = 2P_1$ lie on $C + P_1$, but are situated on different sides of C. Hence C and $C + P_1$ intersect at least once, and so, by symmetry, C_+ and $C_+ + P_1$ also have a non-empty intersection, $I_2(P_1)$ say. If P_2 describes $I_2(P_1)$, then $P_3 = P_2 - P_1$ runs over a second set, $I_3(P_1)$ say. Since $P_3 + P_1 = P_2$ belongs to $C_+ + P_1$, P_3 lies on C_+. *Hence both sets* $I_2(P_1)$ *and* $I_3(P_1)$ *are subsets of* C_+.

Lemma 2: *All points of* $I_2(P_1)$ *and* $I_3(P_1)$ *lie at the same distance from the line* L_1.

Proof: Denote by $\delta(P)$ the distance of P from L_1, so that

$$\delta(P_2) = \delta(P_3)$$

for corresponding points P_2 and $P_3 = P_2 - P_1$ of $I_2(P_1)$ and $I_3(P_1)$. Let the assertion be false. There exist then two points P_2' and P_2'' of $I_2(P_1)$ and the corresponding points $P_3' = P_2' - P_1$ and $P_3'' = P_2'' - P_1$ of $I_3(P_1)$ such that

$$\delta(P_2') = \delta(P_3') < \delta(P_2'') = \delta(P_3'').$$

Since C_+ is a convex arc, $\delta(P)$ increases on C_+ from the value 0 at $P = P_1$ to a certain maximum value, and then decreases again to the value 0 at $P = -P_1$. Hence P_2'', P_3'' lie on the arc of C_+ bounded by P_2', P_3', while $P_1, -P_1$ lie outside this arc. From the construction, the two lines L_2 through P_2', P_2'', and L_3 through P_3', P_3'', are parallel. Therefore, by the convexity of C_+, P_1 and $-P_1$ lie in the parallel strip bounded by L_2 and L_3. This is, however, impossible because the line segment $P_1, -P_1$ is parallel to, but twice as long as, the line segments from P_2' to P_3', or from P_2'' to P_3''.

Since $\delta(P)$ is constant in $I_2(P_1)$ and $I_3(P_1)$, only the following two cases arise.

a) $I_2(P_1)$ consists of a single point P_2, $I_3(P_1)$ of a single point $P_3 = P_2 - P_1$. The line segment P_2P_3 is parallel and of equal length to the segment OP_1, and so $OP_1P_2P_3$ is a parallelogram. The line segment may possibly form part of C_+, but then no larger segment containing it has this property.

b) $I_2(P_1)$ contains at least two different points P_2', P_2'', and $I_3(P_1)$ contains at least the corresponding points $P_3' = P_2' - P_1$, $P_3'' = P_2'' - P_1$. All four points P_2', P_2'', P_3', P_3'' lie on *one* line L parallel to L_1; let Σ^* be the smallest line segment on L containing them. By the convexity of C_+, Σ^* is a subset of this arc. There exists then a longest line segment Σ contained in C_+ and itself containing Σ^*. This segment Σ is of greater length than OP_1 since Σ^* has this property. It is now clear that $I_2(P_1)$ consists of all points P_2 of Σ for which $P_3 = P_2 - P_1$ also lies on Σ, and that $I_3(P_1) = I_2(P_1) - P_1$. If $P_2, P_3 = P_2 - P_1$ is any pair of corresponding points in $I_2(P_1)$ and $I_3(P_1)$, then $OP_1P_2P_3$ is a parallelogram. By what

has already been proved, the area of this parallelogram depends on P_1, but not on P_2 and P_3; denote it by $A(P_1)$. Write further $\Lambda(P_1)$ for any lattice of basis P_1, P_2 where P_2 belongs to $I_2(P_1)$. Then

$$d(\Lambda(P_1)) = A(P_1).$$

Example: Let K_0 be again the unit square

$$|x_1| \leqq 1, \quad |x_2| \leqq 1.$$

It suffices, by symmetry, to consider points

$$P_1 = (1, \eta), \quad 0 \leqq \eta \leqq 1,$$

which lie on the side $x_1 = 1$ of K_0. Choose for P_+ the semiplane $y \geq \eta x$. Then, if $\eta \neq 0$, $I_2(P_1)$ consists of the single point $P_2 = (0, 1)$, and $I_3(P_1)$ of the single point $P_3 = (-1, 1-\eta) = P_2 - P_1$. If, however, $\eta = 0$, then $I_2(P_1)$ is the line segment of all points $P_2 = (\xi, 1)$ where $0 \leq \xi \leq 1$, and $I_3(P_1)$ is the adjoining line segment of all points $P_3 = (\xi', 1)$ where $-1 \leq \xi' \leq 0$. In both cases,

$$d(\Lambda(P_1)) = A(P_1) = 1,$$

independent of the choice of P_1.

§ 3. The critical lattices of K.

By MINKOWSKI [2], the following result holds:

Lemma 3: *Let Λ be any critical lattice of the convex domain K. Then Λ contains three points P_1, P_2, P_3 on C such that (i) P_1, P_2 is a basis of Λ, and (ii) $OP_1P_2P_3$ is a parallelogram of area $d(\Lambda) = \Delta(K)$. Conversely, if P_1, P_2, P_3 are three points on C such that $OP_1P_2P_3$ is a parallelogram, then the area of this parallelogram is not less than $\Delta(K)$, and it is equal to $\Delta(K)$ if and only if the lattice of basis P_1, P_2 is critical.*

This lemma, together with the results of last paragraph, leads immediately to the following construction of the critical lattices of K.

Lemma 4: *Denote by Π the set of all points P_1 on C for which $A(P_1)$ assumes its smallest value, A say* [3]. *Let further $\{\Lambda\}$ be the set of all lattices $\Lambda(P_1)$ where P_1 runs over Π. Then $\{\Lambda\}$ is identical with the set of all critical lattices of K.*

Example: Let K_0 be again the unit square

$$|x_1| \leqq 1, \quad |x_2| \leqq 1.$$

Then $\Pi = C$, and *all* lattices $\Lambda(P_1)$, where P_1 is an arbitrary point on C, are critical. We shall see that a similar result holds for all irreducible convex domains.

For parallelograms with centre at O, Theorem 1 has already been proved

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[2] Diophantische Approximationen, § 4. See also my paper Proc. London Math. Soc. (2) **49** 137, 158—159 (1946).

[3] That such a minimum value is attained, follows immediately from Lemma 3 and the existence of critical lattices.

in the corollary to Lemma 1. In the next three lemmas, this case is excluded, and it is assumed that K *is a convex domain which is not a parallelogram.*

Lemma 5: *Let P_1 be any point on C such that there exists a critical lattice Λ of K containing P_1. Denote by P_2 and P_3 two further points of Λ on C such that $OP_1P_2P_3$ is a parallelogram of area $\Delta(K)$. Then all points of the line segment P_2P_3 different from P_2 and P_3 are inner points of K, and so Λ is the only critical lattice of K containing P_1.*

Proof: Denote by L^* any tac line of C at P_1, by L^{**} the line through P_2 and P_3, by $-L^*$ and $-L^{**}$ the lines symmetrical to L^* and L^{**} in O, and by K^* the parallelogram bounded by the four lines L^*, L^{**}, $-L^*$, $-L^{**}$. If at least one inner point of the segment P_2P_3 lies on L^{**}, then L^{**} is a tac line of C, and so K is contained in K^* as a subset. Now Λ is K^*-admissible, hence a critical lattice of K^*, whence

$$\Delta(K^*) = \Delta(K).$$

Therefore, by Lemma 1, K^* coincides with K, contrary to hypothesis. This proves the first part of the assertion. The second part also holds since $I_2(P_1)$ reduces to the single point P_2.

Corollary: *Every critical lattice Λ of K has just six points*

$$P_1, P_2, P_3, P_4 = -P_1, P_5 = -P_2, P_6 = -P_3$$

on C. These points divide C into six arcs $A_1, \ldots, A_6$, none of which is a line segment.

Lemma 6: *Let Λ and Λ^* be two different critical lattices of K; let $P_1, \ldots, P_6$ be the six points of Λ on C; and let $A_1, \ldots, A_6$ be the six arcs into which these points divide C. Then Λ^* has just one point P_i^* on each arc A_i.*

Proof: Write $\{X, Y\} = x_1y_2 - x_2y_1$ for the determinant of any two points $X = (x_1, x_2)$ and $Y = (y_1, y_2)$, and choose the indices such that, if C is described in positive direction, the points of Λ on C are met in the order

$$P_1, P_2, P_3, P_4 = -P_1, P_5 = -P_2, P_6 = -P_3,$$

or in a cyclical permutation therefrom. Further denote by $A_1, \ldots, A_6$ the six arcs

$$\overset{\frown}{P_1P_2}, \overset{\frown}{P_2P_3}, \ldots, \overset{\frown}{P_5P_6}, \overset{\frown}{P_6P_1}$$

of C bounded by these points. By Lemma 3, P_1 and P_2 form a basis of Λ, and so

(a): $$\{P_1, P_2\} = \Delta(K),$$

since the determinant on the left is positive.

One may assume that Λ^* contains a point P_1^* of A_1; this must be an *inner* point of A_1, since Λ^* is different from Λ (Lemma 5). None of the three arcs A_6, A_1, A_2 is a line segment; hence, by the convexity of C, P_1^*

is also an *inner* point of the *triangle* with vertices at P_1, $P_1 + P_2$, P_2. Therefore, if P_1^* is written as

$$P_1^* = sP_1 + tP_2,$$

then s and t satisfy the inequalities,

$$0 < s < 1,\ 0 < t < 1,\ s + t > 1,$$

whence, by (a),

$$\begin{aligned} 0 < \{P_1^*, P_2\} &= s\,\Delta(K) < \Delta(K), \\ \{P_1^*, P_3\} &= (s+t)\,\Delta(K) > (K), \\ 0 < \{P_1^*, P_4\} &= t\,\Delta(K) < (K). \end{aligned}$$

Hence the line

(b): $$\{P_1^*, X\} = \Delta(K)$$

intersects C in at least one point P_2^* of A_2 and at least one points P_3^* of A_3, and both points are inner points of these arcs. There cannot be more than *one* such point of intersection on each arc A_2 and A_3, since, by Lemma 5, Λ^* has just two points on C satisfying (b).

Lemma 7: *Let Λ, Λ^*, Λ^{**} be three different critical lattices of K, and let P_l, P_l^*, P_l^{**} $(l = 1, 2, \ldots, 6)$ be their points on C, the indices being chosen such that (i) the points P_l follow one another in their natural order if C is described in positive direction, and (ii) the two points P_l^* and P_l^{**} lie on the arc A_l bounded by P_l and P_{l+1} (P_7 is to mean the same as P_1). If P_1^* separates P_1 from P_1^{**} on A_1, then, for $l = 2, \ldots, 6$, P_l^* likewise separates P_l from P_l^{**} on A_l.*

Proof: Assume the assertion is false. Let then λ with $2 \leq \lambda \leq 6$ be the smallest index for which P_λ^* does not separate P_λ from P_λ^{**} on A_λ. Then $P_{\lambda-1}^{**}$ and P_λ^{**} are two consecutive points of Λ^{**} on C such that the arc joining them contains no point of Λ^*, in contradiction to the last lemma.

§ 4. The critical lattices of an irreducible convex domain.

The last lemmas lead to a particularly simple result if K is irreducible. I have proved elsewhere [4]) the following result:

Lemma 8: *If K is an irreducible star domain, and if P is any point on C, then there exists at least one critical lattice of K containing P.*

On combining this result with the Lemmas 5—7, one finds:

Lemma 9: *Let K be an irreducible convex domain which is not a parallelogram. Then to every point P_1 on C, there exists a unique critical lattice $\Lambda = \Lambda(P_1)$ containing P_1. This lattice has just six points $P_l = P_l(P_1)$ $(l = 1, \ldots, 6)$ on C. Let $A_1, \ldots, A_6$ be the six arcs into which these points divide C; denote further by P_1^* a variable point on A_1, and by $^* = P_l(P_1^*)$ for $l = 2, \ldots, 6$ the other five points of $\Lambda(P_1^*)$ on C. If P_1^**

[4]) Proc. Royal Acad. Amsterdam, **49**, 331—343 (1946), Theorem C.

describes A_1 continuously in positive direction, then P_l^*, for $l = 2, \ldots, 6$, describes A_l in the same manner.

Proof: Choose the indices in the same way as in the last proofs. The line,

$$L(P_1^*): \qquad \{P_1^*, X\} = \Delta(K),$$

has two, and by Lemma 5 only two, points of intersection with C, namely P_2^* and P_3^*. When P_1^* describes A_1 continuously, then this line changes in a continuous manner, and so the same is true for P_2^* and P_3^*. Further, if P_1^* runs over A_1 in positive direction, then, by Lemma 7, P_2^* and P_3^* do the same on A_2 and A_3.

By means of this lemma, one can construct all irreducible convex domains. In a further note, I shall apply this construction.

§ 5. Lemmas on reducible domains.

In the paper already mentioned, I proved the following results:

Lemma 10: *Let K be a reducible star domain and P a point on C such that no critical lattice of K contains P. Then there exists a star domain H contained in K, but not containing the two points $\pm P$ symmetrical in O, for which $\Delta(H) = \Delta(K)$* [5].

Lemma 11: *Let K be a star domain, and let Π be the set of all points on C which belong to at least one critical lattice of K. Then Π is a closed set. The set Π^* of all points of C which do not belong to Π is therefore open and consists of an enumerable set of open arcs on C* [6].

Lemma 12: *If K is a convex domain, and if every point of C belongs to Π, then K is irreducible* [7].

§ 6. The main lemma.

The following lemma forms the basis for the proof of Theorem 1:

Lemma 13: *Let K be a reducible convex domain. Then there exists a* **convex** *domain H contained in, but different from, K such that $\Delta(H) = \Delta(K)$.*

Proof: Divide the points P on C into two classes A and B, according as to whether P is, or is not, an *inner* point of a *line segment contained in* C.

By the hypothesis and by the Lemmas 11 and 12, Π^* is not the null set and so contains at least one arc of C. Assume, firstly, that at least one point P of Π^* is of class B. By Lemma 10, there exists a star domain H' contained in K, but not containing the two points $\pm P$, such that $\Delta(H') = \Delta(K)$. Draw a line L' through P which has no point in common with H' and is not a tac line of C at P. Then a line L parallel to L', and separating L' from O, can be chosen which also has no points in common

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[5] l.c. [4], proof of Theorem C.
[6] l.c. [4], special case of Theorem B.
[7] l.c. [4], Theorems D and E.

with H'; denote by $-L$ the line symmetrical to L in O, and by H the part of K between L and $-L$. Then H is a convex domain containing H', and is itself a proper subset of K. Evidently $\Delta(H') \leq \Delta(H) \leq \Delta(K)$, whence $\Delta(H) = \Delta(K)$.

Secondly, let all points of Π^* be of class A, and let P be one of these points. The tac line to C at P is unique, and its intersection with C is a line segment, Γ say. Then P is an inner point of Γ, and the two endpoints of Γ are of class B, hence belong to Π. There exists therefore a largest sub-segment Γ_1 of Γ such that, (*i*) P is an inner point of Γ_1; (*ii*) all inner points of Γ_1 belong to Π^*; (*iii*) the two endpoints P_1, P'_1 of Γ_1 belong to Π.

Denote by Λ, Λ' the two critical lattices of K containing P_1 and P'_1, respectively, and by P_l, P'_l $(l = 2, \ldots, 6)$ the other points of these two lattices on C. Let the notation be again such that if C is described in positive direction, then the points P_l, and similarly the points P'_l, follow one another in the order of their indices; denote further by $A_1, \ldots, A_6$ the arcs

$$\overset{\frown}{P_1 P_2}\ ,\ \overset{\frown}{P_2 P_3}, \ldots, \overset{\frown}{P_5 P_6}\ ,\ \overset{\frown}{P_6 P_1}$$

of C. By the hypothesis about Γ_1, P'_1 belongs either to A_1 or to A_6; assume, without loss of generality, that P'_1 lies on A_1 and so is an inner point of this arc. By Lemma 6, P'_l is then, for $l = 2, \ldots, 6$, an inner point of A_l, and so the arc

$$\Gamma_l = \overset{\frown}{P_l P'_l} \qquad (l = 2, \ldots, 6)$$

is a subarc of A_l.

By construction, the endpoints of all arcs Γ_l, where $l = 1, 2, \ldots, 6$, belong to Π. On the other hand, it is immediately clear from Lemma 7 that the inner points of these arcs belong to Π^*; for this is the case for Γ_1. But then the inner points of all arcs Γ_l are of class A, and so all six arcs Γ_l are line segments.

One shows now, just as in the proof of Lemma 6, that P'_1, P'_2 and P'_3 are inner points of the triangles with vertices at $P_1, P_1 + P_2, P_2$, at P_2, $P_2 + P_3, P_3$, and at P_3, $-P_1 + P_3$, $-P_1$, respectively. The three points

$$Q_1 = P'_1 - P_1,\ Q_2 = P'_2 - P_2,\ Q_3 = P'_3 - P_3$$

are therefore inner points of the triangles with vertices at O, P_2, P_3, at O, P_3, $-P_1$, and at O, $-P_1$, $-P_2$, respectively. By the assumed choice of indices, the radius vector from O to Q_1 changes therefore into that from O to Q_3 by a rotation in the *positive* sense of *less* than 180°, and so finally,

$$\{Q_1, Q_3\} > 0.$$

Next, the general point P_l^* of the arc Γ_l, where $l = 1, 2, 3$, is of the form

$$P_l^* = t_l P'_l + (1 - t_l) P_l = P_l + t_l Q_l, \qquad \text{where } 0 \leq t_l \leq 1.$$

Choose, in particular, $t_1 = t_2 = t_3, = t$ say. Then

$$P_1^* + P_3^* = P_2^*,$$

and so $OP_1^* P_2^* P_3^*$ is a parallelogram with three vertices on C. By the hypothesis and by Lemma 3, this parallelogram is of area

$$\{P_1^*, P_3^*\} > \Delta(K) \qquad \text{for } 0 < t < 1.$$

Now,

$$\{P_1^*, P_3^*\} = \{P_1 + tQ_1,\ P_3 + tQ_3\}, \qquad = f(t) \text{ say,}$$

can be written as

$$f(t) = a + bt + ct^2, \quad \text{where} \quad c = \{Q_1, Q_3\}.$$

Hence, by the construction of the points P_l and P'_l, and by (a),

$$f(0) = f(1) = \Delta(K),\ f''(t) = 2c > 0,$$

and so there exists a number τ with $0 < \tau < 1$ such that $f(t)$ assumes a minimum value at $t = \tau$ satisfying

$$f(\tau) < \Delta(K).$$

The lattice of basis $P_1 + \tau Q_1$, $P_3 + \tau Q_3$, which is K-admissible, is therefore of determinant less than $\Delta(K)$, which is impossible.

This proves that all points of Π^* cannot be of class A, and so completes the proof.

§ 7. Proof of Theorem 1.

Let K be a reducible convex domain, hence, by Lemma 1, not a parallelogram. By the last lemma, the set P of all convex domains H contained in, but different from, K and satisfying $\Delta(H) = \Delta(K)$, is not the null set. The elements H of P are all of area greater than $\Delta(K)$. For if $\tilde{\omega}$ is any parallelogram $OP_1P_2P_3$ with three vertices of the boundary of H, then $\tilde{\omega}$ is a subset of H, and so by Lemma 3,

$$V(H) \geqq V(\tilde{\omega}) \geqq \Delta(K),$$

where $V(J)$ denotes the area of J.

Let now

$$v = \operatorname*{fin\,inf}_{H \text{ in } P} V(H)$$

be the lower bound of $V(H)$ extended over all elements of P; evidently

$$v \geqq \Delta(K) > 0.$$

An infinite sequence of elements $H_1, H_2, H_3, \ldots$, of P not necessarily all different, can be chosen such that

$$\lim_{n \to \infty} V(H_n) = v.$$

All elements of this sequence are convex domains contained in K; hence, by a selection theorem of W. BLASCHKE [8]), a suitable subsequence,

$$K_1 = H_{n_1},\ K_2 = H_{n_2},\ K_3 = H_{n_3}, \ldots \qquad (n_1 < n_2 < n_3 < \ldots)$$

tends of a convex domain, K' say. As we show now, this convex domain has the required properties.

For firstly [9])

$$V(K') = \lim_{n \to \infty} V(K_n) = v.$$

Secondly, also [10])

$$\Delta(K') = \lim_{n \to \infty} \Delta(K_n) = \Delta(K).$$

Thirdly, K' is a subset of K. I assert that K' is irreducible. If this were false, then, by Lemma 13, there would exist a convex domain K'' contained in, but different from, K' such that $\Delta(K'') = \Delta(K') = \Delta(K)$. This is, however, impossible, since by the construction,

$$V(K'') < V(K') = v,$$

contrary to the definition of v. This completes the proof.

September 5, 1946.

Mathematics Department, Manchester University.

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[8]) KREIS und KUGEL (Leipzig 1916), 62.

[9]) l.c. [8]), 61.

[10]) Theorem 9 of my paper "Lattice points in n-dimensional star bodies I". Proc. Royal Society, A, 187 (1946), 151—187.

Mathematics. — *On the area and the densest packing of convex domains.* By K. MAHLER (Manchester). (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of December 21, 1946.)

In the preceding paper "On irreducible convex domains" [1], I studied the critical lattices of convex domains in the (x_1, x_2)-plane and proved that every such domain contains an irreducible convex domain of equal determinant.

Of these results, applications are made in the present paper, which deals with two closely allied problems:

Problem I: If $V(K)$ and $\Delta(K)$ denote the area and determinant of a convex domain K, *to find the lower bound of*

$$Q(K) = \frac{V(K)}{\Delta(K)}$$

extended over all convex domains K.

Problem II: About every point P of a lattice Λ as its centre describe a convex domain $K(P)$ congruent to K and with the same orientation, but assume that no two domains $K(P)$ overlap. Choose Λ such that the ratio of the area covered by the domains $K(P)$ to the whole plane assumes its largest value, $q(K)$ say. *To find the lower bound of* $q(K)$ *extended over all convex domains.*

MINKOWSKI established the close connection between $Q(K)$ and $q(K)$ and obtained the upper bounds for $Q(K)$ and $q(K)$, and some lower bound for $Q(K)$. Also Problem II has been considered before [2], but no solution seems to have so far been given. I have not succeeded in solving either of the two problems. But I show in this paper how they can be reduced to a question in the calculus of variations. I prove further that this variation problem does admit of a best possible solution in form of an irreducible convex domain, and that this solution is *not* an ellipse, contrary to what might be expected.

All the first paragraphs deal with Problem I; the application to Problem II is made at the end of this paper.

§ 1. Formulation of the problem.

Let

$$x_1' = \alpha x_1 + \beta x_2, \quad x_2' = \gamma x_1 + \delta x_2$$

be any affine transformation of determinant

$$d = \alpha\delta - \beta\gamma > 0,$$

---

[1] Quoted as ICD. Compare this paper for all the definitions and lemmas.

[2] See W. BLASCHKE, Differentialgeometrie II, § 27, problem 17.

and let K be any convex domain [3]) in the (x_1, x_2)-plane. When (x_1, x_2) describes K, then (x'_1, x'_2) describes a second convex domain K'. As is well known [4]), the areas $V(K)$, $V(K')$ and the determinants $\Delta(K)$, $\Delta(K')$ satisfy the equations

$$V(K') = d\,V(K), \quad \Delta(K') = d\,V(K).$$

Hence the quotient,

$$Q(K) = \frac{V(K)}{\Delta(K)},$$

is an absolute invariant,

$$Q(K') = Q(K), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

for all affine transformations.

An upper bound for $Q(K)$ is given by MINKOWSKI's classical theorem on lattice points in convex domains, viz.

$$Q(K) \leqslant 4;$$

the equality sign holds only for parallelograms and certain classes of hexagons [5]).

It is the lower bound for $Q(K)$ with which this paper is concerned. A trivial lower bound for $Q(K)$, namely

$$Q(K) \geqslant 1,$$

follows immediately from the obvious inequality $V(K) \geq \Delta(K)$ [6]). Elsewhere, I proved the much better inequality [7]),

$$Q(K) \geqslant \sqrt{12},$$

but this is also not the exact lower bound for $Q(K)$.

In order to obtain the exact lower bound for $Q(K)$ in the set of all convex domains, the following restrictions on K may be imposed without loss of generality:

(A): *K is not a parallelogram*; for otherwise $Q(K) = 4$, and this is not the smallest possible value for $Q(K)$, since, e.g. for an ellipse

$$Q(K) = \frac{2\pi}{\sqrt{3}} < 4.$$

(B): $\Delta(K) = 1$; this condition may be enforced by means of a suitable similar transformation, on account of (1).

---

[3]) As in ICD, all convex domains are assumed symmetrical in $O = (0, 0)$.

[4]) The first equation is classical; for the second one see Theorem 16 of my paper "Lattice points in n-dimensional star bodies I". Proc. Royal Society A, **187** (1946), 151—187.

[5]) Geometrie der Zahlen, §§ 34—35.

[6]) ICD, § 7.

[7]) See my paper, "On the theorem of MINKOWSKI–HLAWKA", which is to appear in DUKE's Journal.

(C): The boundary C of K contains the six points,

$$\left.\begin{aligned} P_1' = (\sqrt[4]{\tfrac{4}{3}}, 0),\ P_2' = (\sqrt[4]{\tfrac{1}{12}}, \sqrt[4]{\tfrac{3}{4}}),\ P_3' = (-\sqrt[4]{\tfrac{1}{12}}, \sqrt[4]{\tfrac{3}{4}}), \\ P_4' = -P_1,\ P_5' = -P_2,\ P_6' = -P_3, \end{aligned}\right\} \quad (2)$$

and these points are the points of a critical lattice Λ' of K of basis P'_1, P'_2. For assume that the conditions (A) and (B) hold, and choose any critical lattice of K. Then, by *ICD*, § 3, this lattice has just six points on C, such that three of them together with the origin form the vertices of a parallelogram. Since the lattice is of unit determinant, it can be transformed into Λ' by means of an affine transformation with $d = 1$.

We can now restate our problem as follows:

Problem 1': *To find a convex domain K of minimum area satisfying the three conditions* (A), (B), (C). *Its area gives the required lower bound for* $Q(K) = V(K)$ [8]).

§ 2. Proof that the lower bound is attained.

Denote by H' the hexagon with the six vertices (2), and by H'' the polygon

$$P_1' P_1'' P_2' P_2'' P_3' P_3'' P_4' P_4'' P_5' P_5'' P_6' P_6'',$$

where $P_1'', \ldots, P_6''$ are the points

$$\left.\begin{aligned} P_1'' = (\sqrt[4]{\tfrac{27}{4}}, \sqrt[4]{\tfrac{3}{4}}),\ P_2'' = (0, \sqrt[4]{12}),\ P_3'' = (-\sqrt[4]{\tfrac{27}{4}}, \sqrt[4]{\tfrac{3}{4}}), \\ P_4'' = -P_1'',\ P_5'' = -P_2'',\ P_6'' = -P_3''. \end{aligned}\right\} \quad (3)$$

If K is any convex domain satisfying the conditions (A), (B), (C), then it contains H' as a subset, and is itself contained in H''. Denote by Σ the set of all such convex domains.

As already mentioned in § 1, $Q(K) \geq 1$ for all convex domains, and so

$$V(K) \geqq 1$$

for all elements K of Σ. Hence the lower bound

$$Q = \underset{K \text{ in } \Sigma}{\text{l.b.}}\ V(K)$$

extended over all elements of Σ is a positive number, and is in fact also the lower bound of $Q(K)$ extended over all convex domain. Evidently $Q < 4$.

Definition: *A convex domain K is called extreme if* $Q(K) = Q$.

Theorem 1: *There exists an extreme convex domain.*

Proof: Choose an infinite sequence

$$K_1, K_2, K_3, \ldots \ldots \ldots \ldots \quad (4)$$

of elements of Σ, not all necessarily different, such that

$$\lim_{n \to \infty} V(K_n) = Q.$$

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[8]) That the area of K attains its lower bound, is proved in the next paragraph, and is here already taken for granted.

All these convex domains K_n are subsets of the bounded polygon H''. Hence, by the selection theorem of BLASCHKE [9]), it is possible to choose an infinite subsequence,

$$K_{n_1}, K_{n_2}, K_{n_3} \ldots$$

of (4) which converges to a convex domain, K say. Then, firstly,

$$V(K) = Q.$$

Secondly, it is obvious that K has the properties (A) and (C). Thirdly, it has also the property (B), since [10])

$$\Delta(K) = \lim_{r \to \infty} \Delta(K_{n_r}) = 1.$$

Hence K is an extreme convex domain, and the assertion is proved.

Theorem 2: *Every extreme convex domain is irreducible.*

Proof: If K is reducible, then, by *ICD*, Lemma 13, a convex domain K' contained in, but different from, K can be found such that $\Delta(K') = \Delta(K)$. Hence $Q(K) > Q(K') \geq Q$, and so K is not extreme.

§ 3. A parameter representation of K.

The last result allows us to restrict the convex domains to be considered still further and to restate the problem as follows:

Problem 1'': *To find an irreducible convex domain K of minimum area Q satisfying the three conditions (A), (B), (C).*

For the investigation of this problem, we apply Lemma 9 of *ICD*:

"Let K be an irreducible convex domain which is not a parallelogram. Then to every point P_1 on C, there exists a unique critical lattice $\Lambda = \Lambda(P_1)$ containing P_1. This lattice has just six points $P_l = P_l(P_1)$ $(l = 1, 2, \ldots, 6)$ on C. Let $A_1, \ldots, A_6$ be the six arcs into which these points divide C; denote further by P_1^* a variable point on A_1, and by $P_l^* = P_l(P_1)$ for $l = 2, \ldots, 6$ the other five points of $\Lambda(P_1^*)$ on C. If P_1^* describes A_1 continuously in positive direction, then P_l^*, for $l = 2, \ldots, 6$, describes A_l in the same manner."

This lemma leads to the following parameter representation of the boundary C of K:

Let $P = (x_1, x_2)$ be the general point of C. Then denote by t a parameter which runs from 0 to 2π when P runs in positive direction over C from $P_1' = (\sqrt[4]{\tfrac{1}{3}}, 0)$ back to P_1'; thus

$$x_1 = x_1(t), \quad x_2 = x_2(t)$$

are functions of t defined for $0 \leq t \leq 2\pi$ in the first instance. So as to simplify the considerations, extend these two functions to all real values of t by the periodicity condition,

$$x_1(t + 2\pi) = x_1(t), \quad x_2(t + 2\pi) = x_2(t).$$

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[9]) W. BLASCHKE, Kreis und Kugel, 62.

[10]) Theorem 9 of my paper, l.c. [4]).

Denote further by

$$P(t) = (x_1(t), x_2(t))$$

the point on C of parameter t, and by $\Lambda(t)$ the critical lattice of K containing $P(t)$. It is clearly possible to choose the parameter t in such a way that the six points of $\Lambda(t)$ on C are just given by

$$P\left(t + \frac{h\pi}{3}\right), \text{ where } h = 0, 1, \ldots, 5;$$

in particular, it is necessary that

$$P\left(\frac{h\pi}{3}\right) = P'_h \qquad (h = 1, 2, \ldots, 6).$$

Since $\Lambda(t)$ is critical, the quadrilateral

$$OP(t)\,P\left(t + \frac{\pi}{3}\right)P\left(t + \frac{2\pi}{3}\right)$$

is a parallelogram of area $\Delta(K) = 1$; hence

$$P(t) - P\left(t + \frac{\pi}{3}\right) + P\left(t + \frac{2\pi}{3}\right) = 0, \qquad \left\{P(t), P\left(t + \frac{\pi}{3}\right)\right\} = 1.$$

The first condition is equivalent to the functional equations,

$$x_1(t) - x_1\left(t + \frac{\pi}{3}\right) + x_1\left(t + \frac{2\pi}{3}\right) = 0,$$

$$x_2(t) - x_2\left(t + \frac{\pi}{3}\right) + x_2\left(t + \frac{2\pi}{3}\right) = 0,$$

which have the general solution,

$$x_1(t) = a_1(t)\cos t + b_1(t)\sin t; \quad x_2(t) = a_2(t)\cos t + b_2(t)\sin t, \qquad (5)$$

where

$$a_1(t), b_1(t), a_2(t), b_2(t) \text{ are functions of } t \text{ of period } \frac{\pi}{3}.$$

The second condition is equivalent to the equation,

$$x_1(t)\,x_2\left(t + \frac{\pi}{3}\right) - x_1\left(t + \frac{\pi}{3}\right)x_2(t) = 1;$$

on substituting the expressions (5) and simplifying, this equation takes the form,

$$a_1(t)\,b_2(t) - a_2(t)\,b_1(t) = +\sqrt{\tfrac{4}{3}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The conditions $P\left(\frac{h\pi}{3}\right) = P'_h$ give the initial values,

$$a_1(0) = b_2(0) = +\sqrt[4]{\tfrac{4}{3}}, \qquad a_2(0) = b_1(0) = 0. \quad . \quad . \quad . \quad (7)$$

Next, the condition that C is a convex curve is equivalent to the inequality,

$$\begin{vmatrix} a_1(t_1)\cos t_1 + b_1(t_1)\sin t_1, & a_2(t_1)\cos t_1 + b_2(t_1)\sin t_1, & 1 \\ a_1(t_2)\cos t_2 + b_1(t_2)\sin t_2, & a_2(t_2)\cos t_2 + b_2(t_2)\sin t_2, & 1 \\ a_1(t_3)\cos t_3 + b_1(t_3)\sin t_3, & a_2(t_3)\cos t_3 + b_2(t_3)\sin t_3, & 1 \end{vmatrix} \geqq 0 \quad (8)$$

$$\text{if } 0 \leqq t_1 < t_2 < t_3 < 2\pi.$$

If $a_1(t)$, $b_1(t)$, $a_2(t)$, $b_2(t)$ have second derivatives, then this inequality implies that

$$\begin{vmatrix} \frac{d}{dt}\{a_1(t)\cos t + b_1(t)\sin t\}, & \frac{d}{dt}\{a_2(t)\cos t + b_2(t)\sin t\} \\ \frac{d^2}{dt^2}\{a_1(t)\cos t + b_1(t)\sin t\}, & \frac{d^2}{dt^2}\{a_2(t)\cos t + b_2(t)\sin t\} \end{vmatrix} \geqq 0. \quad (8')$$

I have not succeeded in expressing either of these two formulae in a more convenient form. (See, however, § 5.)

Finally, an explicit value for the area $V(K)$ of K is found in the following way, under the assumption that $a_1(t)$, $b_1(t)$, $a_2(t)$, $b_2(t)$ are differentiable:

In the integral,

$$V(K) = \frac{1}{2}\int_0^{2\pi} \left\{ x_1(t)\frac{dx_2(t)}{dt} - x_2(t)\frac{dx_1(t)}{dt} \right\} dt,$$

the integrand may be written as,

$$x_1(t)\frac{dx_2(t)}{dt} - x_2(t)\frac{dx_1(t)}{dt} =$$

$$= \{a_1(t)b_2(t) - a_2(t)b_1(t)\} + \{A(t)\cos^2 t + B(t)\cos t \sin t + C(t)\sin^2 t\},$$

where

$$A(t) = a_1(t)\frac{da_2(t)}{dt} - a_2(t)\frac{da_1(t)}{dt},$$

$$B(t) = a_1(t)\frac{db_2(t)}{dt} - b_2(t)\frac{da_1(t)}{dt} + b_1(t)\frac{da_2(t)}{dt} - a_2(t)\frac{db_1(t)}{dt},$$

$$C(t) = b_1(t)\frac{db_2(t)}{dt} - b_2(t)\frac{db_1(t)}{dt}.$$

Since by (6),

$$\frac{1}{2}\int_0^{2\pi} \{a_1(t)b_2(t) - a_2(t)b_1(t)\}\, dt = \frac{2\pi}{\sqrt{3}},$$

evidently,

$$V(K) = \frac{2\pi}{\sqrt{3}} + \frac{1}{2}\int_0^{2\pi} \left\{ A(t)\cos^2 t + B(t)\cos t \sin t + C(t)\sin^2 t \right\} dt.$$

The integral on the right can be much simplified since $A(t)$, $B(t)$, $C(t)$ are periodic functions of period $\frac{\pi}{3}$. To this purpose, replace t under the integral sign by

$$t,\ t+\frac{\pi}{3},\ t+\frac{2\pi}{3},$$

and take the arithmetical means. Since

$$\cos^2 t+\cos^2\left(t+\frac{\pi}{3}\right)+\cos^2\left(t+\frac{2\pi}{3}\right)=$$

$$=\sin^2 t+\sin^2\left(t+\frac{\pi}{3}\right)+\sin^2\left(t+\frac{2\pi}{3}\right)=\frac{3}{2}$$

and

$$\cos t\sin t+\cos\left(t+\frac{\pi}{3}\right)\sin\left(t+\frac{\pi}{3}\right)+\cos\left(t+\frac{2\pi}{3}\right)\sin\left(t+\frac{2\pi}{3}\right)=0,$$

this leads to the formula,

$$V(K)=\frac{2\pi}{\sqrt{3}}+I(K), \qquad (9)$$

where

$$I(K)=\frac{1}{4}\int_0^{2\pi}\{A(t)+C(t)\}\,dt,$$

hence,

$$I(K)=\frac{1}{4}\int_0^{2\pi}\left\{a_1(t)\frac{da_2(t)}{dt}-a_2(t)\frac{da_1(t)}{dt}+b_1(t)\frac{db_2(t)}{dt}-b_2(t)\frac{db(t)}{dt}\right\}dt. \qquad (10)$$

We see then that Problem 1″ is essentially equivalent [11]) to the following

Problem 1′: *To find four functions* $a_1(t)$, $b_1(t)$, $a_2(t)$, $b_2(t)$ *of period* $\frac{\pi}{3}$ *satisfying the conditions* (6), (7), (8), *and giving the integral* $I(K)$ *in* (10) *a smallest value.*

§ 4. **The integrals of the** EULER **differential equations.**

There is no difficulty in applying the classical EULER–LAGRANGE method to Problem 1′, omitting, however, the inequality condition (8).

Write a_h, b_h instead of $a_h(t)$, $b_h(t)$, and use a dot for the differential coefficients with respect to t; denote further by λ a suitable function of t. Then the EULER equations for the function

$$F(a_1, b_1, a_2, b_2)=\{a_1\dot{a}_2-\dot{a}_1 a_2+b_1\dot{b}_2-\dot{b}_1 b_2\}+\lambda\{a_1 b_2-a_2 b_1-\sqrt{\tfrac{4}{3}}\},$$

[11]) It is not a priori evident that the boundary of an extreme convex domain has everywhere a tangent, thus that $a_1(t)$, $b_1(t)$, $a_2(t)$, $b_2(t)$ are differentiable for all t.

i.e. the differential equations,

$$\frac{\partial F}{\partial a_h}-\frac{d}{dt}\frac{\partial F}{\partial \dot{a}_h}=0, \qquad \frac{\partial F}{\partial b_h}-\frac{d}{dt}\frac{\partial F}{\partial \dot{b}_h}=0 \qquad (h=1,2),$$

are as follows:

$$2\dot{a}_1+\lambda b_1=2\dot{b}_1-\lambda a_1=2\dot{a}_2+\lambda b_2=2\dot{b}_2-\lambda a_2=0 \quad . \quad . \quad . \quad (11)$$

On eliminating λ from the first or the last two equations (11), we get

$$a_1\dot{a}_1+b_1\dot{b}_1=0, \qquad a_2\dot{a}_2+b_2\dot{b}_2=0,$$

whence, on integrating for t,

$$a_1^2+b_1^2=\gamma_1^2, \qquad a_2^2+b_2^2=\gamma_2^2, \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where γ_1, γ_2 are independent of t. If, on the other hand, λ is eliminated from the first and the third equation, then

$$\frac{\dot{a}_1}{b_1}=\frac{\dot{a}_2}{b_2}, \quad \text{or} \quad \frac{\dot{a}_1}{\sqrt{\gamma_1^2-a_1^2}}=\frac{\dot{a}_2}{\sqrt{\gamma_2^2-a_2^2}},$$

whence, on integrating again for t,

$$\cos^{-1}\frac{a_1}{\gamma_1}=\cos^{-1}\frac{a_2}{\gamma_2}-\Gamma, \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where Γ is a further number independent of t.

The two equations (12) and (13) imply that there is an angle Θ such that

$$\left.\begin{aligned} a_1&=\gamma_1\cos\Theta, & b_1&=\gamma_1\sin\Theta,\\ a_2&=\gamma_2\cos(\Theta+\Gamma), & b_2&=\gamma_2\sin(\Theta+\Gamma). \end{aligned}\right\} \quad . \quad . \quad . \quad (14)$$

On substituting these values in (6),

$$a_1 b_2-a_2 b_1=\gamma_1\gamma_2\sin\Gamma=+\sqrt{\tfrac{4}{3}}. \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Further, from (5),

$$x_1=a_1\cos t+b_1\sin t=\gamma_1\cos(t-\Theta),$$
$$x_2=a_2\cos t+b_2\sin t=\gamma_2\cos(t-\Theta-\Gamma),$$

and so,

$$\gamma_2 x_1\cos\Gamma+\gamma_2\sqrt{\gamma_2^2-x_1^2}\sin\Gamma=\gamma_1 x_2,$$

whence from (15),

$$\gamma_2^2 x_1^2-2\gamma_1\gamma_2 x_1 x_2\cos\Gamma+\gamma_1^2 x_2^2=\tfrac{4}{3}. \quad . \quad . \quad . \quad . \quad (16)$$

Since

$$\sin\Gamma\neq 0, \quad |\cos\Gamma|<1.$$

this is the equation of an ellipse E which evidently has the properties,

$$\Delta(E)=1, \quad V(E)=Q(E)=\frac{2\pi}{\sqrt{3}}.$$

For instance, the circle Z,

$$x_1^2 + x_2^2 = \sqrt{\tfrac{4}{3}}, \quad \ldots\ldots\ldots \quad (17)$$

obtained for

$$\gamma_1 = \gamma_2 = \sqrt[4]{\tfrac{4}{3}}, \ \Gamma = \frac{\pi}{2},$$

is of this kind; it passes through the six points P'_h.

§ 5. A property of ellipses.

Theorem 3: *No ellipse is an extreme domain.*

Proof: By affine invariance, it suffices to prove the assertion for the circle Z defined in (17), i.e. for the functions

$$a_1(t) \equiv b_2(t) \equiv 1, \ a_2(t) \equiv b_1(t) \equiv 0 \text{ identically in } t.$$

Denote by ε a small positive number, and consider the neighbouring domain K_ε belonging to the functions,

$$\left.\begin{aligned} a_1(t) &= \sqrt[4]{\tfrac{4}{3}}\,(1+\varepsilon \sin 6t)^{-1}, & b_1(t) &= -\varepsilon(\cos 6t - 1), \\ a_2(t) &= 0, & b_2(t) &= \sqrt[4]{\tfrac{4}{3}}\,(1+\varepsilon \sin 6t). \end{aligned}\right\} \quad (18)$$

These functions satisfy both the identity (6) and the initial conditions (7). Further, on substituting in (8′), this determinant can be developed into a power series

$$\sqrt{\tfrac{4}{3}} + \sum_{n=1}^{\infty} u_n(t)\,\varepsilon^n$$

in ε which converges absolutely and uniformly in t if ε is sufficiently small. Moreover, the coefficients $u_n(t)$ are continuous functions of t. The determinant is therefore positive for sufficiently small positive ε, and so K_ε is then a convex domain.

On substituting the functions (18) into the integral (10) for $I(K_\varepsilon)$, this integral becomes,

$$I(K_\varepsilon) = \sqrt[4]{\tfrac{27}{4}}\,\varepsilon \int_0^{2\pi} \{-\varepsilon + \varepsilon \cos 6t - \sin 6t\}\,dt = -\sqrt[4]{108}\,\pi\varepsilon^2 < 0.$$

Therefore from (9),

$$V(K_\varepsilon) < \frac{2\pi}{\sqrt{3}} = V(Z), \ Q(K_\varepsilon) < Q(Z),$$

as asserted.

Corollary: *The lower bound of $Q(K)$ extended over all convex domains K is smaller than* $\dfrac{2\pi}{\sqrt{3}}$.

§ 6. **Another form of the variation problem.**

Problem 1 can be expressed in many other ways as a problem in the calculus of variations. One particularly simple formulation is as follows:

Assume that

$$\Delta(K) = 2,$$

and that the boundary C of K passes through the six points,

$$p_1 = (2, 0),\ p_2 = (1, 1),\ p_3 = (-1, 1),\ p_4 = -p_1,\ p_5 = -p_2,\ p_6 = -p_3;$$

this is permitted since

$$p_1 + p_3 = p_2, \quad \{p_1, p_2\} = 2.$$

Denote by

$$P_1 = (x_1', x_2'), \quad P_2 = (x_1, x_2), \quad P_3 = (x_1''', x_2''')$$

three points of C on the arcs p_1p_2, p_2p_3, p_3p_1, respectively, which belong to the same critical lattice of K. Then x_2 is a single-valued continuous function of x_1 for $-1 \leq x_1 \leq 1$ such that

$$x_2 = 1 \text{ for } x_1 = -1 \text{ and } x_1 = 1.$$

The conditions,

$$P_1 + P_3 = P_2, \quad \{P_1, P_2\} = 2$$

are satisfied by chosing,

$$P_1 = \left(\frac{2}{x_2} + \frac{1-\varrho}{2} x_1, \frac{1-\varrho}{2} x_2\right), \quad P_3 = \left(-\frac{2}{x_2} + \frac{1+\varrho}{2} x_1, \frac{1+\varrho}{2} x_2\right),$$

where $\varrho = \varrho(x_1)$ is a continuous function of x_1; on identifying P_2 with p_2 or p_3, one finds that

$$\varrho(-1) = -1, \quad \varrho(1) = 1.$$

There are further some rather complicated conditions involving the first and second derivatives of $x_2(x_1)$ and $\varrho(x_1)$ which express that C is convex.

A simple calculation leads now to the integral

$$V(K) = \int_{-1}^{+1} \left\{ \frac{3+\varrho^2}{2} (x_2 - x_1 x_2') + 4 \frac{x_2'}{x_2} \varrho + 2\varrho' \right\} dx_1 \qquad \left(x_2' = \frac{dx_2}{dx_1}\right)$$

for $V(K)$. I omit the discussion of EULER's equations which gives the same results as the other method.

Final remark: It seems highly probable from the convexity condition, that the boundary of an extreme convex domain consists of line segments and arcs of hyperbolae. So far, however, I have not succeeded in proving this assertion.

§ 7. **The relation to Problem II.**

Let K be a convex domain, and let Λ and $\lambda = 2\Lambda$ be two lattices such that

λ consists of the points $2P$ where P belongs to Λ.

Denote by $K(P)$ the convex domain of all points

$$P + X \text{ where } X \text{ belongs to } K,$$

by

$$\Sigma = \sum_{P \text{ in } \lambda} K(P)$$

the join of all domains

$$K(P) \text{ where } P \text{ belongs to } \lambda,$$

and by Σ_R the set of all points $X = (x_1, x_2)$ of Σ which belong to the square Z_R:

$$|x_1| \leqq R, \ |x_2| \leqq R.$$

By MINKOWSKI [12]), the following results hold:

(1) The ratio,

$$\frac{V(\Sigma_R)}{V(Z_R)} = \frac{V(\Sigma_R)}{4R^2}$$

of the areas of Σ_R and Z_R tends of a limit, $q(K, \Lambda)$ say, as R tends to infinity.

(2) When P and P' run over all pairs of different elements of λ, then no two domains $K(P)$ and $K(P')$ are overlapping if and only if Λ is K-admissible.

(3) If Λ is K-admissible, then,

$$q(K, \Lambda) = \frac{V K)}{d(\lambda)} = \frac{V(K)}{4\, d(\Lambda)}.$$

Since

$$d(\Lambda) \geqq \Delta(K)$$

for all K-admissible lattices, with equality only if K is critical, the lower bound of $q(K, \Lambda)$ extended over all admissible lattices Λ, say $q(K)$, is thus given by

$$q(K) = \frac{V(K)}{4\,\Delta(K)} = \tfrac{1}{4} Q(K).$$

Hence the two problems I and II are completely equivalent.

We see, in particular, from the results proved earlier that there exists a convex domain (viz., an extreme domain) such that

$$q(K) < \frac{\pi}{\sqrt{12}},$$

and that this domain is *not* an ellipse.

Mathematics Department, Manchester University.

September 29, 1946.

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[12]) Diophantische Approximationen, 82—90. MINKOWSKI considers the case of three dimensions; but the ideas are the same for the plane. His notation is different from the one used here.

Zoology. — *The Rhizocephalan parasites of the crab Chlorodiella nigra (Forsk.)*. By H. BOSCHMA.

(Communicated at the meeting of January 25, 1947.)

In the course of his studies on parasites of Crustacea GIARD expressed as his opinion that in each group of parasites every form occurring on a different species of host is a representative of a separate species. Consequently GIARD and his collaborator BONNIER gave specific names to several Rhizocephala occurring on hosts of which previously no parasites of the group had become known (GIARD, 1886, 1887, 1888; BONNIER, 1887; GIARD and BONNIER, 1890). This procedure was severely criticized by DELAGE (1884) who was convinced that *Sacculina carcini* Thomps. was the only species of the genus. SMITH (1906) in his studies on the Rhizocephala of the Gulf of Napels also came to the conclusion that all the described species of the genus *Sacculina* should be regarded as synonyms of *S. carcini*. These two authors did not sufficiently take into account the researches of KOSSMANN (1872), who demonstrated that several species of Sacculinidae may be characterized by specific peculiarities of the shape of the genital organs and of the excrescences of the external cuticle.

Even when Rhizocephala of various hosts present striking differences one might be inclined to regard these differences not as specific characters of the parasites themselves, but as peculiarities induced by the host. This might explain why in many cases closely allied species of crabs may be infested with parasites showing the same specific characters, e.g., the various European species of *Inachus* and *Macropodia*, parasitized by *Drepanorchis neglecta* (Fraisse).

When, however, a certain species of crab may be infested by more than one species of parasite it is evident that the differences of these parasites are not caused by influences of the host, but constitute definite specific characters of the parasites themselves. A striking example of a crab with four different species of Rhizocephala is that of *Chlorodiella nigra* (Forsk.). The four parasites of this crab may be defined in the following way:

Drepanorchis villosa (V. K. and B., 1925) (diagnosis compiled from VAN KAMPEN and BOSCHMA, 1925, and BOSCHMA, 1931*b*): Male genital organs distinctly curved, vasa deferentia rather narrow, testes enlarged into voluminous sacs which are in close contact, but remain completely separated. Male organs of approximately equal size. Colleteric glands with a small number of canals (less than 10 in longitudinal sections of the most strongly branched region). External cuticle densely covered with hairs or elongate papillae which have a length of 7 to 15 μ, they consist of the

same kind of chitin as that of the main layers. These excrescences possess some minute lateral hairs. Retinacula with 8 to 14 spindles on a common basal part, the spindles have a length of about 14 μ.

Type on *Chlorodiella nigra* (Forsk.); type-locality Jiddah, Red Sea. The species is not known to occur on other hosts.

Sacculina phacelothrix Boschma, 1931*a* (diagnosis copied from BOSCHMA, 1937): Male genital organs in the posterior part of the body, outside the visceral mass, completely separated. Testes more or less globular, rather abruptly passing into the vasa deferentia; the latter are comparatively wide. Colleteric glands with few canals (less than 10 in longitudinal sections of the most strongly divided part), the canals neatly arranged in a single row parallel to the surface of the visceral mass. External cuticle of the mantle with excrescences of a hyaline kind of chitin, differing from that of the main layers of the cuticle. These excrescences are composed of groups of spines which in their basal part usually are not united, but they may be combined on a very little developed basal part. The spines have a length of 15 to 30 μ, they may possess numerous minute lateral hairs. Internal cuticle of the mantle with retinacula which are more or less regularly distributed on its surface. Each retinaculum consists of a basal part and 3 to 5 spindles; the latter have a length of 9 μ approximately.

Type on *Chlorodiella nigra* (Forsk.); type-locality Trincomalee, Ceylon. The species is not known to occur on other hosts.

Loxothylacus variabilis Boschma, 1940: Male genital organs of equal size or left small and right large. Curvature of male organs distinct, narrow or wide, or male organs slightly bent, or male organs practically straight. Colleteric glands with a moderate to fairly large number of branched canals. External cuticle densely covered with comparatively small hairs (minimum and maximum measurements 6 and 52 μ). Between these hairs there are larger spines in far smaller numbers (minimum and maximum measurements 30 and 186 μ). The excrescences have undivided tips or are irregularly divided into smaller branches. Retinacula with 1 to 5 spindles which may show barbs and vary in length from 9 to 13 μ.

Type on *Chlorodiella nigra* (Forsk.); type-locality near Koepang, Timor. The species moreover is known as a parasite of *Actaea rüppellii* (Krauss) and of another, unidentified Xanthid crab.

Loxothylacus vepretus nov. spec.: Male genital organs of unequal size, the one much larger than the other. The smaller testes usually is straight or slightly curved, the larger testis usually shows a more or less distinct curve but may be straight. Colleteric glands with a fairly large number of canals (25 to 50 canals in the most strongly branched region of these glands). External cuticle with excrescences consisting of spines of a hyaline kind of chitin, differing in structure from that of the main layers of the cuticle. The spines are arranged into groups, the individual spines remain isolated or are combined into compounds with a common basal part. The length of the excrescences varies from 15 to 60 μ. Retinacula with 3 to 7

spindles united on a common basal part, the length of the spindles amounts to 9 to 12 μ.

Type on *Chlorodiella nigra* (Forsk.), type-locality Red Sea (material collected by KOSSMANN, now in the collection of the Zoological Institute at Heidelberg). The species is not known to occur on other hosts.

A more elaborate description of the material of the new species follows here.

Five specimens of *Loxothylacus vepretus*, all of which were parasites of *Chlorodiella nigra*, were studied in some detail:

No. 1092 A (holotype). Red Sea, measurements 6 × 5 × 2 mm.
No. 403 A. Massaua, Red Sea, measurements 4 × 3 × 2 mm.
No. 720 A. Obi latoe, East Indies, measurements 4 × 3 × 1½ mm.
No. 720 B. Obi latoe, East Indies, measurements 5 × 4 × 2 mm.
No. 1092 B. Red Sea, measurements 6 × 5 × 2 mm.

It may be noted that in *Loxothylacus vepretus* the most striking character of the genus is well pronounced in the specimens examined, as the visceral mass is attached to the mantle at a considerable distance from the stalk, which is especially evident in longitudinal sections (figs. 1*c*, 2*a*, 3*a*).

The larger of the two male organs has a vas deferens which from the narrow male genital opening gradually increases in size. On its inner wall there are several ridges so that its lumen here is divided into numerous cavities. The vas deferens gradually passes into the testis which in its ventral part has a comparatively thick wall and consequently a rather narrow lumen. Towards the dorsal part of the testis its wall becomes thinner and here usually the testis shows a curve in an anterior direction. The smaller of the two male organs has a similar structure, in a less pronounced manner; usually it does not show a distinct curve in an anterior direction.

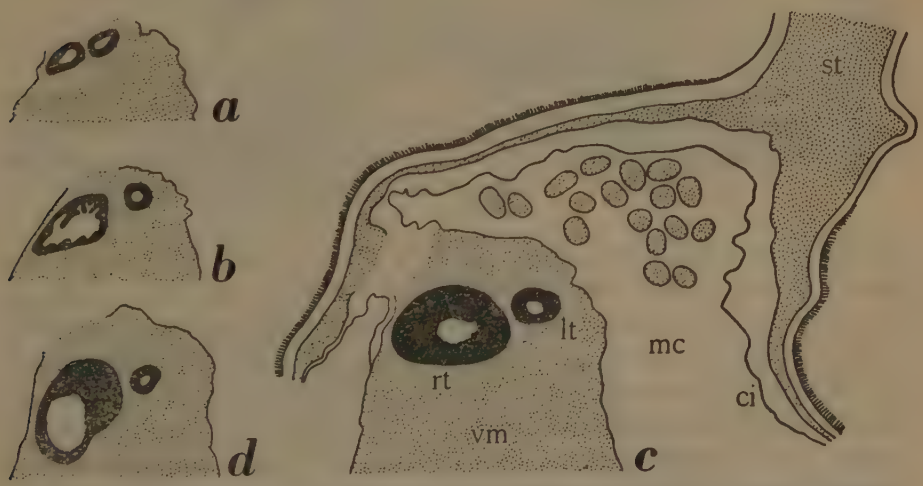

Fig. 1. *Loxothylacus vepretus*, specimen no. 1092 A, longitudinal sections; *a*, through the vasa deferentia, each following section from a more dorsal part. *ci*, internal cuticle; *lt*, left testis; *mc*, mantle cavity; *rt*, right testis; *st*, stalk; *vm*, visceral mass. × 45.

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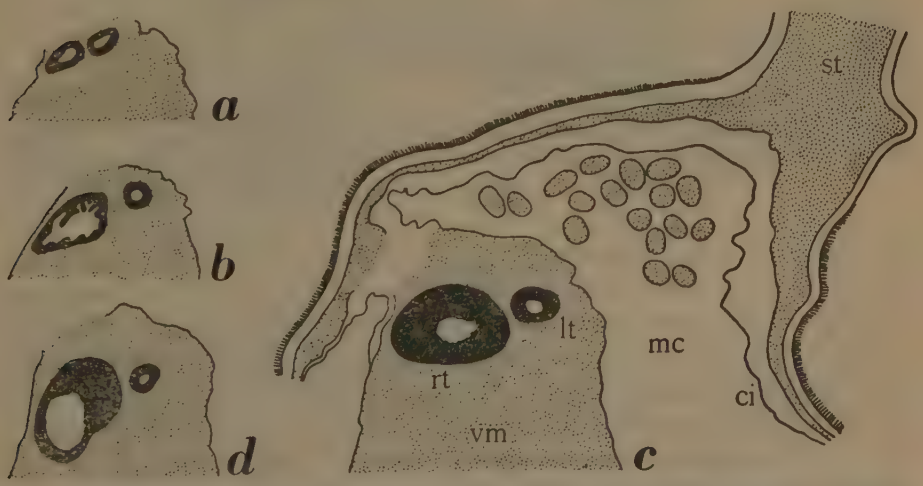

Fig. 1. *Loxothylacus vepretus*, specimen no. 1092 A, longitudinal sections; *a*, through the vasa deferentia, each following section from a more dorsal part. *ci*, internal cuticle; *lt*, left testis; *mc*, mantle cavity; *rt*, right testis; *st*, stalk; *vm*, visceral mass. × 45.

As in the various specimens there is a certain amount of individual variation in the shape of the male genital organs, the chief particulars of these organs in the five specimens may be briefly described here.

No. 1092 A (longitudinal sections). In their ventral part the vasa deferentia have an approximately equal size (fig. 1*a*). Gradually towards a more dorsal plane the right vas deferens considerably increases in size, whilst on its inner wall there occur numerous ridges which protrude towards the centre; the left vas deferens does not become appreciably larger (fig. 1*b*). In the region of the stalk the vasa deferentia have passed into the testes of which the right is much larger than the left (fig. 1*c*). Both testes have a rather thick wall, at least for the greater part of their length. In their dorsal (closed) extremity the wall of the testes becomes thinner. In this part the right testis is slightly curved towards the anterior region of the body, whilst the left testis remains practically straight (fig. 1*d*).

No. 403 A. The male organs practically have the same shape and structure as those of the former specimen. Two sections, one from the ventral part of the male organs (fig. 2*a*) and one from the dorsal part (fig. 2*b*)

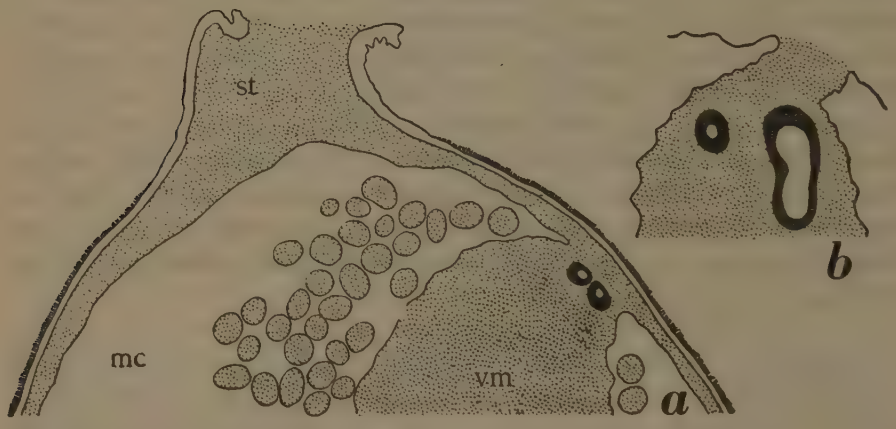

Fig. 2. *Loxothylacus vepretus*, specimen no. 403 A, longitudinal sections; *a*, through the stalk, *b*, through the dorsal part of the testes. *mc*, mantle cavity; *st*, stalk; *vm*, visceral mass. × 45.

are represented here. Here again the right testis (at the right side of the figure) is much larger than the left. Its curvature in the extreme dorsal part is somewhat more pronounced than in the former specimen (fig. 2*b*).

No. 720 A. In this specimen the male organs have a straight course. Fig. 3*a* shows a section from the region of the stalk (in sections of the vicinity of the represented part the stalk inserts on the part of the mantle drawn in the upper part of the figure). Here the ventral parts of the vasa deferentia are shown. A section from a more dorsal region contains the two testes (fig. 3*b*), the right (at the left side of the figure) again is of much larger size than the left. The extremity of the larger testis is shown

in fig. 3c, it does not show a tendency for a curvature in an anterior direction. The smaller testis does not extend as far dorsally.

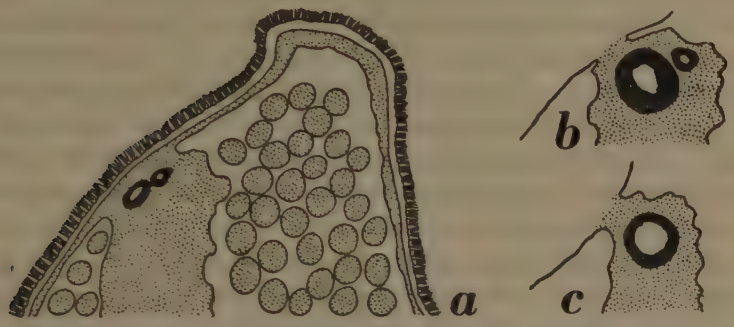

Fig. 3. *Loxothylacus vepretus*, specimen no. 720 A, longitudinal sections; *a*, through the vasa deferentia, each following section from a more dorsal part. × 45.

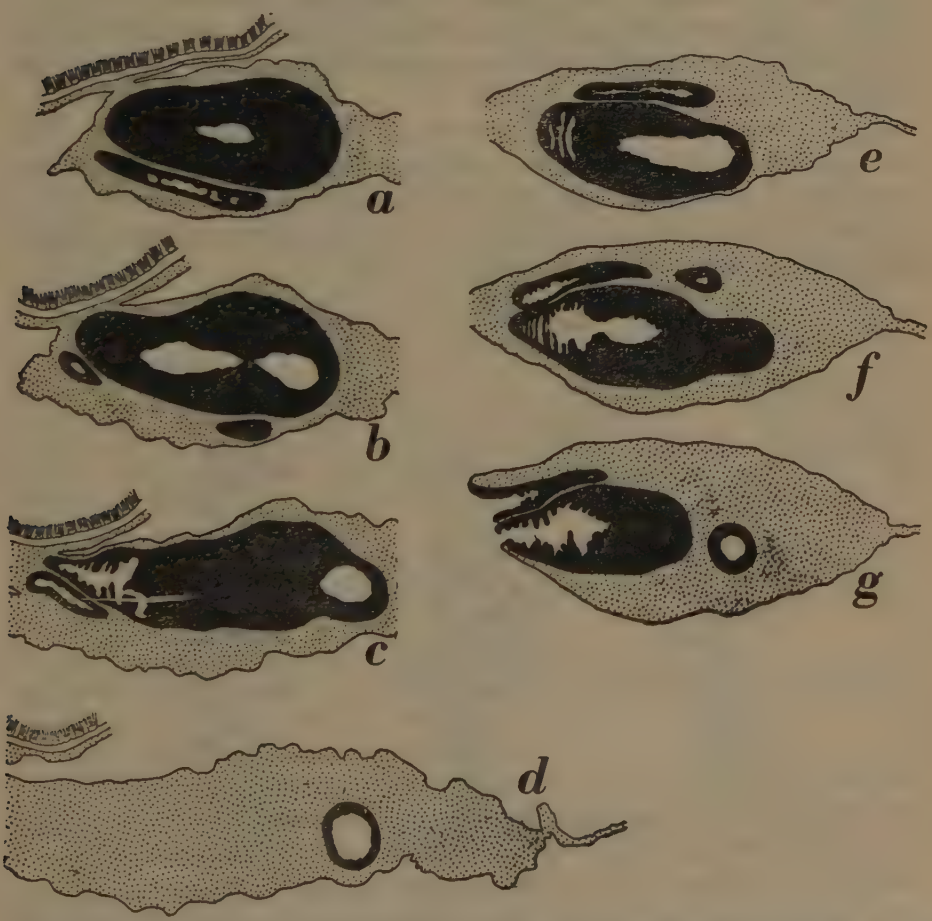

Fig. 4. *Loxothylacus vepretus*, transverse sections; *a*—*d*, specimen no. 720 B; *e*—*g*, specimen no. 1092 B. *a* and *e*, from the posterior part of the body, each following section from a more anterior part. × 45.

Transverse sections of the body very distinctly show the differences in shape and size of the male genital organs. It appears from these sections that not only the larger testis may have a pronounced curve in an anterior direction but that also the smaller testis has a not altogether straight course.

No. 720 B. Especially in the larger male organ the vas deferens has a great number of ridges on its inner wall, resulting in a rather irregular cavity (fig. 4*c*). The larger testis for the greater part of its extent has a very thick wall (fig. 4*a*, *b*), towards its dorsal region the wall becomes thinner (fig. 4*c*), whilst the organ shows a pronounced curve in an anterior direction so that it penetrates for some distance into the visceral mass (fig. 4*d*). The smaller male organ has an irregular lumen (fig. 4*a*), its extreme ventral and dorsal parts are slightly bent in an anterior direction (fig. 4*b*).

No. 1092 B. The shape and the course of the male genital organs correspond in every detail with those of the former specimen. Fig. 4*e* shows a section through the posterior part of the male organs, fig. 4*f* and *g* represent slightly more anterior parts. The male genital openings are shown in fig. 4*g*, in which also the curved anterior part of the larger testis is to be seen. The smaller male organ has a similar slight curve as that of the former specimen (fig. 4*f*).

Fig. 5 shows sections of the colleteric glands of three specimens. The canal system is well developed, in the most strongly branched region of these glands in the three specimens the number of canals amounts to 40 (fig. 5*a*), 25 (fig. 5*b*) and 50 (fig. 5*c*). The canals form a rather compact mass, they are not arranged in rows.

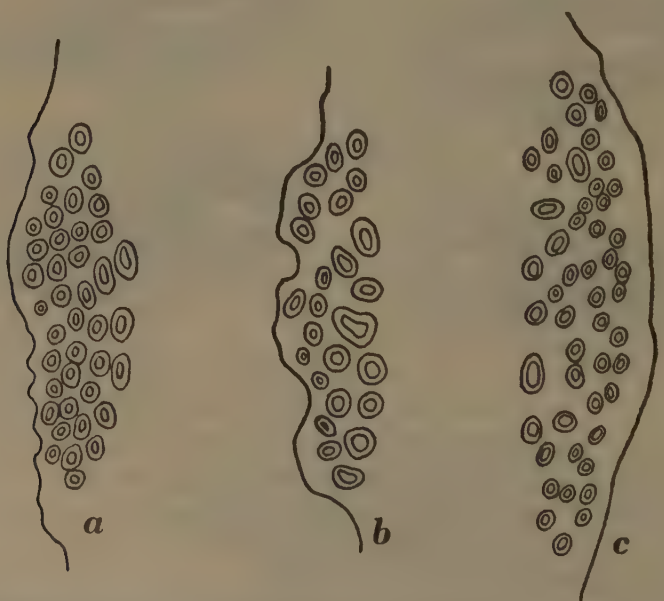

Fig. 5. *Loxothylacus vepretus*, longitudinal sections of colleteric glands of three specimens showing the most strongly branched region. Posterior part in the upper side of the figures. *a*, specimen no. 403 A, *b*, specimen no. 720 A, *c*, specimen no. 1092 A. × 128.

The excrescences of the external cuticle of the mantle in each specimen vary to a certain degree as in some parts of the mantle they have a different size and structure from that in other regions. On the whole they consist of groups of spines which may be united on a common basal part. The chitin of the excrescences differs from that of the main layers of the cuticle by its being more hyaline and harder.

In the specimens 1092 A and B the excrescences of the external cuticle are rather small and slender, their length varies from 20 to 40 μ (fig. 6 *a–d*). In these specimens many spines remain isolated though there is a tendency to form small groups of 2 to 3 spines each. In specimen 720 A the excrescences may reach a length of 60 μ, then they form groups of spines united on a basal part (fig. 6 *e–f*). In some parts of the mantle the spines remain more or less isolated (fig. 6*g*), then they also are somewhat smaller (length

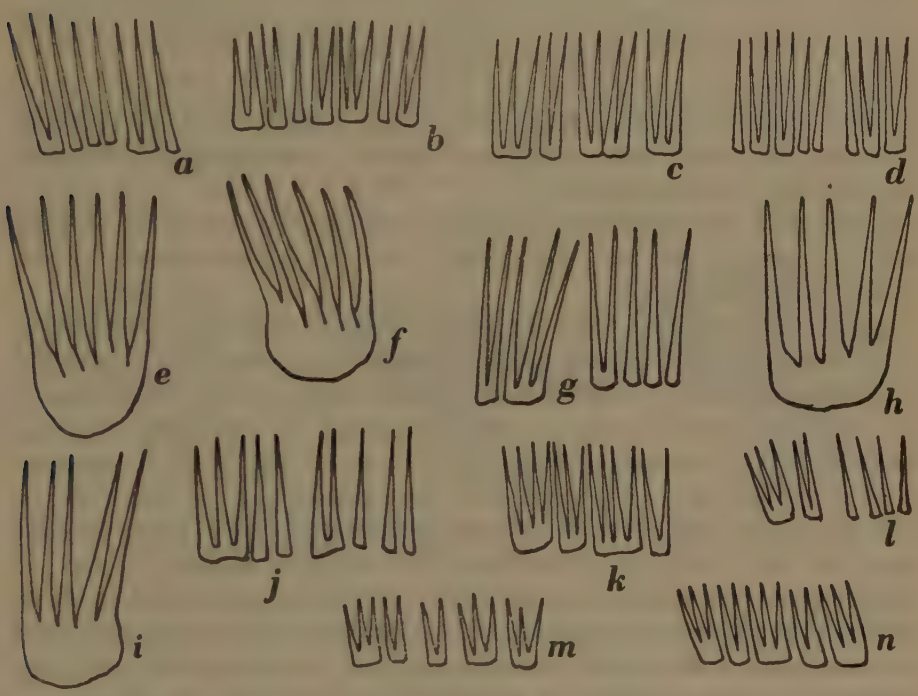

Fig. 6. *Loxothylacus vepretus,* excrescences of the external cuticle. *a—b,* specimen no. 1092 A; *c—d,* specimen no. 1092 B; *e—g,* specimen no. 720 A; *h—j,* specimen no. 720 B; *k—n,* specimen no. 403 A. × 530.

about 38 μ). Specimen 720 B has excrescences of similar structure and size. Here again the larger spines are united on common basal parts (fig. 6 *h–i*), the length of these is about 60 μ. When the excrescences are smaller (fig. 6 *j*, about 30 μ) the spines usually remain more or less isolated. In specimen 403 A the excrescences are smaller and more slender than those of the former specimens. They vary in size and in shape in a corresponding manner, here again the spines may be united into compounds

with a common basal part, or may remain more or less isolated (fig. 6 *k–n*). In this specimen the length of the excrescences varies from 15 to 30 μ.

In specimens 1092 A and B the internal cuticle of the mantle bears distinct retinacula (fig. 7). These excrescences are more or less regularly

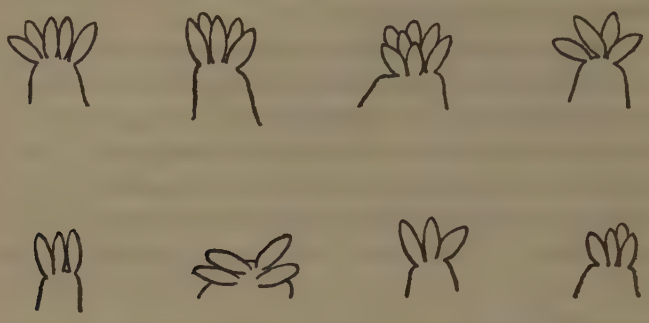

Fig. 7. *Loxothylacus vepretus*, retinacula. Upper row, specimen no. 1092 A; lower row, specimen no. 1092 B. × 530.

distributed over the surface of the cuticle, they consist of a basal part and 3 to 7 spindles, which have a length of 9 to 12 μ. No barbs could be seen on the sides of these spindles.

In the other specimens undoubtedly also retinacula occur on the internal cuticle; they were, however, not found on the parts of the internal cuticle examined for this purpose.

Besides *Loxothylacus vepretus* four species of the genus are known which possess excrescences of a similar shape, viz., *L. aristatus, L. setaceus, L. desmothrix,* and *L. strandi* (cf. BOSCHMA, 1936). By their characters of the male genital organs and of the excrescences of the mantle three of these are easily to be distinguished from *L. vepretus,* the fourth, *L. aristatus,* in its characters shows a great deal of resemblance to *L. vepretus*. In the only known specimen of *L. aristatus* the male organs are of approximately equal size. As in the known specimens of *L. vepretum* the male organs are decidedly unequal this might constitute a distinct specific character. It is, however, not quite certain that this peculiarity may be regarded as a constant specific character as in another species of the genus, *L. variabilis,* there occur specimens with male organs of equal size and structure and specimens with highly different male organs (BOSCHMA, 1940). The excrescences of the external cuticle of the mantle in the two species show differences which really may be considered as of specific value. In *L. vepretus* these excrescences consist of a small number of spines, whilst their size does not exceed 60 μ; in *L. aristatus* the excrescences in their most fully developed form consist of numerous spines, they may have a length of 85 μ. Especially the fact that in the latter species the excrescences may form compounds consisting of a great number of spines united on a well developed basal part points to a specific distinction between *L. aristatus* and *L. vepretus*.

The four species of Rhizocephala known to infest *Chlorodiella nigra* have quite different characters.

Loxothylacus variabilis is distinct from all other Sacculinidae by its peculiar excrescences of the external cuticle: very long spines on a surface densely covered with much shorter spines.

Drepanorchis villosa differs from the three other species by the excrescences of the external cuticle which consist of rather long papillae not differing in structure from the chitin of the main layers. Its retinacula have numerous spindles of a very slender shape. Moreover the male organs differ from those of the other species as the testes are enlarged into rather wide sacs with thin walls. Not too much stress may be laid on the generic characters, as the available specimens are no more in an altogether excellent state of preservation.

Sacculina phacelothrix has excrescences of the external cuticle of a shape and size as they may occur in *Loxothylacus vepretus*. Here the generic distinction already shows that the two species are distinct. In a previous paper (BOSCHMA, 1940) I could show that specimens of certain species of *Loxothylacus* may present characters regarded as generic characters of *Sacculina* (more or less straight male genital organs), but *Sacculina phacelothrix* undoubtedly is a representative of the latter genus. In this species the male genital organs are found outside the visceral mass (fig. 8). Here

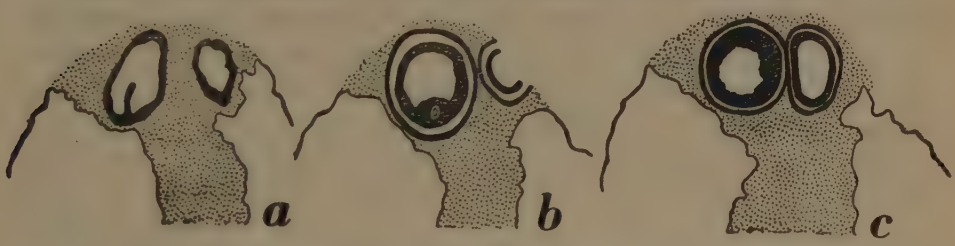

Fig. 8. *Sacculina phacelothrix*, type specimen, longitudinal sections; *a*, through the vasa deferentia, each following section from a more dorsal part. × 45.

the vasa deferentia are comparatively wide and possess a few ridges only on their inner wall (fig. 8*a*), each vas deferens is connected with its testis by a narrow canal with a chitinous wall (fig. 8*b*, this canal is visible in the wall of the testis which is surrounded by a muscular layer), the two testes are not strongly differing in size (fig. 8*c*). In *Loxothylacus vepretus* the male genital organs have a completely different structure, so that already on this account the two species can be easily distinguished.

As the four species of Sacculinidae occurring on *Chlorodiella nigra* show specific characters in a so strongly different manner we may safely conclude that these characters are not influenced by the host but are inherent to the species.

LITERATURE.

BONNIER, J., 1887. Catalogue des Crustacés Malacostracés receuillis dans la Baie de Concarneau. Bull. scient. Nord France et Belg. (2), vol. 10.

BOSCHMA, H., 1931*a*. Rhizocephala. Papers from Dr. Th. Mortensen's Pacific Expedition 1914—16, LV. Vidensk. Medd. Dansk naturk. Foren., vol. 89.

———, 1931*b*. Die Rhizocephalen der Siboga-Expedition. Supplement. Siboga-Exp., monogr. 31 bis.

———, 1936. Notes on some Rhizocephala of the Genus Loxothylacus. Festschrift Embrik Strand, vol. 1.

———, 1937. The Species of the Genus Sacculina (Crustacea Rhizocephala). Zool. Meded., vol. 19.

———, 1940. Biological Results of the Snellius Expedition, VIII. Some Rhizocephala of the Genus Loxothylacus. Temminckia, vol. 5.

DELAGE, Y., 1884. Evolution de la Sacculine (Sacculina carcini Thomps.) Crustacé endoparasite de l'ordre nouveau des Kentrogonides. Arch. Zool. exp. gén. (2), vol. 2.

GIARD, A., 1886. De l'influence de certains parasites Rhizocéphales sur les caractères sexuels de leur hôte. C. R. Ac. Sc. Paris, vol. 103.

———, 1887. La castration parasitaire et son influence sur les caractères extérieurs du sexe male chez les Crustacés Décapodes. Bull. scient. Nord France et Belg. (2), vol. 10.

———, 1888. Le laboratoire de Wimereux en 1888 (recherches fauniques). Bull. scient. France et Belg. (3), vol. 1.

GIARD, A., and J. BONNIER, 1890. Prodrome d'une monographie des Epicarides du Golfe de Naples. Bull. scient. France et Belg. (4), vol. 1.

KAMPEN, P. N. VAN, and H. BOSCHMA, 1925. Die Rhizocephalen der Siboga-Expedition. Siboga-Exp., monogr. 31 bis.

KOSSMANN, R., 1872. Beiträge zur Anatomie der schmarotzenden Rankenfüssler. Inaug.-Diss., Würzburg (also in Arb. zool.-zoot. Inst. Würzburg, vol. 1, 1874).

SMITH, G., 1906. Rhizocephala. Fauna und Flora des Golfes von Neapel, monogr. 29.

Mathematics. — *A note on 0-dimensional spaces.* By J. DE GROOT. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of January 25, 1947.)

1. *Introduction.* An 0-dimensional space is, as is well known, a space in which any point has arbitrarily small neighbourhoods with vacuous boundary, or — which is the same thing — in which for any point p and any neighbourhood $U \supset p$ may be found a both open and closed set S with $p \subset S \subset U$.

Among spaces in general these 0-dimensional spaces naturally have a special place: for 0-dimensional spaces some problems may easily be solved, while other ones — by total lack of connection — may present particular difficulties. From this special place among spaces of higher dimension the 0-dimensional spaces derive the right of a separate treatment. This is, however, not intended by this note. We only want to justify the particular position of the 0-dimensional spaces by one special question, which we may put as follows: *which axioms must we impose on a general space* (space of neighbourhoods or topological space) *such that this space be 0-dimensional and metrisable* (= separable metric) *and how far is it easier to prove the metrization for these spaces than for general separable spaces?*

In 2. we shall give the answer to this question and offer a proof of the metrization of the 0-dimensional separable spaces, which is much simpler than that of URYSOHN's well-known theorem of metrization for separable spaces in general.

In 3. we shall consider a totally different problem, connected with 0-dimensional spaces; here we shall study the *space of quasicomponents* $Q(M)$, *corresponding with an arbitrary separable space* M. This space $Q(M)$ is always of dimension 0. Some properties and examples will be mentioned without comment. The more complicated proofs and an extensive discussion will be given on another occasion. Incidentally a so far unsolved problem of KURATOWSKI's, put in 1938 (comp. [2]), will be solved by the study of the space of quasicomponents.

2. *Definition and metrization of 0-dimensional separable spaces.* In a (non-vacuous) point-set N we consider a certain system $\{S\}$ of subsets $S \subset N$ — called *lump-sets* —: this system $\{S\}$ satisfies the conditions

1°. N itself and the vacuous set are contained in $\{S\}$;

2°. if $\{S\}$ contains S, then $\{S\}$ also contains $N - S$;

3°. if S_1 and S_2 belong to $\{S\}$, then the intersection $S_1 \cdot S_2$ and the sum $S_1 + S_2$ also belong to $\{S\}$.

If in N is defined a system of lump-sets $\{S\}$ and if every S is understood to be a neighbourhood of every point n belonging to S, then $N = N_{|S|}$ is called an 0-dimensional topological space. Moreover two 0-dimensional topological spaces $N_{|S_1|}$ and $N_{|S_2|}$, corresponding with a point-set N, are considered to be identical: $N_{|S_1|} \equiv N_{|S_2|} \equiv N$, if for any point $n \subset N$ and any S_1 with $n \subset S_1$ it is possible to find an S_2 with $n \subset S_2 \subset S_1$, and conversely (interchanging the indices 1 and 2).

Let $N_{|S|}$ be an 0-dimensional topological space and let $\{U\}$ be a system of subsets of N such that for any point $n \subset N$ and any S containing n may be found a U with $n \subset U \subset S$; and conversely, that for any $U \supset n$ may be found an S with $n \subset S \subset U$. In this case $\{U\}$ is called a *base* of $N_{|S|}$.

$\{S\}$ itself, but possibly also subsystems of $\{S\}$ are e.g. bases (*lump-bases*) of $N_{|S|}$.

Two spaces N and N' are called *topologically equivalent* if there exists a one to one mapping f of N on N': $f(N) = N'$, such that a base $\{S\}$ of N is mapped on a base $\{f(S)\}$ of N'.

An 0-dimensional topological space $N = N_{|S|}$ is called by definition an 0-*dimensional separable space,* if the following two conditions are satisfied:

(α) (HAUSDORFF *axiom*) For two different points n_1 and n_2 of N there exist disjunct lump-sets S_1 and S_2:

$$n_1 \subset S_1 \quad , \quad n_2 \subset S_2 \quad , \quad S_1 \cdot S_2 = 0$$

(β) (*axiom of countability*) Among the different bases of N there exists a *countable base* $\{S_i\}$ $(i = 1, 2, \ldots)$ [1].

Theorem. *An 0-dimensional separable space N is metrisable; in particular N is topologically equivalent with a subset of the discontinuum of* CANTOR [2].

Proof. Be $\{S_j\}$ $(j = 1, 2, \ldots)$ a countable base of N; moreover $\{S_j\}$ will be a sub-system of a system of lump-sets $\{S\}$, which defines N. Each point $n \subset N$ we map on one point $n' = f(n)$ of the real axis, defined by:

$$f(n) = n' = \sum_{i=1}^{\infty} \frac{\delta_i}{3^i} \qquad \begin{pmatrix} \delta_i = 0 & \text{if} & n \not\subset S_i \\ \delta_i = 2 & \text{if} & n \subset S_i \end{pmatrix} \quad . \quad . \quad . \quad . \quad (1)$$

The set N' of the points n' is apparently a subset of the discontinuum D. We shall prove that N' is topologically equivalent with N. $f(N) = N'$ is one to one, for, according to (α), for different points n_1 and n_2 exist disjunct lump-sets S_k and S_l such that n_1' and n_2' determine different points.

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[1]) It is easy to prove that in this case $\{S\}$ contains a countable sub-system which is a countable base (and therefore even a lumpbase) of $N_{|S|}$.

[2]) The discontinuum is, as known, the set of those real numbers of the interval [0,1] which may be expanded in the triadic number system using only the digits 0 and 2.

We shall prove that f is a topological mapping. For fixed arbitrary m all points of the shape

$$V_m = \sum_{i=1}^{m} \frac{\delta_i}{3^i} + \sum_{j=1}^{\infty} \frac{\varepsilon_j}{3^{m+j}} \qquad \begin{pmatrix} \delta_i = 0, \text{ if } n \notin S_i;\ \delta_i = 2, \text{ if } n \subset S_i \\ \varepsilon_j = 0, 2. \end{pmatrix} \quad (2)$$

form a neighbourhood $V_m = V_m(n')$ of n' in N'. The system $\{V_m\}$ $(m = 1, 2, \ldots)$ is apparently a system of neighbourhoods of n'. Let S_{a_i} $(i = 1, 2, \ldots, k)$, resp. S_{b_i} $(i = 1, 2, \ldots, l)$ be all those sets of S_i $(i = 1, 2, \ldots, m)$ which contain, resp. do not contain n. Because of the definition of f and because of (2) now apparently

$$f^{-1}(V_m) = \left[\prod_{i=1}^{k} S_{a_i}\right] \cdot \left[N - \sum_{i=1}^{l} S_{b_i}\right] \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If we put

$$f^{-1}(V_m) = U_m,$$

then, according to 2. 1°, 2° and 3°, $\{U_m\}$ is a sub-system of $\{S\}$. n has a system of neighbourhoods $\{S_{r_i}\}$ $(i = 1, 2, \ldots)$. Now obviously $n \subset U_{r_i} \subset S_{r_i}$ $(i = 1, 2, \ldots)$. From this follows that the countable system of *all* sets U_m corresponding with *all* points $n \subset N$ is a base of N. Therefore N and N' are topologically equivalent, q.e.d.

Remark 1. By the afore-said not only the metrization of the 0-dimensional separable spaces has been shown, but also the topological equivalence with a subset of the discontinuum, which fact was shown first — in another way — by SIERPIŃSKI.

Remark 2. It is obvious that our definition of 0-dimensional separable spaces, which differs from the usual one, indeed corresponds with the usually given definition (viz. as a separable space (i.e., a normal space with countable base, comp. [1]), where every point has arbitrarily small neighbourhoods with vacuous boundary). For this well-known definition one may give the same proof of metrization that we gave.

3. *Definition and properties of the space of quasicomponents $Q(M)$ of a separable space M.*

Let M be an arbitrary separable space and let $\{U\}$ be a (for instance countable) base of M, consisting of open sets (the term "base" in this case is used in the usual sense of the word to characterize M and must not be taken for the special base mentioned in 2.).

We consider a system of lump-sets $\{S\}$ of M, *whose topology is weaker than that of M*, i.e., for each point $m \subset M$ and each $S \supset m$ may be found a U of $\{U\}$ with $m \subset U \subset S$ (but not necessarily conversely). — There always exist such lump-systems $\{S\}$; for instance the system consisting of M and the vacuous set is such a lump-system. — Any S of such a $\{S\}$ is q.q. a set open in M; therefore, since $M - S$ also belongs to $\{S\}$, $M - S$

is also open and therefore S is likewise closed, in other words: *the lump-systems* $\{S\}$, *whose topology is weaker than that of* M, *consist of* (*in* M) *both open and closed sets.*

In the following we shall consider only such lump-systems and may call an (in M) both open and closed set a *lump-set* by definition. — From 2. we already know when to call two lump-systems $\{S\}$ and $\{S'\}$ topologically equivalent. — The topology of $\{S\}$ is *weaker* than that of $\{S'\}$ (or that of $\{S'\}$ is stronger than that of $\{S\}$), if for each point $m \subset M$ and each $S \supset m$ may be found an S' with $m \subset S' \subset S$, but not necessarily conversely. — A lump-system $\{S^*\}$ which is stronger or just as strong (topologically equivalent) as any other lump-system (but weaker than the base $\{U\}$ of M) consists of all lump-sets of M, or of a sub-system of $\{S^*\}$ which is topologically equivalent with $\{S^*\}$. If $\{U\}$ is not stronger, but topologically equivalent with $\{S^*\}$, M is 0-dimensional (and conversely). In the other extreme case that $\{S^*\}$ consists only of M and the vacuous set, M is connected. $\{S^*\}$ leads to a new space $Q(M)$: two points of M are in the same class only if both of them either belong or do not belong to S^* for any $S^* \subset \{S^*\}$. Thus the points of M are divided into disjunct classes. These classes Q are called *quasicomponents* (HAUSDORFF [4] [3]) and are taken as points q of the space of quasicomponents $Q(M)$. The set $Q(M)$ of the points q becomes a space by considering the system $\{S^*\}$ as a system of neighbourhoods of $Q(M)$ (while at the same time one has to consider the sets Q as points q). It is easy to prove that $Q(M)$ *is always an 0-dimensional regular space* (regular in the sense of [1]). Further *the 0-dimensional separable spaces* M *are exactly those separable spaces which are identical with their spaces of quasicomponents* $Q(M)$.

For $Q(M)$ to be separable it is obviously only necessary that $Q(M)$ has a countable base, in other words $\{S^*\}$ contains a countable sub-system which is at the same time a base of $Q(M)$. We shall now, by means of a general example, show that such a countable base of $Q(M)$ does not always exist.

M by definition is called *totally disconnected,* if the quasicomponents Q considered as subsets of M, all consist of one point. This does not necessarily, however, infer that M is of dimension 0, since the base $\{U\}$ of M may be stronger than the base $\{S^*\}$ of $Q(M)$. There exist (MAZURKIEWICZ [5], SIERPIŃSKI [3]) totally disconnected n-dimensional ($n \geqq 0$) separable spaces M. Now the following theorem holds true.

Theorem. *Any totally disconnected* n*-dimensional* ($n > 0$) *separable space* M *has a space of quasicomponents* $Q(M)$ *without a countable base.*

Proof. Suppose $Q(M)$ did have a countable base; then the system $\{S^*\}$, defining $Q(M)$, would satisfy 2. (α) and (β) ((α) because $Q(M)$ is regular) and $Q(M)$ would, according to the theorem and remark 2 of 2.,

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[3]) The quasicomponent $Q \subset M$, corresponding with a point a of M, may be shortly defined as the intersection of all those both open and closed subsets of M, which contain a.

be an 0-dimensional separable metrisable space (in the usual sense of the word), contrary to the given dimension n.

Remark. By this theorem we have solved a problem put by KURATOWSKI in 1938 and communicated by SIERPIŃSKI [2]. This problem ran as follows: is it possible for any separable metric space M to find a countable number of both open and closed subsets of M such that any both open and closed subset of M is the finite or infinite sum of a suitably chosen number of the first-mentioned subsets? It is easy to see that this question is identical with the following one: has any separable M a $Q(M)$ with countable base? It follows from our last mentioned theorem that this question must be answered in the negative.

Now the following further question arises: when *is* $Q(M)$ separable, in other words, when does $Q(M)$ have a countable base? To me this question seems to present great difficulties.

I am, however, able to answer the following question: *when is* $Q(M)$ *a compact separable space*? For this the following condition is necessary and sufficient: the intersection of any decreasing sequence of lump-sets of M

$$S_1 \supset S_2 \supset S_3 \ldots$$

has a non-vacuous intersection in M. This expresses nothing else than the compactness of $Q(M)$. Thus we have the following

Theorem. *A separable space* M *has a compact 0-dimensional separable space* $Q(M)$ *of quasicomponents, only if* $Q(M)$ *is compact.*

The rather complicated proof of this theorem will be given on another occasion. Special cases of this theorem (for which were given sufficient conditions that were, however, not necessary) were already known: for compact separable M the theorem has been proved by L. E. J. BROUWER [6] (for compact M the space of components and the space of quasicomponents coincide), whereas in FREUDENTHAL [7], and in [8] the compactness of M is replaced by the double, weaker, condition: M semicompact and $Q(M)$ compact. It is now clear from our theorem that it is unnecessary to ask that M be semicompact.

REFERENCES.

1. P. ALEXANDROFF und H. HOPF, Topologie I, Berlin (1935).
2. W. SIERPIŃSKI, Fund. Math. **30**, 129—131 (1938).
3. ———, Fund. Math. **2**, 81 (1921).
4. F. HAUSDORFF, Grundzüge der Mengenlehre, First Edition, Leipzig (1914).
5. ST. MAZURKIEWICZ, Fund. Math. **10**, 311 (1927).
6. L. E. J. BROUWER, Versl. Kon. Akad. Amsterdam, March 26, (1910).
7. H. FREUDENTHAL, Neuaufbau der Endentheorie, (1941).
8. J. DE GROOT, Top. Studiën, Thesis, Groningen (1942).

Mathematics. — *Sur les systèmes d'équations aux dérivées partielles, qui, comme les systèmes normaux, comportent autant d'équations que de fonctions inconnues* (première communication). By A. FINZI. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of December 21, 1946.)

Introduction.

1.

Une classe de systèmes d'équations aux dérivées partielles pour lesquels on connaît depuis très longtemps un théorème d'existence valable dans le champ analytique est celle des systèmes qui

a) comportent autant d'équations que de fonctions inconnues

b) satisfont à une condition de résolubilité par rapport à certaines de leurs dérivées de l'ordre le plus élevé, qui sera précisée au N. 3.

Pour de tels systèmes, que, selon une dénomination adoptée par plusieurs auteurs, nous appellerons systèmes normaux, le théorème de CAUCHY–KOWALEVSKY est valable.

Soit

$$F_i = 0 \qquad (i = 1, 2, \ldots n)$$

un système normal d'ordre h, de n fonctions inconnues φ_1, φ_2 ... φ_n de $m + 1$ variables x^0, x^1 ... x^m.

Le théorème de CAUCHY–KOWALEVSKY nous assure l'existence d'une solution et d'une seule lorsqu'on connaît, sur une hypersurface de l'espace des variables x, les inconnues et leurs dérivées jusqu'à l'ordre $h - 1$.

Le théorème n'est plus valable relativement à des choix particuliers de l'hypersurface et des données initiales (hypersurface caractéristique). Dans ce cas le nombre des dérivées, qui peuvent être données arbitrairement, diminue, tandis que l'unicité de la solution n'est plus assurée, en général.

La considération des hypersurfaces caractéristiques est d'une très grande importance dans l'étude de la propagation des ondes [1]).

Des systèmes plus généraux que les systèmes normaux ont été étudiés par MERAY, DELASSUS, RIQUIER, GUNTHER, JANET [2]).

Il est bien connu, d'autre part, qu'on doit à M. CARTAN [3]) une théorie générale des systèmes d'équations de PFAFF et l'on sait que tout système

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[1]) LEVI-CIVITA, Caratteristiche dei sistemi differenziali a propagazione ondosa, Bologna 1931 (ou, en traduction française par M.M. Brelot, Paris, Alcan, 1932).

[2]) On peut trouver une exposition de ces théories et une vaste bibliographie dans l'article de M. JANET: Les systèmes d'équations aux dérivées partielles. Mémorial des sciences math. 1927.

[3]) E. CARTAN, Ann. Ec. Norm. 1901 t. 18 et 1904 t. 21.

d'équations aux dérivées partielles peut se mettre sous la forme d'un système de PFAFF.

Toutes ces recherches s'appliquent à des systèmes à un nombre quelconque d'équations et d'inconnues et, dans ce mémoire, je me suis proposé d'étudier le cas, particulièrement remarquable, de systèmes qui comprennent autant d'équations que de fonctions inconnues.

Je dois citer, à ce propos, une note de M. HADAMARD [4]) et aussi un mémoire de M. JANET [5]), dont toutefois je n'ai eu connaissance qu'après avoir obtenu les résultats que j'expose dans mon mémoire.

Dans son mémoire M. JANET donne, pour les systèmes du premier ordre, satisfaisant à une certaine autre condition, les théorèmes démontrés ici au § I, II, XI.

2.

Etant donné un système (1) non normal à autant d'équations que d'inconnues d'ordre h, je démontre avant tout qu'on peut construire un nouveau système (1^1) d'ordre $h^{(1)}$ (généralement plus élevé que h) qui est vérifié par toute solution de (1).

Réciproquement, si le système (1^1) est normal, ses solutions vérifient le système (1), si toutefois les valeurs initiales des φ et de leurs $h^{(1)}-1$ premières dérivées satisfont à certaines conditions convenables.

Si (1^1) lui aussi n'est pas normal, nous pouvons de nouveau procéder de la même manière et construire ainsi successivement de nouveaux systèmes d'ordre $h^{(2)}$, $h^{(3)}$, ...; supposons donc que nous avons abouti, après l fois, à construire un système normal d'ordre $h^{(l)}$.

Ce dernier sera encore vérifié par toute solution des (1) et, d'autre part, on montre que si l'on impose à ses valeurs initiales (c.à.d. à celles des φ et de leurs premières $h^{(l)}-1$ dérivées) $(h^{(l)}-h)n+l$ conditions convenablement choisies, ses solutions vérifient aussi le système donné (1).

Les caractéristiques de (1) sont ensuite les mêmes, en général, que celles du système normal.

Les systèmes de ce type ont donc, du point de vue de l'existence des solutions, une manière de se comporter assez semblable aux systèmes normaux, parce qu'une de leurs solutions est, en général, déterminée lorsque sont données, sur la variété initiale, les φ et certaines de leurs dérivées.

Considérons maintenant, au contraire, le cas où, en répétant le procédé indiqué, on n'arrive pas à un système normal: j'ai démontré au § VI que si les hn premiers systèmes trouvés ne sont pas normaux, nécessairement les F et leurs dérivées sont liées par une identité

$$\Psi(F_1, F_2, \ldots, F_n, \ldots x) = 0.$$

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[4]) J. HADAMARD, Bulletin Soc. Math. de France 1906 p. 48.

[5]) M. JANET, Journal de math. p. 339, 1929.

Dans un tel cas le théorème d'existence a un caractère totalement différent: on pourra assigner, sur la variété initiale, les φ et leurs premières $h-1$ dérivées, assujetties à des conditions convenablement choisies, mais la solution ne sera plus déterminée et on pourra lui imposer dans tout le domaine d'existence une condition arbitraire d'ordre h par rapport aux φ

$$Z=0.$$

De ce point de vue, le système donné peut donc être comparé à un système de $n-1$ équations à n inconnues.

Revenant ensuite aux systèmes du premier type, je donne au § IX une condition, sous laquelle, pour déterminer la solution, il n'est plus nécessaire de connaître, sur la variété initiale, les φ et leurs premières $h^{(l)}-1$ dérivées, mais il suffit, comme dans le cas normal, de connaître, outre les φ, leurs $h-1$ premières dérivées: à cette condition satisfont en particulier les systèmes d'équations linéaires par rapport aux inconnues et leurs dérivées partielles et ceux pour lesquels $l=1$.

A titre d'exemple j'ai étudié dans le cours de ce mémoire deux systèmes particuliers non normaux: les équations de l'applicabilité des surfaces et celles de la relativité généralisée.

J'ai obtenu ainsi, pour ces dernières équations, une démonstration générale de l'existence des solutions, que je crois la première donnée jusqu'ici.

3.

Le théorème de CAUCHY–KOWALEVSKY et la théorie des caractéristiques doivent être regardés comme classiques. Je vais toutefois les rappeler rapidement au *n.* 3, 4, 5 afin d'introduire les notations dont je ferai usage dans la suite de mon mémoire.

Si nous supposons que les dérivées de l'ordre le plus élevé interviennent linéairement un système de n équations $F_1, F_2 \ldots, F_n$ d'ordre h, de n fonctions inconnues $\varphi_1, \varphi_1, \ldots, \varphi_n$ de $m+1$ variables indépendantes $x^0, x^1, \ldots, x^m$, est du type

$$\left.\begin{aligned} F_l \equiv \Sigma E^{lj}_{h_0 h_1 \ldots h_m} \frac{\partial^h \varphi_j}{\partial x^{0^{h_0}} \partial x^{1^{h_1}} \ldots \partial x^{m^{h_m}}} + M_l = 0 \\ (h_0 + h_1 + \ldots = h_m = h) \end{aligned}\right\} \qquad (1)$$

les E et les M étant des expressions qui ne contiennent pas de dérivées d'ordre h.

On dit que le système est normal par rapport à x^0 s'il peut être résolu par rapport aux $\frac{\partial^h \varphi_j}{\partial x^{0^h}}$, c'est à dire s'il peut être mis sous la forme

$$\frac{\partial^h \varphi_j}{\partial x^{0^h}} = P(\varphi, \psi, x) \qquad (2)$$

où l'on désigne par ψ les dérivées des φ dont l'ordre, par rapport à x^0, est inférieur à h.

La condition nécessaire pour que le système soit normal, est que le déterminant

$$\| E^{ij}_{h00\ldots0} \|$$

soit différent de zéro.

Dans une telle hypothèse supposons données les hn fonctions de $x^1, x^0, \ldots x^m$ auxquelles se réduisent sur l'hyperplan

$$x^0 = a \tag{3}$$

les φ et leurs premières $h-1$ dérivées.

Nous pourrons alors calculer, sur $x^0 = a$, par de simples dérivations, toutes les quantités qui interviennent dans les seconds membres des équations (2) et les $\frac{\partial^h \varphi_j}{\partial x^{0h}}$ seront par conséquent déterminées.

En dérivant ensuite les (2) par rapport à x^0 nous pourrons obtenir les $\frac{\partial^{h+1} \varphi_j}{\partial x^{0h+1}}$ en fonction des dérivées d'un ordre, par rapport à x^0, ne dépassant pas h. Nous pourrons calculer ensuite de la même manière les dérivées d'un ordre aussi élevé que nous voudrons et construire ainsi les développements en séries des fonctions φ.

Il a été démontré que ces développements ont certainement un rayon de convergence différent de zéro et fournissent ainsi une solution des équations (1).

Il en résulte donc que, si les fonctions φ et leurs premières $h-1$ dérivées sont données sur l'hyperplan (3), le système considéré admet une solution analytique et une seule.

La proposition d'existence dont nous venons d'indiques la démonstration, constitue précisément le célèbre théorème de CAUCHY–KOWALEVSKY.

4.

Lorsque

$$\| E^{ij}_{h00\ldots0} \| = 0 \tag{4}$$

le théorème de CAUCHY–KOWALEVSKY ne peut plus être appliqué: on dit dans ce cas, que l'hyperplan $x^0 = a$ est une caractéristique du système donné.

Pour simplifier le problème nous supposerons que les mineurs d'ordre $n-1$ du déterminant (4) ne s'annulent pas au même point de l'hyperplan $x = a$.

Désignons par $\Delta^{ij}_{h(n-1)00\ldots0}$ les compléments algébriques des $E^{ij}_{h00\ldots0}$[6]). D'après l'hypothèse que nous avons faite, dans la combinaison

$$\Gamma = \Sigma_i \, \Delta^{i1}_{h(n-1)00\ldots0} \, F_i = 0 \tag{5}$$

[6]) Nous comprendrons plus loin la raison de cette notation.

les $\frac{\partial^h \varphi_j}{\partial x^{0^h}}$ seront éliminées: nous ne pourrons donc plus donner sur l'hyperplan $x^0 = a$ les φ et leurs premières $h-1$ dérivées d'une manière complétement arbitraire, puisque (5) constitue une condition qui doit être satisfaite.

On voit immédiatement que réciproquement, lorsque cette condition est satisfaite, la solution n'est pas déterminée d'une manière univoque.

En effet on a seulement $n-1$ équations indépendantes pour déterminer les $\frac{\partial^h \varphi_j}{\partial x^{0^h}}$. Supposons qu'il en soit ainsi des équations $F_2 = 0$, $F_3 = 0$, $F_4 = 0, \ldots, F_n = 0$, nous pourrons leur associer l'équation

$$\frac{\partial \Gamma}{\partial x^0} = 0.$$

Celle ci contiendra en général, outre les $\frac{\partial^h \varphi_j}{\partial x^{0^h}}$, leurs dérivées premières per rapport à $x^1, x^2, \ldots x^m$, et cela introduira des éléments arbitraires dans le calcul des $\frac{\partial^h \varphi_j}{\partial x^{0^h}}$.

De la même manière on pourra déduire des équations

$$\frac{\partial F_2}{\partial x^0} = 0, \quad \frac{\partial F_3}{\partial x^0} = 0, \ldots, \quad \frac{\partial F_n}{\partial x^0} = 0, \quad \frac{\partial^2 \Gamma}{\partial x^{0^2}} = 0$$

les $\frac{\partial^{h+1} \varphi_i}{\partial \varphi^{0^{h+1}}}$ en introduisant de nouveaux éléments arbitraires.

Des considérations analogues sont valables pour le calcul des dérivées successives.

On peut dire que, dans le cas où l'hyperplan (3) est une caractéristique, deux solutions distinctes du système peuvent avoir, le long de cet hyperplan, un contact d'ordre fini aussi élevé qu'on veut.

Il faut noter enfin que dans le cas actuel, la convergence des développements en série ainsi construits n'est plus assurée a priori: SOPHIE KOWALEVSKY elle-même en a donné des exemples [7]).

Dans le cas où tous les mineurs du déterminant (4) d'ordre supérieur à $n-t$ s'annulent simultanément sur l'hyperplan (3), les considérations du même ordre, qui peuvent être développées, sont seulement un peu plus compliquées.

Nous verrons aux paragraphes suivants l'importance du cas où un certain déterminant Ω, qui généralise le déterminant (4), s'annule identiquement d'une manière que nous préciserons.

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[7]) Journal f. r. u. angew. Math. 80 (1875), 22.

5.

Il est naturel de chercher si l'on peut choisir les données initiales sur une hypersurface quelconque au lieu de les choisir sur l'hyperplan (3). Soit

$$z^0(x^0, x^1, \ldots, x^m) = 0 \tag{6}$$

l'équation de cette hypersurface.

Cette importante généralisation du théorème d'existence peut être obtenue de la façon suivante:

Associons à z^0 m autres fonctions indépendantes $z^1, z^2, \ldots, z^m$ et effectuons la transformation

$$\begin{pmatrix} x^0, x^1, \ldots x^m \\ z^0, z^1, \ldots z^m \end{pmatrix}$$

qui changera le système par rapport aux variables x en un système par rapport aux variables z et l'hypersurface (6) en l'hyperplan

$$z^0 = 0.$$

Si le système transformé est normal par rapport à z^0, le théorème du numéro 3 nous assure qu'il possède une solution et une seule correspondant aux valeurs initiales données sur (6).

Nous avons donc à exprimer la condition de normalité par rapport à z^0. Posons

$$\frac{\partial z^0}{\partial x^i} = p_i.$$

On peut écrire

$$\frac{\partial^h \varphi_j}{\partial x^{0^{h_0}} \partial x^{1^{h_1}} \ldots \partial x^{m^{h_m}}} = \frac{\partial^h \varphi_j}{\partial z^{0^h}} p_0^{h_0} p_1^{h_1} \ldots p_m^{h_m} + \ldots$$

en mettant en évidence seulement les dérivées par rapport à z^0 d'ordre h. Le coefficient de $\dfrac{\partial^h \varphi_j}{\partial z^{0^h}}$ dans la i-ème équation du système transformé sera donc

$$\Sigma_{h_0 h_1 \ldots h_m} E^{ij}_{h_0 h_1 \ldots h_m} p_0^{h_0} p_1^{h_1} \ldots p_m^{h_m}$$

et la condition de normalité sera par suite exprimée par

$$\Omega = \| \Sigma_{h_0 h_1 \ldots h_m} E^{ij}_{h_0 h_1 \ldots h_m} p_0^{h_0} p_1^{h_1} \ldots p_m^{h_m} \| \neq 0.$$

Quand, au contraire, en correspondance aux valeurs initiales données, la (6) satisfait à l'équation

$$\Omega = 0$$

(équation des surfaces caractéristiques) on obtient des résultats analogues à ceux établis au numéro 4.

Les considérations de ces paragraphes peuvent être facilement étendues au cas où le système n'est pas linéaire par rapport aux dérivées d'ordre le plus élevé h: si l'on désigne alors plus généralement par $E^{i\,j}_{h_0 h_1 \ldots h_m}$ au lieu simplement du coefficient de $\dfrac{\partial^h \varphi_j}{\partial x^{0 h_0} \partial x^{1 h_1} \ldots \partial x^{m h_m}}$ dans F_i, la dérivée de F_i par rapport à $\dfrac{\partial^h \varphi_j}{\partial x^{0 h_0} \partial x^{1 h_1} \ldots \partial x^{m h_m}}$, le déterminant Ω, qui exprime, lorsqu'il n'est pas nul, la condition de normalité, conserve son aspect formel.

Mathematics. — *Sur les systèmes d'équations aux dérivées partielles, qui, comme les systèmes normaux, comportent autant d'équations que de fonctions inconnues* (deuxième communication). By A. FINZI. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of December 21, 1946.)

§ I.

Considérations préliminaires. — Une propriété générale.

Supposons que le système donné

$$F_i = 0$$

ne soit pas normal (quelque soit l'hypersurface $z^0(x^0, x^1 \ldots x^m) = 0$).

En vertu de l'observation faite à la fin du numéro 5 il n'est pas nécessaire de supposer, dans les considérations qui vont suivre, que les dérivées des inconnues d'ordre h interviennent linéairement.

En conséquence de l'hypothèse que nous venons de faire, le déterminant Ω relatif à un tel système

$$\begin{vmatrix} \Sigma E^{11}_{h_0 h_1 \ldots h_m} p_0^{h_0} p_1^{h_1} \ldots p_m^{h_m} \ldots \Sigma E^{1n}_{h_0 h_1 \ldots h_m} p_0^{h_0} p_1^{h_1} \ldots p_m^{h_m} \\ \Sigma E^{21}_{h_0 h_1 \ldots h_m} p_0^{h_0} p_1^{h_1} \ldots p_m^{h_m} \ldots \Sigma E^{2n}_{h_0 h_1 \ldots h_m} p_0^{h_0} p_1^{h_1} \ldots p_m^{h_m} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \Sigma E^{n1}_{h_0 h_1 \ldots h_m} p_0^{h_0} p_1^{h_1} \ldots p_m^{h_m} \ldots \Sigma E^{nn}_{h_0 h_1 \ldots h_m} p_0^{h_0} p_1^{h_1} \ldots p_m^{h_m} \end{vmatrix}$$

doit s'annuler identiquement par rapport à tous les arguments p dont il dépend.

Le développement de Ω montre qu'il s'agit d'une forme homogène dans les p de degré nh dont les coefficients doivent être identiquement nuls par rapport aux x et éventuellement (dans le cas non linéaire) par rapport aux φ et à leurs dérivées.

Supposons (le cas contraire sera examiné au paragraphe XI) que les mineurs d'ordre $n-1$ de Ω ne sont pas tous identiquement nuls.

Le complément algébrique de l'élément de Ω, placé dans la i-ième ligne et j-ième colonne est à son tour une forme homogène par rapport aux p de degré $h(n-1)$: désignons par

$$\Delta^{ij}_{r_0 r_1 \ldots r_m} \quad (r_0 + r_1 + \ldots + r_m = (n-1)h)$$

le coefficient de $p_0^{r_0} p_1^{r_1} \ldots p_m^{r_m}$.

Dans le déterminant Ω le coefficient de $p_0^{s_0} p_1^{s_1} \ldots p_m^{s_m}$ (avec

$s_0 + s_1 + \ldots + s_m = hn$) est donné, comme on le voit immédiatement en développant selon les éléments de la première colonne, par

$$\Sigma E^{i1}_{h_0 h_1 \ldots h_m} \Delta^{i1}_{r_0 r_1 \ldots r_m}$$

où la sommation doit être étendue à toutes les valeurs des h et des r pour lesquelles

$$h_0 + r_0 = s_0 ; h_1 + r_1 = s_1 ; \ldots ; h_m + r_m = s_m \tag{7}$$

alors que l'indice i varie de 1 à n.

On devra donc avoir

$$\Sigma E^{i1}_{h_0 h_1 \ldots h_m} \Delta^{i1}_{r_0 r_1 \ldots r_m} = 0.$$

Démontrons maintenant pour le système (1) la proposition suivante: *Il existe une combinaison linéaire entre les dérivées d'ordre $h(n-1)$ des équations du système dans laquelle les coefficients des dérivées des inconnues φ d'ordre le plus élevé hn sont identiquement nuls.*

La combinaison qui jouit de la propriété annoncée est la suivante:

$$\Phi_1 = \Sigma \Delta^{i1}_{r_0 r_1 \ldots r_m} \frac{\partial^{h(n-1)} F_i}{\partial x^{0 r_0} \partial x^{1 r_1} \ldots \partial x^{m r_m}} = 0 \quad (r_0 + r_1 + \ldots + r_m = (n-1)h). \tag{8}$$

Pour démontrer notre proposition, ayant posé encore

$$h_0 + r_0 = s_0 ; h_1 + r_1 = s_1 ; \ldots ; h_m + r_m = s_m, \tag{7}$$

notons en premier lieu que le coefficient d'une dérivée de l'ordre le plus élevé $\dfrac{\partial^{hn} \varphi_j}{\partial x^{0 s_0} \partial x^{1 s_1} \ldots \partial x^{m s_m}}$ dans l'équation

$$\frac{\partial^{h(n-1)} F_i}{\partial x^{0 r_0} \partial x^{1 r_1} \ldots \partial x^{m r_m}} = 0$$

est égal au coefficient de $\dfrac{\partial^h \varphi_j}{\partial x^{0 h_0} \partial x^{1 h_1} \ldots \partial x^{m h_m}}$ en $F_i = 0$, c'est à dire à $E^{ij}_{h_0 h_1 \ldots h_m}$.

Il en résulte que le coefficient de $\dfrac{\partial^{hn} \varphi_j}{\partial x^{0 s_0} \partial x^{1 s_1} \ldots \partial x^{m s_m}}$ dans l'équation (8) est donné par

$$\Sigma E^{ij}_{h_0 h_1 \ldots h_m} \Delta^{i1}_{r_0 r_1 \ldots r_m},$$

la sommation étant étendue toujours à toutes les valeurs des h et des r qui vérifient (7), alors que l'indice i varie de 1 à n.

Si $j = 1$ une telle expression coïncide avec le coefficient de $p_0^{s_0} p_1^{s_1} \ldots p_m^{s_m}$ dans le développement de Ω, et par suite elle est identiquement nulle.

Nous pouvons donc assurer que, dans la combinaison considérée, les coefficients des dérivées de l'ordre le plus élevé de φ_1 sont nuls.

Montrons maintenant que les coefficients des dérivées de φ_j $(j = 2, 3, \ldots, n)$ sont aussi nuls.

Le coefficient de $\dfrac{\partial^{hn}\varphi_j}{\partial x^{0^{s_0}}\partial x^{1^{s_1}}\ldots\partial x^{m^{s_m}}}$ s'obtient en substituant $E^{ij}_{h_0h_1\ldots h_m}$ à $E^{i1}_{h_0h_1\ldots h_m}$ dans celui de $\dfrac{\partial^{hn}\varphi_1}{\partial x^{0^{s_0}}\partial x^{1^{s_1}}\ldots\partial x^{m^{s_m}}}$ c.à.d. dans

$$\Sigma E^{i1}_{h_0h_1\ldots h_m}\Delta^{i1}_{r_0r_1\ldots r_m}.$$

Il est donc égal en général au coefficient de $p_0^{s_0}p_1^{s_1}\ldots p_m^{s_m}$ dans le développement du déterminant qui s'obtient à partir de Ω en substituant à la première colonne la j-ième.

Mais un tel déterminant aura deux colonnes égales et sera par conséquent identiquement nul; tous les coefficients de la forme dans le p de degré hn obtenue en le développant, seront donc nuls.

Notre proposition est donc complètement démontrée.

Il serait facile de démontrer que, inversement, s'il existe une combinaison linéaire entre les dérivées des équations d'un système dans laquelle les coefficients des dérivées des inconnues de l'ordre le plus élevé sont nuls, le système ne peut pas être normal.

§ II.

Existence des solutions pour une première classe de systèmes non normaux.

Supposons, ce qui est certainement permis, que la caractéristique de la matrice, qui s'obtient en supprimant en Ω la première ligne, soit, en général, pour des valeurs données des p, égale à $n-1$.

Associons à $\Phi_1 = 0$ les dérivées d'ordre $h(n-1)-1$ de $F_2 = 0$, $F_3 = 0, \ldots, F_n = 0$ par rapport à une des variables, par exemple x^0: nous obtenons ainsi un nouveau système de n équations d'ordre $hn-1 = h^{(1)}$

$$\left.\begin{array}{c}\dfrac{\partial^{h^{(1)}-h}F_i}{\partial x^{0^{h^{(1)}-h}}} = 0 \qquad (i = 2, 3, \ldots, n),\\ \Phi_1 = 0\end{array}\right\}. \tag{1^1}$$

Le déterminant $\Omega^{(1)}$ relatif au système (1^1) aura $n-1$ de ses lignes proportionelles à celles de Ω.

Si ce système est normal, une de ses solutions sera déterminée quand on se donnera les $h^{(1)}n$ fonctions, auxquelles se réduisent, sur $x^0 = a$, les φ et leurs premières $h^{(1)}-1$ dérivées par rapport à x^0.

Imposons à ces fonctions la condition de rendre nulles, sur $x^0 = a$, F_2, $F_3, \ldots, F_n$ ainsi que leurs premières $h^{(1)}-h-1$ dérivées.

En vertu des premières $n-1$ équations (1^1) on aura alors, quelque soit x^0,

$$F_i = 0 \qquad (i = 2, 3, \ldots, n),$$

et la dernière des (1^1) sera donc équivalente à une équation différentielle homogène en F_1 d'ordre $h^{(1)} - h + 1$.

Imposons aux mêmes fonctions la condition de rendre nulles F_1 et ses premières $h^{(1)} - h$ dérivées sur $x^0 = a$.

En vertu de la dernière des (1^1), on aura alors, quelque soit x^0,

$$F_1 = 0,$$

et la solution de (1^1) vérifiera aussi le système donné (1).

Il est évident que, réciproquement, chaque solution de (1) vérifiera aussi (1^1). Ce raisonnement serait en défaut si le coefficient de $\dfrac{\partial^{h^{(1)}-h+1} F_1}{\partial x^{0\,h^{(1)}-h+1}}$ dans Φ_1, qui est donné par $\Delta^{11}_{h(n-1)00\ldots0}$, était nul.

Mais nous pourrons, dans une telle hypothèse, chercher à être ramenés au cas où le coefficient est différent de zéro en substituant dans les calculs à F_1 une autre des F_i ou aussi en adoptant comme coefficient dans (8) $\Delta^{ij}_{r_0 r_1 \ldots r_m}$ au lieu de $\Delta^{11}_{r_0 r_1 \ldots r_m}$.

Si en opérant ainsi en n'arrive pas au but, c'est que nécessairement

$$\Delta^{ij}_{h(n-1)00\ldots0} = 0 \qquad (i, j = 1, 2, \ldots, n).$$

Dans ce cas la caractéristique de Ω, en correspondant à l'hyperplan portant les données.

$$x^0 = a, \tag{3}$$

est inférieure à $n-1$, et puisque $\Omega^{(1)}$ a $n-1$ lignes proportionelles à celles de Ω, l'hyperplan (3) devra être une caractéristique pour le système (1^1).

Ce cas étant exclu, nous pourrons donc supposer

$$\Delta^{11}_{h(n-1)00\ldots0} \neq 0.$$

Les considérations précédentes s'étendent facilement au cas, où la surface portante est une variété quelconque

$$z^0(x^0, x^1, \ldots, x^m) = 0. \tag{6}$$

§ III.

Hypersurfaces caractéristiques pour les systèmes considérés au paragraphe précédent.

Nous avons vu que quand les valeurs initiales des inconnues et de leurs premières $h^{(1)} - 1$ dérivées satisfont à $(h^{(1)} - h)n + 1$ conditions convenables, la solution du système normal (1^1) vérifie aussi le système (1).

Du fait que d'autre part chaque solution de (1) vérifie le système (1^1) on peut conclure que le système donné admet en général une et une seule

solution quand les valeurs initiales des φ et de leurs premières $h^{(1)} - 1$ dérivées sont données et satisfont aux conditions indiquées.

Au § 9 nous montrerons que, en réalité, pour déterminer la solution des (1) il suffit de donner seulement, comme dans le cas d'un système normal, les φ et leurs premières $h - 1$ dérivées assujetties cependant à la condition nécessaire (5).

Nous pouvons dire que dans le cas actuel, comme dans le cas d'un système normal, la solution est déterminée d'une manière univoque, si l'on connaît sur une hypersurface, les inconnues et certaines de leurs dérivées.

En outre, lorsque, correspondant à certaines valeurs initiales, on a construit un système de développements en séries entières qui satisfait formellement à (1), on peut affirmer que les développements convergent et fournissent par conséquent la solution du système. En effet tels développements satisfont aussi le système normal (1^1) et par conséquent, d'après le théorème de CAUCHY–KOWALEVSKY, doivent posséder un rayon de convergence différent de zéro.

Voyons maintenant comment on peut étendre au cas actuel le concept de surface caractéristique.

Nous supposons que, en correspondance aux valeurs initiales satisfaisantes les $(h^{(1)} - h)n + 1$ conditions indiquées, la surface portante (6) est caractéristique pour le système (1^1).

Ces valeurs initiales devront aussi vérifier la condition nécessaire qui s'obtient en éliminant entre les équations (1^1) les dérivées des φ par rapport à z^0 d'ordre $h^{(1)}$; il y aura alors des solutions distinctes de (1^1) ayant sur (6) un contact d'ordre aussi élevé qu'on voudra. Si l'on n'a pas

$$\Sigma A^{11}_{r_0 r_1 \dots r_m} p_0^{r_0} p_1^{r_1} \dots p_m^{r_m} = 0$$

ces solutions devront aussi vérifier les (1).

Les caractéristiques du système (1^1), obtenu par la méthode indiquée, fournissent donc en général, pour le système non normal (1), l'extension du concept de surface caractéristique d'un système normal.

Correspondant à telles hypersurfaces, les valeurs initiales, c'est-à-dire les φ et leurs premières $h^{(1)} - 1$ dérivées, doivent être assujetties à $(h^{(1)} - h)n + 2$ conditions et, d'autre part, la connaissance de ces valeurs ne détermine plus d'une manière univoque la solution.

§ IV.

Exemple.

Considérons à titre d'exemple un système non normal de deux équations linéaires du premier ordre à deux fonctions inconnues de deux variables indépendantes x^0 et x^1

$$\left.\begin{aligned} a_1 \frac{\partial u}{\partial x^0} + b_1 \frac{\partial u}{\partial x^1} + c_1 \frac{\partial v}{\partial x^0} + d_1 \frac{\partial v}{\partial x^1} + \dots = 0 \\ a_2 \frac{\partial u}{\partial x^0} + b_2 \frac{\partial u}{\partial x^1} + c_2 \frac{\partial v}{\partial x^0} + d_2 \frac{\partial v}{\partial x^1} + \dots = 0 \end{aligned}\right\} \qquad (9)$$

où nous n'avons pas indiqué les termes qui ne contiennent pas de dérivation.

Comme on doit avoir identiquement

$$\Omega = \begin{vmatrix} a_1 p_0 + b_1 p_1 & c_1 p_0 + d_1 p_1 \\ a_2 p_0 + b_2 p_1 & c_2 p_0 + d_2 p_2 \end{vmatrix} = 0,$$

on trouve les relations

$$\left.\begin{aligned} &a_1 c_2 - a_2 c_1 = 0, \\ &a_1 d_2 - a_2 d_1 + b_1 c_2 - b_2 c_1 = 0, \\ &b_1 d_2 - b_2 d_1 = 0 \end{aligned}\right\}. \tag{10}$$

Les (10) étant vérifées on aura ou $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{d_1}{d_2}$ — dans ce cas la première ligne de Ω est identiquement proportionelle à la seconde, et le système donné, s'il n'est pas incompatible, est équivalent à une équation différentielle et à une relation en termes finis — ou — et c'est le cas le plus remarquable — $\frac{a_1}{c_1} = \frac{a_2}{c_2} = \frac{b_1}{d_1} = \frac{b_2}{d_2}$ — et alors la première colonne de Ω est identiquement proportionelle à la seconde.

Dans la relation

$$c_2 F_1 - c_1 F_2 = 0 \tag{11}$$

les dérivées de u et v par rapport à x^0 sont éliminées: (11) constitue donc une condition nécessaire pour les valeurs de u et de v données sur $x^0 = a$. Dans l'équation

$$c_2 \frac{\partial F_1}{\partial x^0} + d_2 \frac{\partial F_1}{\partial x^1} - c_1 \frac{\partial F_2}{\partial x^0} - d_1 \frac{\partial F_2}{\partial x^1} = 0$$

les coefficients de $\frac{\partial^2 u}{\partial x^{2'}}, \frac{\partial^2 u}{\partial x^0 \partial x^1}, \frac{\partial^2 u}{\partial x^{1^2}}$ sont respectivement égaux aux premiers membres de (10), alors que ceux de $\frac{\partial^2 v}{\partial x^{0^2}}, \frac{\partial v^2}{\partial x^0 \partial x^1}, \frac{\partial^2 v}{\partial x^{1^2}}$ sont identiquement nuls: les dérivées secondes de deux fonctions inconnues sont donc éliminées.

Supposons maintenant que le système du premier ordre

$$F_2 = 0,$$

$$c_2 \frac{\partial F_1}{\partial x^0} + d_2 \frac{\partial F_1}{\partial x^1} - c_1 \frac{\partial F_2}{\partial x^0} - d_1 \frac{\partial F_2}{\partial x^1} = 0$$

soit normal.

On déduit de ce système la relation

$$a_2 \frac{\partial F_1}{\partial x^0} + b_2 \frac{\partial F_1}{\partial x^1} = 0.$$

En vertu de cette relation, si F_1 est nulle sur $x^0 = a$, elle est toujours

nulle et la solution du système normal vérifie donc aussi le système donné. Il résulte maintenant de (11), puisque

$$F_2 = 0,$$

que F_1 doit être nulle pour les valeurs initiales données.

Il s'ensuit que le système donné admet une solution et une seule lorsque la condition nécessaire (11) est satisfaite.

§ V.

Généralisation des résutats des §§ II *et* III.

Si le système (1^1) n'est pas, non plus, normal, nous pouvons répéter le procédé indiqué au § I. Nous obtenons de cette manière une seconde équation $\Phi_2 = 0$ d'ordre $h^{(2)}$.

Cette équation est constituée par une combinaison entre les dérivées des F, d'un ordre non supérieur à $h^{(2)} - h + 2$ et dans laquelle les coefficients des dérivées des φ d'ordre $h^{(2)} + 2$ et $h^{(2)} + 1$ sont nuls: ces dérivées sont par conséquent éliminées.

Associons à $\Phi_2 = 0$ les dérivées par rapport à x^0 d'ordre $h^{(2)} - h$ des $F_2 = 0$, $F_3 = 0, \ldots, F_n = 0$: nous obtenons ainsi un troisième système (1^2).

Supposons que, répétant l fois ce procédé, nous parvenions à construire un système normal, à savoir

$$\left.\begin{array}{c} \dfrac{\partial^{h^{(l)}-h} F_i}{\partial x^{0\,h^{(l)}-h}} = 0 \qquad (i = 2, 3, \ldots; n), \\ \Phi_l = 0 \end{array}\right\}, \tag{1^l}$$

Φ_l étant une combinaison entre les dérivées des F d'un ordre ne dépassant pas $h^{(l)} - h + l$ et dans laquelle les dérivées des φ d'ordre

$$h^{(l)} + l,\ h^{(l)} + l - 1, \ldots, h^{(l)} + 1$$

sont éliminées.

En raisonnant comme au § II on démontre que quand les φ et leurs premières $h^{(l)} - 1$ dérivées rendent nulles sur $x^0 = a$ $F_2, F_3, \ldots, F_n$ ainsi que leurs premières $h^{(l)} - h - 1$ dérivées, et en outre F_1 et ses premières $h^{(l)} - h + l - 1$ dérivées, la solution du système normal (1^l) vérifie aussi le système (1). Cette conclusion n'est plus valable lorsque le coefficient de $\dfrac{\partial^{h^{(l)}-h+l} F_1}{\partial x^{0\,h^{(l)}-h+l}}$ dans Φ_l devient nul (coefficient donné par $(\Delta^{11}_{h(n-1)00\ldots0})^l$) mais on peut se passer d'une telle hypothèse, s'il n'est pas

$$\Delta^{ij}_{h(n-1)00\ldots0} = 0 \qquad (i, j = 1, 2, \ldots, n). \tag{9}$$

Dans un tel cas (3) doit donc être caractéristique pour le système normal (1^l).

Les considérations précédentes peuvent s'étendre au cas où les valeurs initiales sont données sur une hypersurface

$$z^0(x^0, x^1, \ldots, x^m) = 0. \tag{6}$$

Les caractéristiques du système normal (1^l) pour lesquelles on n'a pas

$$\Sigma \Delta^{11}_{r_0 r_1 \ldots r_m} p_0^{r_0} p_1^{r_1} \ldots p_m^{r_m} = 0$$

fournissent ensuite, pour le système donné non normal, l'extension du concept de surface caractéristique.

Mathematics. — *Einige einfache Anwendungen der areolären Ableitungen und -Derivierten.* By J. RIDDER. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of January 25, 1947.)

In der Arbeit: *Ueber areolär-harmonische Funktionen* [1]) führten wir areoläre Ableitungen und -Derivierte ein; jede mit -1 multiplizierte areoläre Ableitung ist ein Beispiel eines generalisierten Laplaceschen Operators [2]). Hier lassen wir einige weitere Anwendungen dieser Ableitungen und Derivierten folgen.

§ 1. Definition 1. Die (endlich- und reellwertige) Intervallfunktion $\Phi(J)$ sei definiert für jedes Intervall J, das samt seinem Rande zu einem Bereiche B der xy-Ebene gehört. *Die obere und untere Derivierte von* $\Phi(J)$ *in einem Punkte* $(x, y) \in B$, $D^{+}_{(x,y)}\Phi(J)$ bzw. $D^{-}_{(x,y)}\Phi(J)$, seien definiert als

$$\limsup_{m(J)\to 0} \frac{\Phi(J)}{m(J)} \text{ bzw. } \liminf_{m(J)\to 0} \frac{\Phi(J)}{m(J)},$$

wobei J ein (achsenparalleles) Quadrat darstellt, das (x, y) im Innern oder auf dem Rande enthält. Sind sie einander gleich, so definiere ihr gemeinsamer Wert *die Ableitung*, $D_{(x,y)}\Phi(J)$, von $\Phi(J)$ — in (x, y).

Definition 2. Hat die reellwertige Funktion $u(x, y)$ in einem Bereiche B stetige partielle Ableitungen nach x und nach y, so seien *obere und untere areoläre Derivierte von* $u(x, y)$ *im Punkte* $(x, y) \in B$, $D^{+}_{(x,y)}\Phi_u(J)$ bzw. $D^{-}_{(x,y)}\Phi_u(J)$, die obere bzw. die untere Derivierte in (x, y) der zugehörigen Intervallfunktion $\Phi_u(J) \equiv \int_{R(J)} \frac{\partial u}{\partial n}\, ds$; dabei ist $R(J)$ der Rand von J, und $\frac{\partial u}{\partial n}$ die nach der inneren Normale genommenen Ableitung von $u(x, y)$. Sind obere und untere areoläre Derivierte einander gleich, so definiere ihr gemeinsamer Wert *die areoläre Ableitung*, $D_{(x,y)}\Phi_u(J)$, *von* $u(x, y)$ *in* (x, y).

Aus Satz 14 und der Bemerkung zu Satz 7ter in der zitierten Arbeit folgt:

Zu jeder in einem beschränkten Dirichletschen Bereiche B beschränkten und nach Lebesgue messbaren (reellwertigen) Funktion $\varphi(\xi, \eta)$ gibt es

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[1]) Siehe Acta math. **78** (1946), S. 205—289.

[2]) Siehe loc. cit. [1]), S. 285.

eine und nur eine in B definierte (reellwertige) Funktion $u(\xi, \eta)$, welche auf dem Rande von B Grenzwerte $\equiv 0$ hat, und deren extreme areoläre Derivierte in B existieren, beschränkt und fast überall gleich $\varphi(\xi, \eta)$ sind. In jedem Punkte $(x, y) \in B$ ist

$$u(x, y) = \frac{1}{2\pi} \iint_B g(\xi, \eta; x, y) \cdot \varphi(\xi, \eta)\, d\xi\, d\eta;$$

dabei ist $g(\xi, \eta; x, y)$ die Greensche Funktion von B für den Punkt (x, y).

Daraus lässt sich ableiten [3]) der

Satz 1. $g(\xi, \eta; x, y)$ sei die Greensche Funktion eines beschränkten Dirichletschen Bereiches B, mit Rand C, für den Punkt $(x, y) \in B$. Es seien weiter:

(1) L_2 die Klasse der in B definierten, komplexwertigen, nach Lebesgue messbaren Funktionen $f(\xi, \eta)$, für die $\iint_B |f(\xi, \eta)|^2\, d\xi d\eta$ existiert;

(2) $\mathfrak{D}$ die grösste Teilklasse von L_2, deren jede Funktion auf C Grenzwerte $\equiv 0$ hat, während Real- und Imaginärteil einer solchen Funktion in B beschränkte extreme areoläre Derivierte haben;

(3) $\mathfrak{R}$ die grösste Teilklasse von L_2, deren jede Funktion in B beschränkt ist; zwei Funktionen von $\mathfrak{R}$, welche nur in den Punkten einer Menge vom Lebesgueschen Masse Null verschieden sind, betrachten wir als identisch.

Der Operator T, für die zu $\mathfrak{D}$ gehörenden Funktionen $u + iv$ in fast allen Punkten von B definiert durch die Relation

$$D_{(x,y)}\, \Phi_u(J) + i D_{(x,y)}\, \Phi_v(J) = \varphi(x, y) + i\psi(x, y),$$

liefert eine eineindeutige Abbildung von $\mathfrak{D}$ auf $\mathfrak{R}$; der inverse Operator von T ist ein Integraloperator mit symmetrischem Kern von Hilbert–Schmidt-Typus [4]) und mit dem Bereich $\mathfrak{R}$:

$$u(x, y) + iv(x, y) = T^{-1}[\varphi(x, y) + i\psi(x, y)] = \frac{1}{2\pi} \iint_B g(\xi, \eta; x, y) \cdot [\varphi(\xi, \eta) + i\psi(\xi, \eta)]\, d\xi\, d\eta.$$

Die Operatoren T und T^{-1} sind essentially, selbst-adjungiert [5]) in dem aus der Klasse L_2 in bekannter Weise hervorgehenden Hilbertschen Raum.

Das Analogon von Satz 1 gilt in jedem beschränkten Dirichletschen Bereiche des dreidimensionalen euklidischen Raumes.

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[3]) Vergl. den Beweis von Theorem 3.13 in Stone, Linear transformations in Hilbert space, Am. Math. Soc. Coll. Publ., New-York 1932. Man beachte, dass der Wert der Greenschen Funktion $g(\xi, \eta; x, y)$ von B für den Punkt (x, y) in jedem Punkte $(\xi, \eta) \in B$ zwischen Null und $\log K + \log \frac{1}{\varrho(\xi, \eta; x, y)}$ liegt; dabei ist K eine (von B abhängige) Konstante $\geqq 1$.

[4]) Siehe die Definition in Stone, loc. cit. [3]), S. 101.

[5]) Siehe die Definition in Stone, loc. cit. [3]), S. 51 (Def. 2.12).

§ 1bis. Definition 3. *Die zweite areoläre Ableitung* $u^{(2)}(x, y)$ oder $D_{(x,y)}\,\Phi_{u^{(1)}}(J)$ einer in einem Bereiche B definierten, reellwertigen Funktion $u(x, y)$ sei die areoläre Ableitung der (ersten) areolären Ableitung $u^{(1)}(x, y)$ oder $D_{(x,y)}\Phi_u(J)$. Im allgemeinen sei *die k-te areoläre Ableitung* $u^{(k)}(x, y)$ einer Funktion $u(x, y)$ die areoläre Ableitung der $(k-1)$-ten areolären Ableitung (k ganz und $\geqq 2$).

Hat eine reellwertige Funktion $u(x, y)$ in einem Bereiche B erste bis $(p-1)$-te areoläre Ableitungen (p ganz und $\geqq 2$), und sind die partiellen Ableitungen von $u^{(p-1)}(x, y)$ nach x und nach y in B vorhanden und stetig, so lassen sich in B *p-te obere und untere areoläre Derivierte* definieren als obere bzw. untere areoläre Derivierte von $u^{(p-1)}(x, y)$.

Aus Satz 27 (nebst letztem Absatz des zugehörigen Par.) und der Bemerkung II zu Satz 23ter in unserer Arbeit: Acta math. **78** (1946) folgt:

Zu jeder in einem beschränkten Dirichletschen Bereiche B beschränkten und nach Lebesgue messbaren (reellwertigen) Funktion $\varphi(\xi, \eta)$ gibt es eine und nur eine in B definierte (reellwertige) Funktion $u(\xi, \eta)$, welche in B beschränkte p-te extreme areoläre Derivierte hat (p ganz und $\geqq 2$), und auf dem Rande — ebenso wie ihre ersten bis $(p-1)$-ten areolären Ableitungen — Grenzwerte $\equiv 0$ hat, während ihre sodann fast überall in B existierende p-te areoläre Ableitung fast überall in B gleich $\varphi(\xi, \eta)$ ist. In jedem Punkte $(x, y) \in B$ ist

$$u(x, y) = \frac{(-1)^{p-1}}{2\pi\, a_p} \iint_B g_p(\xi, \eta; x, y) \cdot \varphi(\xi, \eta)\, d\xi\, d\eta;$$

dabei ist $g_p(\xi, \eta; x, y)$ die Greensche Funktion p-ter Ordnung von B für den Punkt (x, y) [6]; a_p hat den Wert $2^{2(p-1)}\{(p-1)!\}^2$.

Das Beweisverfahren des Satzes 3.13 in Stone, loc. cit. 3) lässt daraus ableiten den

Satz 2. $g_p(\xi, \eta; x, y)$ sei die Greensche Funktion p-ter Ordnung eines beschränkten Dirichletschen Bereiches B, mit Rand C, für den Punkt $(x, y) \in B$. Es seien weiter:

(1) L_2 die Klasse der in B definierten, komplexwertigen, nach Lebesgue messbaren Funktionen $f(\xi, \eta)$, für die $\iint_B |f(\xi, \eta)|^2 d\xi d\eta$ existiert;

(2) $\mathfrak{D}^{(p)}$ die grösste Teilklasse von L_2, deren jede Funktion, ebenso wie ihre existierend anzunehmenden, ersten bis $(p-1)$-ten areolären Ableitungen (p ganz und $\geqq 2$), auf C Grenzwerte $\equiv 0$ hat, während Real- und Imaginärteil einer solchen Funktion in B beschränkte p-te extreme areoläre Derivierte haben [7];

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[6]) Siehe loc. cit. [1]), S. 255 (Def. 8). Für. $p \geqq 2$ hängt die Funktion $g_p(\xi, \eta; x, y)$ stetig von ξ, η, x und y ab; ausserdem ist für $(\xi, \eta) \in B$, $(x, y) \in B$ $g_p(\xi, \eta; x, y) = g_p(x, y; \xi, \eta)$. Zum Beweise benutze man die Darstellung von g_p loc. cit. [1]), S. 256 (Fussn. 63).

[7]) Man hat beim Beweise die Eigenschaft von $\mathfrak{D}^{(p)}$ zu benutzen, dass sie überall dicht liegt in dem von den Funktionen von L_2 gebildeten Hilbertschen Raum. Diese Eigenschaft lässt sich ableiten unter Anwendung von Formel (19), loc. cit. [1]), S. 225.

(3) $\mathfrak{R}$ die grösste Teilklasse von L_2, deren jede Funktion in B beschränkt ist; zwei Funktionen von $\mathfrak{R}$, welche nur in den Punkten einer Menge vom Lebesgueschen Masse Null verschieden sind, betrachten wir als identisch.

Der Operator $T^{(p)}$, für die zu $\mathfrak{D}^{(p)}$ gehörenden Funktionen $u + iv$ in fast allen Punkten von B definiert durch die Relation

$$u^{(p)}(x, y) + i v^{(p)}(x, y) = \varphi(x, y) + i \psi(x, y),$$

liefert eine eineindeutige Abbildung von $\mathfrak{D}^{(p)}$ auf $\mathfrak{R}$; der inverse Operator ist ein Integraloperator mit symmetrischem Kern von Hilbert–Schmidt-Typus [4]) und mit dem Bereich $\mathfrak{R}$:

$$u(x, y) + i v(x, y) = [T^{(p)}]^{-1} [\varphi(x, y) + i \psi(x, y)] = \frac{(-1)^{p-1}}{2\pi a_p} \iint_B g_p(\xi, \eta; x, y) \cdot [\varphi(\xi, \eta) + i \psi(\xi, \eta)] \, d\xi \, d\eta,$$

mit $a_p = 2^{2(p-1)} \cdot \{(p-1)!\}^2$. Die Operatoren $T^{(p)}$ und $[T^{(p)}]^{-1}$ sind ‚essentially' selbst-adjungiert [5]) in dem aus L_2 in bekannter Weise hervorgehenden Hilbertschen Raum.

Das Analogon von Satz 2 gilt in jedem beschränkten Dirichletschen Bereiche des dreidimensionalen euklidischen Raumes.

§ 2. Satz 3. B, L_2, $\mathfrak{D}$, $\mathfrak{R}$ und T mögen die gleiche Bedeutung haben wie in Satz 1. Dann gilt:

α) der Operator T hat abzählbar unendlich viele, von Null verschiedene, isoliert liegende Eigenwerte l_n, die nur reell sein können; $Tf - l \cdot f = 0$ hat für jeden Eigenwert $l = l_n$ ein System von nur endlich vielen, paarweise orthogonalen (und somit linear unabhängigen) Lösungen (Eigenfunktionen);

β) für jeden reellen oder komplexen Wert l, mit $l \neq l_n$ $(n = 1,2, \ldots)$, und $\chi(x, y) \in L_2$ hat $Tf - l \cdot f = \chi$ höchstens eine Lösung $f \in \mathfrak{D}$;

γ) die gemäss α) zu den verschiedenen l_n gehörigen Eigenfunktionen bilden ein vollständiges System im von den Funktionen der Klasse L_2 gebildeten Hilbertschen Raum.

Beweis. Da $T^{-1} f = 0$ nur die Lösung $f \equiv 0$ zulässt, sind die Eigenwerte von T^{-1} von Null verschieden.

Aus der Gleichwertigkeit der Gleichungen

$$T^{-1} f = l \cdot f \quad \text{und} \quad Tf = \frac{1}{l} \cdot f \qquad (l \neq 0)$$

folgt, dass die zu gegebenem $l \neq 0$ gehörenden Eigenfunktionen von T^{-1} gleichzeitig Eigenfunktionen von T für den Eigenwert $\frac{1}{l}$ sind, und umgekehrt.

Nun ist T^{-1} ein selbstadjungierter Operator mit endlicher Norm (in dem

aus L_2 hervorgehenden Hilbertschen Raum). Ihre Eigenwerte sind somit reell, und bilden eine abzählbar unendliche isolierte Menge; zu jedem Eigenwert gehören endlich viele paarweise orthogonale Eigenfunktionen; diese bilden ein vollständiges System (im zugehörigen Hilbertschen Raum) [8]). Da Null kein Eigenwert von T ist, folgen hieraus schon die Behauptungen von Satz 3.

In derselben Weise beweist man den

Satz 4. B, L_2, $\mathfrak{D}^{(p)}$, $\Re$ und $T^{(p)}$ mögen die gleiche Bedeutung haben wie in Satz 2. Dann gilt:

α) der Operator $T^{(p)}$ hat abzählbar unendlich viele, von Null verschiedene, isoliert liegende Eigenwerte l_n, die nur reell sein können; $T^{(p)}f - l \cdot f = 0$ hat für jeden Eigenwert $l = l_n$ ein System von nur endlich vielen, paarweise orthogonalen Lösungen (Eigenfunktionen);

β) für jeden reellen oder komplexen Wert l, mit $l \neq l_n$ $(n = 1, 2, \ldots)$, und $\chi(x, y) \in L_2$ hat $T^{(p)}f - l \cdot f = \chi$ höchstens eine Lösung $f \in \mathfrak{D}^{(p)}$;

γ) die gemäss α) zu den verschiedenen l_n gehörigen Eigenfunktionen bilden ein vollständiges System im zugehörigen Hilbertschen Raum.

Die Analoga der Sätze 3 und 4 gelten im dreidimensionalen euklidischen Raum.

§ 3. Der eindimensionale Fall. Als Analoga zu den areolären Ableitungen und -extremen Derivierten haben wir hier die mit -1 multiplizierten, zweiten Ableitung bzw. zweiten extremen Derivierten; die Existenz der ersten Ableitung wird dabei vorausgesetzt.

Betrachten wir als Greensche Funktion des Intervalles $(0, l)$ für den Punkt $x \in (0, l)$ die Funktion $g(\xi, x)$, definiert durch:

$$g(\xi, x) = \frac{\xi(l-x)}{l}, \text{ falls } 0 < \xi \leqq x,$$

und durch:

$$g(\xi, x) = \frac{x(l-\xi)}{l}, \text{ falls } x < \xi < l.$$

Man beweist dann leicht:

Zu jedem in $(0, l)$ beschränkten und nach Lebesgue messbaren (reellwertigen) Funktion $\varphi(\xi)$ gibt es eine und nur eine in $(0, l)$ definierte (reellwertige) Funktion $u(\xi)$, welche in 0 und l Grenzwerte Null hat, und deren obere und untere Derivierte zweiter Ordnung in $(0, l)$ existieren, beschränkt und fast überall gleich $-\varphi(\xi)$ sind. In jedem Punkte $x \in (0, l)$ ist

$$u(x) = \int_0^l g(\xi, x) \cdot \varphi(\xi)\, d\xi,$$

mit $g(\xi, x)$ als Greensche Funktion von $(0, l)$ für den Punkt x.

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[8]) Siehe Stone, loc. cit. [3]), S. 193 (Th. 5.14).

Die Analoga der Sätze 1 und 3 für den eindim. Fall liegen hiernach auf der Hand.

Als Analoga zu den p-ten areolären Ableitungen und -extremen Derivierten haben wir hier die mit $(-1)^p$ multiplizierten, $2p$-ten Ableitung bzw. $2p$-ten extremen Derivierten [9]).

Betrachten wir als Greensche Funktion p-ter Ordnung des Intervalls $(0, l)$ für den Punkt $x \in (0, l)$ die Funktion $g_p(\xi, x)$, welche folgenden Bedingungen genügt: ihre $2p$-ten extremen Derivierten existieren, und haben den Wert Null in allen von x verschiedenen Punkten von $(0, l)$; in 0 und in l haben g_p und ihre 2-ten, 4-ten bis $(2p-2)$-ten Ableitungen Grenzwerte gleich Null; es ist $\frac{d^{2p-1} g}{d x^{2p-1}}(x+0) - \frac{d^{2p-1} g}{d x^{2p-1}}(x-0) = 1$. g ist durch diese Bedingungen eindeutig bestimmt [10]).

Man beweist wieder leicht:

Zu jedem in $(0, l)$ beschränkten und nach Lebesgue messbaren (reellwertigen) Funktion $\varphi(\xi)$ gibt es eine und nur eine in $(0, l)$ definierte (reellwertige) Funktion $u(\xi)$, welche in 0 und l, ebenso wie ihre 2-ten, 4-ten bis $(2p-2)$-ten Ableitungen, Grenzwerte Null hat, und deren obere und untere Derivierte $2p$-ter Ordnung in $(0, l)$ existieren, beschränkt und fast überall gleich $-\varphi(\xi)$ sind. In jedem Punkte $x \in (0, l)$ ist

$$u(x) = \int_0^l g_p(\xi, x) \cdot \varphi(\xi)\, d\xi,$$

mit $g_p(\xi, x)$ als Greensche Funktion p-ter Ordnung von $(0, l)$ für den Punkt x.

Die Analoga der Sätze 2 und 4 für den eindim. Fall liegen nun wieder auf der Hand.

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[9]) Vergl. loc. cit. [1]), S. 211 u. 262 (Fussn. 68).

[10]) Vergl. Bôcher, Leçons sur les méthodes de Sturm dans la théorie des équations différentielles linéaires, Coll. Borel, Paris 1917, Kap. 5, insbes. S. 100.

Mathematics. — *On some determinants.* By H. J. A. DUPARC. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of January 25, 1947.)

In this paper I evaluate some determinants and give applications of the results.

Theorem 1. *Let* $|e_{rs}|$ *denote the determinant, the element in the* r^{th} *row and* s^{th} *column of which is equal to* e_{rs}. *If none of the sums* $y_r + x_s$ $(r = 1, \ldots, n;\ s = 1, \ldots, n)$ *is equal to zero, we have*

$$\left|\frac{x_r+z_s}{x_r+y_s}\right| = \frac{X\,Y\prod_{s=1}^{n}(z_s-y_s)}{\prod_{r,s=1}^{n}(x_r+y_s)}\left\{1-\sum_{r=1}^{n}\frac{(y_r+x_1)\ldots(y_r+x_n)}{(y_r-z_r)\,q'(y_r)}\right\}, \quad . \quad (1)$$

where

$$X=\prod_{1\leqq s<r\leqq n}(x_r-x_s),\ Y=\prod_{1\leqq s<r\leqq n}(y_r-y_s) \text{ and } q(y)=\prod_{r=1}^{n}(y-y_r).$$

Proof:

$$\left|\frac{x_r+z_s}{x_r+y_s}\right| = \left|1+\frac{z_s-y_s}{x_r+y_s}\right| = A\prod_{s=1}^{n}(z_s-y_s),$$

where

$$A=\left|\frac{1}{z_s-y_s}+\frac{1}{x_r+y_s}\right|$$

can be written in the form

$$\begin{vmatrix} 1 & \frac{1}{y_1-z_1} & \ldots & \frac{1}{y_n-z_n} \\ 1 & \frac{1}{x_1+y_1} & \ldots & \frac{1}{x_1+y_n} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \frac{1}{x_n+y_1} & \ldots & \frac{1}{x_n+y_n} \end{vmatrix}.$$

Developing this determinant in terms of the first row we obtain

$$A=\begin{vmatrix} \frac{1}{x_1+y_1} & \ldots & \frac{1}{x_1+y_n} \\ \cdot & \cdot & \cdot \\ \frac{1}{x_n+y_1} & \ldots & \frac{1}{x_n+y_n} \end{vmatrix} + \sum_{t=1}^{n}\frac{B_t}{y_t-z_t}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where

$$B_t = (-1)^t \left| 1 \; \frac{1}{x_r+y_1} \cdots \frac{1}{x_r+y_{t-1}} \; \frac{1}{x_r+y_{t+1}} \cdots \frac{1}{x_r+y_n} \right| \quad (t=1,\ldots,n).$$

If the product Π' contains only factors such that $1 \leqq s < r \leqq n$, then the first determinant in the right hand side of (2), being the determinant of CAUCHY, may be written in the form

$$\frac{\Pi'(x_r-x_s)\,\Pi'(y_r-y_s)}{\prod_{r,s=1}^{n}(x_r+y_s)} = \frac{X\cdot Y}{\prod_{r,s=1}^{n}(x_r+y_s)}.$$

The determinant B_t is equal to [1])

$$(-1)^t (-1)^{n-1} \frac{X \prod_{\substack{r\neq t\\ s\neq t}}'(y_r-y_s)}{\prod_{\substack{r,s=1\\ s\neq t}}^{n}(x_r+y_s)}$$

$$= (-1)^{t+n-1} \frac{X\cdot Y}{\prod_{r,s=1}^{n}(x_r+y_s)} \; \frac{\prod_{r=1}^{n}(y_t+x_r)}{(-1)^{n-t}\prod_{\substack{r=1\\ r\neq t}}^{n}(y_t-y_r)} = -\frac{XY\prod_{r=1}^{n}(y_t+x_r)}{q'(y_t)\prod_{r,s=1}^{n}(x_r+y_s)}.$$

From (2) we find

$$A = \frac{X\cdot Y}{\prod_{r,s=1}^{n}(x_r+y_s)} \left\{ 1 - \sum_{t=1}^{n} \frac{\prod_{r=1}^{n}(y_t+x_r)}{(y_t-z_t)\,q'(y_t)} \right\},$$

which proves the theorem.

In the applications we distinguish several cases.

I. There exists one and only one number σ with $y_\sigma = z_\sigma$. Then our formula reduces to

$$\left| \frac{x_r+z_s}{x_r+y_s} \right| = \frac{X\cdot Y \prod_{\substack{s=1\\ s\neq\sigma}}^{n}(z_s-y_s)}{q'(y_\sigma)\prod_{\substack{r,s=1\\ s\neq\sigma}}^{n}(x_r+y_s)},$$

since all other terms in the right hand side of (1) vanish.

II. At least two numbers σ and τ exist with $y_\sigma = z_\sigma$ and $y_\tau = z_\tau$. Obviously we have then $\left| \frac{x_r+z_s}{x_r+y_s} \right| = 0$, since this determinant has two equal columns.

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[1]) Confer for instance, J. G. VAN DER CORPUT, Over eenige determinanten, Verh. Kon. Akad. v. Wetensch., Amsterdam, **14**, 39 (1930).

III. For all s $(s=1, \ldots, n)$ we have $y_s \neq z_s$. Since all y_s are different, a polynomial $\varphi(y)$ exists such that

$$z_s = \varphi(y_s) \quad (s=1, \ldots, n).$$

Then $\dfrac{(y_r+x_1)\ldots(y_r+x_n)}{q'(y_r)\,(y_r-z_r)}$ is the residu of the function

$$F(y) = \frac{(y+x_1)\ldots(y+x_n)}{q(y)\;\{y-\varphi(y)\}},$$

taken at the point y_r.

Let $\eta_1, \ldots, \eta_k$ denote the different roots of the equation $y=\varphi(y)$. We first suppose that these roots do not occur amongst the numbers $-x_1, \ldots, -x_n$. Then the only singular points of $F(y)$ are $y_1, \ldots, y_n$, $\eta_1, \ldots, \eta_k$. All these points are different since $y_r \neq y_s$ $(r \neq s)$, and if we had $y_r = \eta_s$, we would obtain

$$z_r = \varphi(y_r) = \varphi(\eta_s) = \eta_s = y_r,$$

while we supposed $y_r \neq z_r$. By the residu theorem we obtain

$$-\sum_{r=1}^{n} \frac{(y_r+x_1)\ldots(y_r+x_n)}{(y_r-z_r)\,q'(y_r)} = \underset{\eta,\infty}{S} \frac{(y+x_1)\ldots(y+x_n)}{q(y)\,\{y-\varphi(y)\}},$$

where $\underset{\eta,\infty}{S}$ denotes the sum of the residues taken at $\eta_1, \ldots, \eta_k$ and infinity. If a root η of $y=\varphi(y)$ occurs in the system $-x_1, \ldots, -x_n$, then η is no singular point of $F(y)$, hence S has not necessarily to be extended over this η. Still in this case we may extend S also over this η, hence over all roots of $y=\varphi(y)$, since the residu of $F(y)$ in this non-singular point η is equal to zero. This proves our

Theorem 2. *If for all s $(s=1, \ldots, n)$ we have $y_s \neq z_s$, then*

$$\left|\frac{x_r+z_s}{x_r+y_s}\right| = \frac{X \cdot Y \prod_{s=1}^{n}(z_s-y_s)}{\prod_{r,s=1}^{n}(x_r+y_s)} \left(1 + \underset{\eta,\infty}{S} \frac{(y+x_1)\ldots(y+x_n)}{q(y)\,(y-\varphi(y))}\right),$$

where $\underset{\eta,\infty}{S}$ denotes the sum of the residues, taken at infinity and at all roots of $y=\varphi(y)$.

We make some applications of theorem 2.

$$|\operatorname{ctg}(a_r+b_s)| = \left|\frac{\operatorname{ctg} a_r \operatorname{ctg} b_s - 1}{\operatorname{ctg} a_r + \operatorname{ctg} b_s}\right| = \left|\frac{\operatorname{ctg} a_r - \operatorname{tg} b_s}{\operatorname{ctg} a_r + \operatorname{ctg} b_s}\right| \prod_{s=1}^{n} \operatorname{ctg} b_s.$$

Here $x_r = \operatorname{ctg} a_r$; $y_r = \operatorname{ctg} b_r$; $z_r = -\operatorname{tg} b_r = -\dfrac{1}{y_r}$, hence $z=\varphi(y)=-\dfrac{1}{y}$.

Consequently the roots of $y=\varphi(y)$ are i and $-i$.

By theorem 2 we obtain

$$|\operatorname{ctg}(a_r+b_s)|=$$
$$=\frac{\Pi'(\operatorname{ctg}a_r-\operatorname{ctg}a_s)\,\Pi'(\operatorname{ctg}b_r-\operatorname{ctg}b_s)\prod_{s=1}^{n}\operatorname{ctg}b_s(-\operatorname{tg}b_s-\operatorname{ctg}b_s)\,P}{\prod_{r,s=1}^{n}(\operatorname{ctg}a_r+\operatorname{ctg}b_s)}$$

where

$$P=1+\underset{l,-l,\infty}{S}\frac{(y+\operatorname{ctg}a_1)\dots(y+\operatorname{ctg}a_n)}{(y-\operatorname{ctg}b_1)\dots(y-\operatorname{ctg}b_n)\left(y+\frac{1}{y}\right)}.$$

The residu at the point i is equal to

$$R_i=\frac{i(\operatorname{ctg}a_1+i)\dots(\operatorname{ctg}a_n+i)}{(i-\operatorname{ctg}b_1)\dots(i-\operatorname{ctg}b_n)\,2i}$$
$$=\tfrac{1}{2}\frac{(\cos a_1+i\sin a_1)\dots(\cos a_n+i\sin a_n)}{(-1)^n(\cos b_1-i\sin b_1)\dots(\cos b_n-i\sin b_n)}\prod_{r=1}^{n}\frac{\sin b_r}{\sin a_r}$$
$$=\tfrac{1}{2}(-1)^n\left\{\cos\sum_{r=1}^{n}(a_r+b_r)+i\sin\sum_{r=1}^{n}(a_r+b_r)\right\}\prod_{r=1}^{n}\frac{\sin b_r}{\sin a_r}.$$

Hence

$$P=1-1+(-1)^n\left\{\cos\sum_{r=1}^{n}(a_r+b_r)\right\}\prod_{r=1}^{n}\frac{\sin b_r}{\sin a_r}$$

and after some reduction we find

$$|\operatorname{ctg}(a_r+b_s)|=\frac{\Pi'\sin(a_r-a_s)\,\Pi'\sin(b_r-b_s)}{\prod_{r,s=1}^{n}\sin(a_r+b_s)}\cos\sum_{r=1}^{n}(a_r+b_r).$$

Herefrom follows further

$$|\operatorname{tg}(a_r+b_s)|=\left|\operatorname{ctg}\left\{\left(\frac{\pi}{4}-a_r\right)+\left(\frac{\pi}{4}-b_s\right)\right\}\right|$$
$$=\frac{\Pi'\sin(a_r-a_s)\,\Pi'\sin(b_r-b_s)}{\prod_{r,s=1}^{n}\cos(a_r+b_s)}\cos\left\{\sum_{r=1}^{n}(a_r+b_r)-n\frac{\pi}{2}\right\}.$$

The determinant $|\sin(a_r+b_s)|=|\operatorname{tg}a_r+\operatorname{tg}b_s|\prod_{r=1}^{n}\cos a_r\cos b_r$ vanishes for $n>2$, since the first factor at the right hand side, being a determinant of the kind $|x_r+y_s|$ is equal to zero for $n>2$. The same is valid for the determinant $|\cos(a_r+b_s)|$.

Theorem 2 enables us also to calculate the determinant

$$|\operatorname{tg}(a_r+b_s)\operatorname{tg}(a_r-b_s)|=\left|\frac{\operatorname{tg}^2a_r-\operatorname{tg}^2b_s}{\operatorname{tg}^2a_r-\operatorname{ctg}^2b_s}\right|\frac{1}{\prod_{s=1}^{n}(-\operatorname{tg}^2b_s)}.$$

If we put

$$x_r = \operatorname{tg}^2 a_r;\ y_r = -\operatorname{ctg}^2 b_r;\ z_r = -\operatorname{tg}^2 b_r;\ \text{hence } z_r = \frac{1}{y_r}\ (r = 1, \ldots, n),$$

we obtain $z = \varphi(y) = \frac{1}{y}$. Then the roots of $y = \varphi(y)$ are $\eta = \pm 1$.

Then

$$X = \frac{\Pi' \sin(a_r + a_s)\, \Pi' \sin(a_r - a_s)}{\prod_{s=1}^{n} \cos^{2(n-1)} a_s};\quad Y = \frac{\Pi' \sin(b_r + b_s)\, \Pi' \sin(b_r - b_s)}{\prod_{s=1}^{n} \sin^{2(n-1)} b_s};$$

$$\prod_{r,s=1}^{n} (x_r + y_s) = \frac{(-1)^{n^2} \prod_{r,s=1}^{n} \cos(a_r + b_s) \cos(a_r - b_s)}{\prod_{s=1}^{n} \cos^{2n} a_s \sin^{2n} b_s};$$

$$\prod_{s=1}^{n} (z_s - y_s) = \frac{\prod_{s=1}^{n} \cos 2 b_s}{\prod_{s=1}^{n} \sin^2 b_s \cos^2 b_s}$$

and

$$F(y) = \frac{y\,(y + \operatorname{tg}^2 a_1) \ldots (y + \operatorname{tg}^2 a_n)}{(y^2 - 1)\,(y + \operatorname{ctg}^2 b_1) \ldots (y + \operatorname{ctg}^2 b_n)}.$$

The residu of $F(y)$ in the point 1 is equal to

$$R_1 = \frac{\sin^2 b_1 \ldots \sin^2 b_n}{2 \cos^2 a_1 \ldots \cos^2 a_n}, \text{ while } R_{-1} = \frac{(-1)^n \cos 2 a_1 \ldots \cos 2 a_n \sin^2 b_1 \ldots \sin^2 b_n}{2 \cos 2 b_1 \ldots \cos 2 b_n \cos^2 a_1 \ldots \cos^2 a_n}$$

and $R_\infty = -1$. From theorem 2 we infer after some reduction

$$|\operatorname{tg}(a_r) + b_s \operatorname{tg}(a_r - b_s)| =$$

$$\frac{\Pi' \{\sin(a_r + a_s) \sin(a_r - a_s) \sin(b_r + b_s) \sin(b_r - b_s)\}}{2 \prod_{r,s=1}^{n} \cos(a_r + b_s) \cos(a_r - b_s)} \left(\prod_{s=1}^{n} \cos 2 b_s + (-1)^n \prod_{s=1}^{n} \cos 2 a_s \right). \quad (3)$$

Replacing a_r by $\frac{\pi}{2} - a_r$ $(r = 1, \ldots, n)$, we find herefrom

$$|\operatorname{ctg}(a_r + b_s) \operatorname{ctg}(a_r - b_s)| =$$

$$\frac{\Pi' \{\sin(a_r + a_s) \sin(a_s - a_r) \sin(b_r + b_s) \sin(b_r - b_s)\}}{2 \prod_{r,s=1}^{n} \sin(a_r + b_s) \sin(a_r - b_s)} \left(\prod_{s=1}^{n} \cos 2 b_s + \prod_{s=1}^{n} \cos 2 a_s \right),$$

while

$$|\sin(a_r + b_s) \sin(a_r - b_s)| = \left| \frac{\cos 2 b_s - \cos 2 a_r}{2} \right| = 0 \text{ for } n > 2,$$

hence

$$|\cos(a_r + b_s) \cos(a_r - b_s)| = 0 \text{ for } n > 2,$$

The determinant

$$|\sec(a_r+b_s)\sec(a_r-b_s)| = \left|\frac{2}{\cos 2a_r+\cos 2b_s}\right|$$

is a determinant of CAUCHY, which easily is found to be equal to

$$\frac{\Pi'\{\sin(a_r+a_s)\sin(a_r-a_s)\sin(b_r+b_s)\sin(b_r-b_s)\}}{\prod_{r,s=1}^{n}\cos(a_r+b_s)\cos(a_r-b_s)}.$$

Hence, replacing a_r by $\frac{\pi}{2}-a_r$ $(r=1,\ldots,n)$, we get

$$|\operatorname{cosec}(a_r+b_s)\operatorname{cosec}(a_r-b_s)| = \frac{\Pi'\{\sin(a_r+a_s)\sin(a_s-a_r)\sin(b_r+b_s)\sin(b_r-b_s)\}}{\prod_{r,s=1}^{n}\sin(a_r+b_s)\sin(a_r-b_s)}.$$

Theorem 2 enables us to calculate the determinant

$$\left|\frac{\operatorname{tg}(a_r+b_s)}{\operatorname{tg}(a_r-b_s)}\right| = \left|\frac{\sin 2a_r+\sin 2b_s}{\sin 2a_r-\sin 2b_s}\right|$$

by putting

$x_r=\sin 2a_r$; $y_r=-\sin 2b_r$; $z_r=\sin 2b_r$, hence $z_r=-y_r$ $(r=1,\ldots,n)$.

Here $z=\varphi(y)=-y$ and the equation $y=\varphi(y)$ has only one root $\eta=0$.

The residu in $y=0$ of the function

$$F(y)=\frac{(y+\sin 2a_1)\ldots(y+\sin 2a_n)}{2y(y+\sin 2b_1)\ldots(y+\sin 2b_n)}$$

is equal to

$$R_0=\frac{\sin 2a_1\ldots\sin 2a_n}{2\sin 2b_1\ldots\sin 2b_n}, \text{ while } R_\infty=-\tfrac{1}{2}.$$

After substitution and reduction we find

$$\left|\frac{\operatorname{tg}(a_r+b_s)}{\operatorname{tg}(a_r-b_s)}\right| = \left.\frac{\Pi'\{\sin(a_r-a_s)\cos(a_r+a_s)\sin(b_s-b_r)\cos(b_s+b_r)\}}{2\prod_{r,s=1}^{n}\sin(a_r-b_s)\cos(a_r+b_s)}\left(\prod_{r=1}^{n}\sin 2a_r+\prod_{r=1}^{n}\sin 2b_r\right)\right\}, \quad (4)$$

hence replacing a_r by $\frac{\pi}{2}-a_r$ and b_r by $-b_r$ $(r=1,\ldots,n)$ we obtain

$$\left|\frac{\operatorname{ctg}(a_r+b_s)}{\operatorname{ctg}(a_r-b_s)}\right| = \left.\frac{\Pi'\{\sin(a_r-a_s)\cos(a_r+a_s)\sin(b_r-b_s)\cos(b_r+b_s)\}}{2\prod_{r,s=1}^{n}\sin(a_r+b_s)\cos(a_r-b_s)}\left(\prod_{s=1}^{n}\sin 2a_s+(-1)^n\prod_{s=1}^{n}\sin 2b_s\right)\right\}. \quad (5)$$

Formulae (4) and (5) may also be derived from (3) directly.

To calculate the determinant

$$\left|\frac{\sin(a_r+b_s)}{\sin(a_r-b_s)}\right| = \left|\frac{\operatorname{tg} a_r+\operatorname{tg} b_s}{\operatorname{tg} a_r-\operatorname{tg} b_s}\right|$$

we put

$x_r = \operatorname{tg} a_r$; $y_r = -\operatorname{tg} b_r$; $z_r = \operatorname{tg} b_r$, hence $z_r = -y_r$ $\quad (r = 1, \ldots, n)$,

which gives $z = \varphi(y) = -y$. The only root of $y = \varphi(y)$ is $\eta = 0$.

The residu in $y = 0$ of the function

$$F(y) = \frac{(y+\operatorname{tg} a_1)\ldots(y+\operatorname{tg} a_n)}{2y(y+\operatorname{tg} b_1)\ldots(y+\operatorname{tg} b_n)}$$

in $y = 0$ is equal to

$$R_0 = \frac{\operatorname{tg} a_1 \ldots \operatorname{tg} a_n}{2 \operatorname{tg} b_1 \ldots \operatorname{tg} b_n}, \text{ while } R_\infty = -\tfrac{1}{2}.$$

After substitution and reduction we find

$$\left|\frac{\sin(a_r+b_s)}{\sin(a_r-b_s)}\right| = \left.\begin{aligned} &= \frac{2^{n-1}\,\Pi'\{\sin(a_r-a_s)\sin(b_s-b_r)\}\prod_{r=1}^{n}\cos a_r\cos b_r}{\prod_{r,s=1}^{n}\sin(a_r-b_s)}\left(\prod_{r=1}^{n}\operatorname{tg} a_r+\prod_{r=1}^{n}\operatorname{tg} b_r\right)\\ &\end{aligned}\right\}. \quad (6)$$

Hence, replacing a_r by $\frac{\pi}{2}-a_r$ and b_r by $-b_r$ $(r = 1, \ldots, n)$, we obtain

$$\left|\frac{\cos(a_r+b_s)}{\cos(a_r-b_s)}\right| = \left.\begin{aligned} &= \frac{2^{n-1}\,\Pi'\{\sin(a_r-a_s)\sin(b_s-b_r)\}\prod_{r=1}^{n}\cos a_r\cos b_r}{\prod_{r,s=1}^{n}\cos(a_r-b_s)}\left(1+(-1)^n\prod_{r=1}^{n}\operatorname{tg} a_r\operatorname{tg} b_r\right)\\ &\end{aligned}\right\}. \quad (7)$$

Finally, replacing b_r by $-b_r$ $(r = 1, \ldots, n)$ in (7), we obtain

$$\left|\frac{\sec(a_r+b_s)}{\sec(a_r-b_s)}\right| = \frac{2^{n-1}\,\Pi'\{\sin(a_r-a_s)\sin(b_r-b_s)\}\prod_{r=1}^{n}\cos a_r\cos b_r}{\prod_{r,s=1}^{n}\cos(a_r+b_s)}\left(1+\prod_{r=1}^{n}\operatorname{tg} a_r\operatorname{tg} b_r\right)$$

and, by the same substitution in (6),

$$\left|\frac{\operatorname{cosec}(a_r+b_s)}{\operatorname{cosec}(a_r-b_s)}\right| = \frac{2^{n-1}\,\Pi'\{\sin(a_r-a_s)\sin(b_r-b_s)\}\prod_{r=1}^{n}\cos a_r\cos b_r}{\prod_{r,s=1}^{n}\sin(a_r+b_s)}\left(\prod_{r=1}^{n}\operatorname{tg} a_r+(-1)^n\prod_{r=1}^{n}\operatorname{tg} b_r\right).$$

The value of the determinant

$$\left|\frac{\sin a_r + \sin b_s}{\sin a_r - \sin b_s}\right| = \left|\frac{\operatorname{tg}\frac{a_r+b_s}{2}}{\operatorname{tg}\frac{a_r-b_s}{2}}\right|$$

follows from (4) by replacing a_r by $\frac{a_r}{2}$ and b_r by $\frac{b_r}{2}$ $(r = 1, \ldots, n)$.

The value of the determinant

$$\left|\frac{\cos a_r + \cos b_s}{\cos a_r - \cos b_s}\right| = \left|\frac{\operatorname{ctg}\frac{a_r+b_s}{2}}{\operatorname{ctg}\frac{a_r-b_s}{2}}\right|$$

follows from (5) by the same substitutions.

The value of the determinant

$$\left|\frac{\operatorname{tg} a_r + \operatorname{tg} b_s}{\operatorname{tg} a_r - \operatorname{tg} b_s}\right| = \left|\frac{\sin(a_r+b_s)}{\sin(a_r-b_s)}\right|$$

follows from (6).

Finally

$$\left|\frac{\operatorname{ctg} a_r + \operatorname{ctg} b_s}{\operatorname{ctg} a_r - \operatorname{ctg} b_s}\right| = (-1)^n \left|\frac{\operatorname{tg} a_r + \operatorname{tg} b_s}{\operatorname{tg} a_r - \operatorname{tg} b_s}\right|,$$

$$\left|\frac{\sec a_r + \sec b_s}{\sec a_r - \sec b_s}\right| = (-1)^n \left|\frac{\cos a_r + \cos b_s}{\cos a_r - \cos b_s}\right|$$

and

$$\left|\frac{\operatorname{cosec} a_r + \operatorname{cosec} b_s}{\operatorname{cosec} a_r - \operatorname{cosec} b_s}\right| = (-1)^n \left|\frac{\sin a_r + \sin b_s}{\sin a_r - \sin b_s}\right|.$$

Theorem 3. *If D_p denotes the determinant, obtained by replacing in the determinant* $\left|\frac{1}{x_r+y_s}\right|$ *of* CAUCHY *the elements of the k^{th} column* $(k = 1, \ldots, p)$ *by* $\left|\frac{x_r+z_k}{x_r+y_k}\right|$, *then*

$$D_p = \frac{X \cdot Y \prod_{s=1}^{p}(z_s - y_s)}{\prod_{r,s=1}^{n}(x_r + y_s)} \left\{1 - S_p \frac{(y+x_1)\ldots(y+x_n)}{q(y)\,(y-\varphi(y))}\right\},$$

where S_p denotes the sum of the residues, taken at the points $y_1, \ldots, y_p$.

Proof.

$$D_p = \left|\frac{1}{x_r+y_1} + \frac{1}{z_1-y_1} \cdots \frac{1}{x_r+y_p} + \frac{1}{z_p-y_p}\,\frac{1}{x_r+y_{p+1}} \cdots \frac{1}{x_r+y_n}\right| \prod_{s=1}^{p}(z_s - y_s)$$

hence

$$D_p = \begin{vmatrix} 1 & \frac{1}{y_1-z_1} & \cdots & \frac{1}{y_p-z_p} & 0 & \cdots & 0 \\ 1 & \frac{1}{x_1+y_1} & \cdots & \frac{1}{x_1+y_p} & \frac{1}{x_1+y_{p+1}} & \cdots & \frac{1}{x_1+y_n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \frac{1}{x_n+y_1} & \cdots & \frac{1}{x_n+y_p} & \frac{1}{x_n+y_{p+1}} & \cdots & \frac{1}{x_n+y_n} \end{vmatrix} \prod_{s=1}^{p} (z_s-y_s). \quad (8)$$

The first factor in the right hand side of (8), if developed in terms of the first row, is found to be equal to

$$\left|\frac{1}{x_r+y_s}\right| + \sum_{h=1}^{p} \frac{(-1)^h}{y_h-z_h} \left| 1 \; \frac{1}{x_r+y_1} \cdots \frac{1}{x_r+y_{h-1}} \; \frac{1}{x_r+y_{h+1}} \cdots \frac{1}{x_r+y_n} \right|$$

$$= \frac{X \cdot Y}{\prod_{r,s=1}^{n} (x_r+y_s)} + \sum_{h=1}^{p} \frac{(-1)^{h+n-1}}{y_h-z_h} \frac{X \cdot \prod'_{\substack{r\neq h \\ s\neq h}} (y_r-y_s)}{\prod_{\substack{r,s=1 \\ r\neq h \\ s\neq h}}^{n} (x_r+y_s)}$$

$$= \frac{X \cdot Y}{\prod_{r,s=1}^{n} (x_r+y_s)} \left\{ 1 - \sum_{h=1}^{p} \frac{(y_h+x_1)\ldots(y_h+x_n)}{q'(y_h)\;(y_h-z_h)} \right\}$$

$$= \frac{X \cdot Y}{\prod_{r,s=1}^{n} (x_r+y_s)} \left\{ 1 - S_p \frac{(y+x_1)\ldots(y+x_n)}{q(y)\,\{y-\varphi(y)\}} \right\}.$$

This proves theorem 3.

Remark. If the determinant D'_p is obtained by replacing in the determinant of CAUCHY the elements of the k^{th} column $(k = p+1, \ldots, n)$ by $\frac{x_r+z_k}{x_r+y_k}$ $(r = 1, \ldots, n)$, then follows immediately from (8)

$$\frac{D_p}{\prod_{s=1}^{p} (z_s-y_s)} + \frac{D'_p}{\prod_{s=p+1}^{n} (z_s-y_s)} = \left|\frac{1}{x_r+y_s}\right| + \frac{\left|\frac{x_r+z_s}{x_r+y_s}\right|}{\prod_{s=1}^{n} (z_s-y_s)} =$$

$$= \frac{X \cdot Y}{\prod_{r,s=1}^{n} (x_r+y_s)} \left(2 + \underset{\eta,\infty}{S} \frac{(y+x_1)\ldots(y+x_n)}{q(y)\;(y-\varphi(y))} \right),$$

which is independant of p.

Mathematics. — *On the theory of simultaneous linear and quadratic representation.* Part III. By F. VAN DER BLIJ. (Communicated by Prof. J. G. VAN DER CORPUT.)

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§ 12. *The ternary system.*

The number of solutions of the system

$$\left.\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= n, \\ x_1 + x_2 + x_3 &= m, \end{aligned}\right\} \quad \ldots\ldots \quad (12.1)$$

can be find by straightforward calculation. Using the results of § 4 we replace the system (12.1) by

$$\left.\begin{aligned} 3(y_1^2 + y_2^2) - (y_1 + y_2)^2 &= 3n - m^2, \\ y_1 + y_2 &\equiv m \pmod 3. \end{aligned}\right\} \quad \ldots\ldots \quad (12.2)$$

Changing this system a little we consider

$$\left.\begin{aligned} y_1^2 + y_1 y_2 + y_2^2 &= \tfrac{1}{2}(3n - m^2) = N, \\ y_1 - y_2 &\equiv m \pmod 3. \end{aligned}\right\} \quad \ldots \quad (12.3)$$

The number of solutions of the equation is (Vid. DICKSON [5])

$$6 \sum_{d \mid N} \left(\frac{d}{3}\right).$$

Theorem 3.

The number of solutions of the system (12.1) *equals*

$$6 \sum_{d \mid N} \left(\frac{d}{3}\right) \text{ if } m \equiv 0 \pmod 3 \text{ and}$$

$$3 \sum_{d \mid N} \left(\frac{d}{3}\right) \text{ if } m \not\equiv 0 \pmod 3.$$

Using the results of § 5 we find that we may replace the system (12.1) by

$$\left.\begin{aligned} z_1^2 + 3 z_2^2 &= 4N, \\ z_1 &\equiv m \pmod 3, \\ z_2 &\equiv z_1 \pmod 2. \end{aligned}\right\} \quad \ldots\ldots \quad (12.4)$$

It may be seen readily that in this way theorem 3 is proved once more.

§ 13. *The system in five variables.*

LIOUVILLE has published many formulae concerning the representation of integers by quaternary quadratic forms. If we use some of his results it is quite easy to calculate the number of solutions of the system

$$\left.\begin{aligned} x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 &= n, \\ x_1 + x_2 + x_3 + x_4 + x_5 &= m, \end{aligned}\right\} \; 5n - m^2 = 2N. \quad \ldots \quad (13.1)$$

By abbreviation we shall write $r(M)$ for the number of solutions of

$$x^2+5(y^2+z^2+t^2)=M. \quad . \quad . \quad . \quad . \quad . \quad (13.2)$$

and $\varrho(M)$ for the number of solutions of

$$x^2+y^2+z^2+5t^2=M. \quad . \quad . \quad . \quad . \quad . \quad (13.3)$$

Further we use the following trivial identity

$$4\{5(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2)-(x_1+x_2+x_3+x_4+x_5)^2\}= \\ =y_1^2+5(y_2^2+y_3^2+y_4^2); \quad . \quad . \quad (13.4)$$

where

$$\left.\begin{aligned} y_1 &= 4x_1-x_2-x_3-x_4-x_5,\\ y_2 &= \quad x_2+x_3-x_4-x_5,\\ y_3 &= \quad x_2-x_3+x_4-x_5,\\ y_4 &= \quad x_2-x_3-x_4+x_5. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (13.5)$$

Now it may be seen readily that the number of solutions of the system (13.1) equals the number of those of

$$y_1^2+5(y_2^2+y_3^2+y_4^2)=8N \quad . \quad . \quad . \quad . \quad . \quad (13.6)$$

which satisfy the congruences

$$\left.\begin{aligned} y_1 \equiv y_2 \equiv y_3 \equiv y_4 \quad & (\mathrm{mod}\ 2),\\ -y_1+y_2+y_3+y_4 \equiv 0 \quad & (\mathrm{mod}\ 4),\\ y_1 \equiv m \quad & (\mathrm{mod}\ 5). \end{aligned}\right\} \quad . \quad . \quad . \quad (13.7)$$

It is quite obvious that the first congruence is satisfied by each solution. If y_1, y_2, y_3 and y_4 are all even, they satisfy the second congruence, if they are all odd, one of each pair of solutions satisfies this congruence. So we deduce that the number of solutions of (13.1) equals, if $m \equiv 0 \pmod 5$

$$\tfrac{1}{2}r(8N)+\tfrac{1}{2}r(2N);$$

and if $m \not\equiv 0 \pmod 5$ it equals

$$\tfrac{1}{4}r(8N)+\tfrac{1}{4}r(2N).$$

First we suppose m to be divisible by 5.

Then we write $N=2^\alpha 5^\beta N_0$, with $\beta>0$ and $(N_0, 10)=1$.

Using $r(5M)=\varrho(M)$ and the formulae of LIOUVILLE [8] we find

$$\left.\begin{aligned} r(2N) &= \varrho(2^{\alpha+1}5^{\beta-1}N_0)=\\ &= \frac{1}{3}\left\{5^\beta+(-1)^{\alpha+1}\left(\frac{N_0}{5}\right)\right\}\{2^{\alpha+2}-(-1)^{\alpha+1}5\}N_0\sum_{d\mid N_0}\left(\frac{d}{5}\right)\frac{1}{d},\\ r(8N) &= \varrho(2^{\alpha+3}5^{\beta-1}N_0)=\\ &= \frac{1}{3}\left\{5^\beta+(-1)^{\alpha+1}\left(\frac{N_0}{5}\right)\right\}\{2^{\alpha+4}-(-1)^{\alpha+1}5\}N_0\sum_{d\mid N_0}\left(\frac{d}{5}\right)\frac{1}{d}. \end{aligned}\right\} \quad (13.8)$$

At last we find that the number of solutions equals

$$5N\left\{1-\left(\frac{2^\alpha N_0}{5}\right)5^{-\beta}\right\}\sum_{d\mid N}\left(\frac{d}{5}\right)\frac{1}{d}. \quad . \quad . \quad . \quad (13.9)$$

If m and N are not divisible by 5 we have

$$r(M) = \tfrac{1}{5}\varrho(M). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.10)$$

Thus the number of solutions is

$$5N \sum_{d \mid N} \left(\frac{d}{5}\right) \frac{1}{d}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.11)$$

So we proved the

Theorem 4.
The number of solutions of the system (13.1) *equals*

$$5AN \sum_{d \mid N} \left(\frac{d}{5}\right) \frac{1}{d},$$

where

$$A = 1 \quad \text{if} \quad N \not\equiv 0 \pmod 5$$

and

$$A = \left\{1 - \left(\frac{N_1}{5}\right) 5^{-\beta}\right\}$$

if

$$5^{\beta} \parallel N, \quad N = 5^{\beta} N_1 \text{ and } \beta > 0.$$

§ 14. *Once more some ternary systems.*

We consider some special values of s_1, s_2 and s_3 such that the number of solutions of the system

$$\left.\begin{aligned} s_1 x_1^2 + s_2 x_2^2 + s_3 x_3^2 &= n, \\ s_1 x_1 + s_2 x_2 + s_3 x_3 &= m, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (14.1)$$

can be found by a straightforward calculation.

First we deduce the formulae according to the theory of § 4. We replace the system by

$$\left.\begin{aligned} (s_1 + s_2 + s_3)(s_1 y_1^2 + s_2 y_2^2) - (s_1 y_1 + s_2 y_2)^2 &= (s_1 + s_2 + s_3) n - m^2, \\ s_1 y_1 + s_2 y_2 &\equiv m \pmod{s_1 + s_2 + s_3}. \end{aligned}\right\} \quad (14.2)$$

Replacing y_2 by $-y_2$ and calculating we find

$$\left.\begin{aligned} s_1(s_2 + s_3) y_1^2 + 2 s_1 s_2 y_1 y_2 + s_2(s_1 + s_3) y_2^2 &= (s_1 + s_2 + s_3) n - m^2, \\ s_1 y_1 - s_2 y_2 &\equiv m \pmod{s_1 + s_2 + s_3}. \end{aligned}\right\} \quad (14.3)$$

If we suppose $s_1 \leq s_2 \leq s_3$, the quadratic form of (14.3) is a reduced one, with the usial definition of reduced quadratic forms.

The determinant of this form equals $s_1 s_2 s_3 (s_1 + s_2 + s_3)$.

Now we consider some examples. First we chose $s_1 = s_2 = 1$, $s_3 = \nu$. We get

$$(\nu + 1) y_1^2 + 2 y_1 y_2 + (\nu + 1) y_2^2 = (\nu + 2) n - m^2, \text{ (with det. } \nu(\nu + 2))$$
$$y_1 - y_2 \equiv m \pmod{(\nu + 2)}.$$

If we take $\nu = 1, 2, 3, 5, 6, 9$ e.g. it is quite easy to calculate the number of solutions; using the well-known theory about the representation of integers by binary quadratic forms.

Further we take $s_1 = 1$, $s_2 = s_3 = \nu$, thus we get

$$2\nu y_1^2 + 2\nu y_1 y_2 + \nu(\nu+1) y_2^2 = (2\nu+1) n - m^2, \text{ (with det. } (2\nu+1)\nu^2)$$
$$y_1 - \nu y_2 \equiv m \pmod{2\nu+1};$$

we may write this

$$2 y_1^2 + 2 y_1 y_2 + (\nu+1) y_2^2 = 2n + \frac{n-m^2}{\nu}, \text{ (with det. } 2\nu+1)$$
$$y_1 - \nu y_2 \equiv m \pmod{2\nu+1}.$$

If we take $\nu = 1, 2, 3, 5, 6, 9$ e.g. we are able to calculate the asked number. At last we chose $s_1 = 1$, $s_2 = 2$, $s_3 = 3$ and we obtain

$$5 y_1^2 + 4 y_1 y_2 + 8 y_2^2 = 6n - m^2, \; y_1 - 2 y_2 \equiv m \pmod 6.$$

§ 15. *Once more some ternary systems* (2).

Now we treat the system of the preceding paragraph with the method of § 5. In five instances we express the number of solutions in a finite sum, using the symbol of JACOBI from the theory of quadratic residues. We deduce the formulae:

$$\frac{s_1}{(s_1+s_2+s_3)(s_2+s_3)} x^2 + \frac{s_2 s_3}{(s_2+s_3)} y^2 = n - \frac{m^2}{s_1+s_2+s_3}, \quad . \quad (15.1)$$
$$x \equiv m \pmod{s_1+s_2+s_3}, \; s_3 y \equiv x \pmod{s_2+s_3},$$

if we chose

$$s_1 = \nu, \; s_2 = s_3 = 1; \quad | \quad s_1 = s_2 = 1, \; s_3 = \nu;$$

we obtain

$$\nu x^2 + (\nu+2) y^2 = 2\{(\nu+2) n - m^2\}, \quad | \quad x^2 + \nu(\nu+2) y^2 = (\nu+1)\{(\nu+2) n - m^2\},$$
$$x \equiv m \pmod{\nu+2}, \quad | \quad x \equiv m \pmod{\nu+2},$$
$$x \equiv y \pmod 2. \quad | \quad x \equiv \nu y \pmod{\nu+1}.$$

First we take $\nu = 2$, thus

$$x^2 + 2y^2 = 4n - m^2;$$
$$x \equiv m \pmod 4, \; x \equiv y \pmod 2.$$

Dividing the number of solutions if m is odd by 2 we may omit the congruences. Introducing $4n - m^2 = 2^a N_0$ where $(N_0, 2) = 1$ and $\varepsilon_1 = 1$ if $m \equiv 0(2)$ and $\varepsilon_1 = \frac{1}{2}$ if $m \equiv 1(2)$ we find:

Theorem 5a. *The number of solutions of the system* (14.1) *with* $s_1 = s_2 = 1$, $s_3 = 2$ *is*

$$2\varepsilon_1 \sum_{d \mid N_0} \left(\frac{-2}{d}\right).$$

After this we consider $\nu = 3$, so we get

$$3x^2 + 5y^2 = 2\{5n - m^2\} = 4N,$$
$$x \equiv m \pmod 5, \; x \equiv y \pmod 2.$$

Let $\varepsilon_2 = 1$ if $m \equiv N \equiv 0 \pmod 5$ and $\varepsilon_2 = \frac{1}{2}$ if m and N are not divisible by 5, further let $4N = 2^\alpha 3^\beta 5^\gamma N_0$ so $\alpha \geq 2$, $(N_0, 30) = 1$.

Theorem 5b. *The number of the solutions of the system* (14.1) *with* $s_1 = s_2 = 1$, $s_3 = 3$ *is*

$$\frac{\alpha - 1}{2} \varepsilon_2 \left\{1 - (-1)^{\alpha+\beta+\gamma} \left(\frac{N_0}{3}\right)\right\}\left\{1 - (-1)^{\alpha+\beta+\gamma} \left(\frac{N_0}{5}\right)\right\} \sum_{d \mid N_0} \left(\frac{d}{15}\right).$$

Now we chose

$$s_1 = s_2 = \nu, \; s_3 = 1. \quad | \quad s_1 = 1, \; s_2 = s_3 = \nu.$$

Thus we obtain

$$\frac{\nu x^2}{(2\nu+1)(\nu+1)} + \frac{\nu y^2}{\nu+1} = \frac{(2\nu+1)n - m^2}{(2\nu+1)}; \quad x \equiv m \pmod{2\nu+1}, \quad y \equiv x \pmod{\nu+1}.$$

$$\frac{\nu^2 x^2}{2\nu(2\nu+1)} + \frac{\nu^2 y^2}{2\nu} = \frac{(2\nu+1)n - m^2}{2\nu+1}; \quad \nu x \equiv m \pmod{2\nu+1}, \quad y \equiv x \pmod 2.$$

We simplify these formulae

$$x^2 + (2\nu+1)y^2 = \frac{\nu+1}{\nu}\{(2\nu+1)n - m^2\}; \quad x \equiv m \pmod{2\nu+1}, \quad y \equiv x \pmod{\nu+1}.$$

$$x^2 + (2\nu+1)y^2 = \frac{2}{\nu}\{(2\nu+1)n - m^2\}; \quad \nu x \equiv m \pmod{2\nu+1}, \quad y \equiv x \pmod 2.$$

First we put $\nu = 2$,

$$x^2 + 5y^2 = 5n - m^2 = 2N, \quad 2x \equiv m \pmod 5.$$

Let $\varepsilon_3 = 1$ if $m \equiv 0(5)$ and $\varepsilon_3 = \frac{1}{2}$ if $m \not\equiv 0(5)$, $N = 2^\alpha 5^\beta N_0$ with $(N_0, 10) = 1$.

Theorem 5c. *The number of solutions of the system* (14.1) *with* $s_1 = 1$, $s_2 = s_3 = 2$ *is*

$$\varepsilon_3 \left(1 - \left(\frac{2^\alpha N_0}{5}\right)\right) \sum_{d \mid N_0} \left(\frac{-5}{d}\right).$$

Now we take $\nu = 3$ and let $\varepsilon_4 = 1$ if $m \equiv 0(7)$ and $\varepsilon_4 = \frac{1}{2}$ if $m \not\equiv 0(7)$.

Theorem 5d. *The number of solutions of the system* (14.1) *with* $s_1 = 1$, $s_2 = s_3 = 3$ *is, if* $7n - m^2 = 6N$,

$$\varepsilon_4 \left\{1 + \left(\frac{N_0}{7}\right)\right\} \sum_{d \mid N_0} \left(\frac{d}{7}\right). \qquad (N = 7^\alpha N_0, (N_0, 7) = 1)$$

At last we consider once more $s_1 = 3$, $s_2 = 2$, $s_3 = 1$. The equation becomes

$$x^2 + 4y^2 = 6n - m^2 = N,$$

accompagnied with the congruences

$$x \equiv m \pmod 6,$$

$$y \equiv x \pmod 3.$$

Let now $N = 2^\alpha N_0$ with odd N_0 and $\varepsilon_5 = 4$ if $m \equiv 0(6)$; $\varepsilon_5 = 2$ if $m \equiv 3(6)$ and $\varepsilon_5 = 1$ if $m \equiv \pm 2(6)$ and $\varepsilon_5 = \frac{1}{2}$ if $(m, 6) = 1$.

Theorem 5e. *The number of the solutions of the system* (14.1) *with* $s_1 = 3$, $s_2 = 2$, $s_3 = 1$ *is*

$$\varepsilon_5 \sum_{d \mid N_0} \left(\frac{-1}{d}\right).$$

There could be given many other examples.

§ 16. *A quaternary system.*

We consider the system (6.1) with $r = 4$. For example we will calculate the number of solutions of this system if $s_1 = 9$, $s_2 = s_3 = s_4 = 1$. So we consider

$$\left.\begin{aligned} 9x_1^2 + x_2^2 + x_3^2 + x_4^2 &= n, \\ 9x_1 + x_2 + x_3 + x_4 &= m. \end{aligned}\right\} \quad \ldots\ldots \quad (16.1)$$

We restrict us to those m which are divisible by 3. We replace this system by

$$\left.\begin{aligned} 12(9x_1^2 + x_2^2 + x_3^2 + x_4^2) - (9x_1 + x_2 + x_2 + x_4)^2 &= 12n - m^2, \\ 9x_1 + x_2 + x_3 + x_4 &= m. \end{aligned}\right\} \quad (16.2)$$

The quadratic equation of (16.2) can be written

$$3(3x_1 - x_2 - x_3 - x_4)^2 + 2(2x_2 - x_3 - x_4)^2 + 6(x_3 - x_4)^2 = 12n - m^2. \quad (16.3)$$

The number of solutions of the system (16.1) equals the number of those solutions of

$$3u_1^2 + 2u_2^2 + 6u_3^2 = 12n - m^2, \quad \ldots\ldots \quad (16.4)$$

to which we can find rational integers x_1, x_2, x_3, x_4 satisfying

$$\begin{aligned} 9x_1 + x_2 + x_3 + x_4 &= m, \\ 3x_1 - x_2 - x_3 - x_4 &= -u_1, \\ 2x_2 - x_3 - x_4 &= -u_2, \\ x_3 - x_4 &= -u_3. \end{aligned}$$

We replace this by the congruences

$$m \equiv u_1 \,(12); \quad u_1 \equiv u_2 \,(3); \quad u_2 \equiv u_3 \,(2). \quad \ldots \quad (16.5)$$

We deduce from (16.4) $u_1 \equiv m\,(2)$ so $u_1^2 \equiv m^2\,(4)$ and $u_2 \equiv u_3\,(2)$ and $3u_1^2 + m^2 \equiv 12n\,(8)$. If m is even, n must be even, else there are no solutions, so we get $m \equiv u_1\,(4)$. If m and n are odd we know that either $m \equiv u_1\,(4)$ or $m \equiv -u_1\,(4)$. Of each two solutions $\pm u_1, u_2, u_3, u_4$ one satisfies the congruence. At last we regard the influence of the remaining congruence restriction $u_1 \equiv u_2 \equiv m \pmod 3$, where we must not forget that the sign of u_1 can be prescribed.

The equation (16.4) can be written as

$$(u_1 + u_2 + u_3)^2 + (u_1 - u_2 + u_3)^2 + (u_1 - 2u_3)^2 = 12n - m^2.$$

Now we research the equation

$$v_1^2 + v_2^2 + v_3^2 = 12n - m^2, \quad \ldots\ldots \quad (16.6)$$

with the restrictions

$$\left.\begin{aligned} v_1+v_2+v_3 &\equiv 3m \pmod 9,\\ v_1-v_2 &\equiv 2m \pmod 6.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (16.7)$$

From $v_1^2+v_2^2+v_3^2+m^2\equiv 0(4)$ we deduce $v_1\equiv v_2\equiv v_3\equiv m(2)$. Now we replace (16.7), m being divisible by 3, by $v_1+v_2+v_3\equiv 0(9)$, $v_1\equiv v_2\,(3)$.

Let now first $v_1\equiv v_2\equiv v_3\equiv 0 \pmod 3$. These are the solutions of

$$w_1^2+w_2^2+w_3^2=\tfrac{1}{9}(12n-m^2)=N, \quad . \quad . \quad . \quad . \quad (16.8)$$

satisfying

$$w_1+w_2+w_3\equiv 0 \pmod 3. \quad . \quad . \quad . \quad . \quad . \quad (16.9)$$

If $N\equiv 1 \pmod 3$ such solutions of (16.8) don't exist.
If $N\equiv 2 \pmod 3$ two of each couple $w_1,\ \pm w_2,\ \pm w_3$ satisfy (16.9).
If $N\equiv 0 \pmod 3$ the number of solutions of (16.8) satisfying (16.9) equals $\frac{1}{4}A_3(N)+\frac{3}{4}A_3(\frac{1}{9}N)$, if $A_3\ (\triangle)$ denote the number of representations of $\triangle$ as a sum of three squares.

Now we consider the solutions with $v_1\equiv v_2\equiv v_3\equiv 1 \pmod 3$, that is to say those solutions of

$$\sum_{\substack{i=1\\ v_i\equiv 1\,(3)}}^{3} v_i^2=12n-m^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.10)$$

which satisfy $v_1+v_2+v_3\equiv 0 \pmod 9$. Let now $v_i=3w_i+1$ then

$$\Sigma v_i=3(\Sigma w_i)+3 \text{ and } \Sigma v_i^2\equiv 6(\Sigma w_i)+3 \pmod 9.$$

If $n\equiv 2 \pmod 3$ we have $12n-m^2\equiv 6(9)$ and $\Sigma v_i^2\equiv 6(9)$ so $\Sigma v_i\equiv 0(9)$.

Then the asked number equals the number of representations of $4n-\frac{1}{3}m^2-1$ as a sum of three (generalized) octagon-numbers. This number $A_3^8(4n-\frac{1}{3}m^2-1)$ has been calculated by STREEFKERK [15], and we will calculate it in part V, too. If $n\not\equiv 2(3)$ there are no solutions of (16.10) satisfying the congruence.

The solutions with $v_1\equiv v_2\equiv v_3\equiv 2 \pmod 3$ can be treated on the analogy of the solutions with $v_1\equiv v_2\equiv v_3\equiv 1 \pmod 3$.

Summary. **Theorem 6.**

The number of solutions of the system (16.1) *equals, if* $n\equiv m\equiv 0 \pmod 2$ *and*

$$\begin{aligned} n&\equiv 1 \pmod 3 && : 0.\\ n&\equiv 2 \pmod 3 && : 2\ A_3^8(4n-\tfrac{1}{3}m^2-1).\\ n&\equiv 0 \pmod 3 \quad N&\equiv 1 \pmod 3 && : 0\ldots(9N=12n-m^2).\\ & \quad N&\equiv 2 \pmod 3 && : \tfrac{1}{2}A_3(N).\\ & \quad N&\equiv \pm 3 \pmod 9 && : \tfrac{1}{4}A_3(N).\\ & \quad N&\equiv 0 \pmod 9 && : \tfrac{1}{4}A_3(N)+\tfrac{3}{4}A_3(\tfrac{1}{9}N).\end{aligned}$$

If $n\equiv m\equiv 1 \pmod 2$ *we must divide these numbers by 2.*

Mathematics. — *Homogeneous Approximations.* By P. MULLENDER. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of January 25, 1947.)

Introduction. Let n be a positive integer greater than 1 and

$$L_k = a_{k1} z_1 + \ldots + a_{kn} z_n \qquad (k = 1, \ldots, n) \ldots \quad (1)$$

a system of n linear forms with variables z and coefficients a satisfying the condition

$$\Delta = \begin{vmatrix} a_{11} \ldots a_{1n} \\ \cdot \quad \cdot \quad \cdot \\ a_{n1} \ldots a_{nn} \end{vmatrix} \neq 0. \quad \ldots \ldots \quad (2)$$

In this note I consider a few inequalities, which can be constructed with these forms, i.a.

$$|L_1 \ldots L_n| \leqq \frac{|\Delta|}{A} \quad \ldots \ldots \quad (3)$$

and

$$|L_1|^\sigma + \ldots + |L_n|^\sigma \leqq \frac{|\Delta|^{\frac{\sigma}{n}}}{B} \text{ with } \sigma > 0. \quad \ldots \quad (4)$$

The problem is: For what values of the constants (here A and B) have these inequalities by arbitrarily chosen coefficients a an infinity of solutions in integral values of the variables z, or (e.g. in the case of the second inequality) only one solution in integral values not all being zero. Those values of the constants I call *allowed*. If in the examples given here an allowed value has been found, it is clear that all lesser values of the constant in question are allowed too. So the problem is completely solved when the upper limit of all allowed values has been determined.

It is possible to approach the solution of the problem from two sides:

A. Bij seeking allowed values of the constants. Then the upper limit must be greater.

B. By seeking non-allowed values. Then the upper limit of the allowed values must be smaller.

The "Geometry of numbers" which has been introduced by MINKOWSKI is mainly directed to solving question *A*. In my dissertation [1]) I have examined results that can be obtained by this method. In the first part of this note I give a summary of these. The second part is devoted to the answering of question *B* for some special problems of approximation.

As will be known, there are various interpretations possible of the above stated problems of approximation. I am going to consider here the following cases:

1. The coefficients a are to be real numbers and the variables z rational integers. Sometimes in this case complex coefficients are also admitted, but then the forms in which those coefficients occur are supposed to appear in conjugate imaginary pairs.

2. The coefficients may be arbitrarily complex and the variables must be integers of a complex algebraic quadratic field $K(i\sqrt{m})$ (m is a positive quadratfrei integer).

In the first case we may speak of "real approximations" and in the second of "complex approximations". In order to treat these cases together I put $\omega = 1$ in the real and $\omega = i\sqrt{m}$ in the complex case. Then we can say in both cases that the variables are to be integers of the field $K(\omega)$.

First Part (Allowed Values).

§ 1. In his "Geometrie der Zahlen" MINKOWSKI deduced from his famous theorem about lattice-points in convex continua a number of results on approximations. BLICHFELDT [2]) improved these results in two cases, the first concerning the simultaneous approximation of $n-1$ real numbers by $n-1$ rational fractions with common divisor and the second concerning the inequality (4) with $\sigma = 2$.

In the first case BLICHFELDT used a generalization of MINKOWSKI's theorem. KOKSMA and MEULENBELD [3]) applied the same ideas to several analogous real and complex problems of approximation. It appears however to be possible to simplify their arguments in several points. In the present dissertation I have carried through these simplifications, by which I have been able to extend the results of KOKSMA and MEULENBELD especially in the complex case.

BLICHFELDT demonstrated the other amelioration of MINKOWSKI's results by a further generalization of the theorem of MINKOWSKI. REMAK [4]) put the argument of BLICHFELDT in an arithmetical form and VAN DER CORPUT and SCHAAKE [5]) extended the result to the case $\sigma > 2$. Besides the last two admitted conjugate complex forms. In my dissertation I have treated the pure complex analogy.

§ 2. Let σ be a positive number and p and q positive integers such that $p + q = n$. Let further be

$$\Lambda^*_{p,q,\sigma} = \frac{q}{\sigma} \frac{(p+q)^{\frac{p+q}{\sigma}}}{(2p)^{\frac{p}{\sigma}}\,(2q)^{\frac{q}{\sigma}}} \int_0^{\frac{2q}{p+q}} (1-x)^{\frac{p}{\sigma}}\, x^{\frac{q}{\sigma}-1} + \frac{p}{\sigma} \int_{\frac{p+q}{2p}}^{1} (1-x)^{\frac{p}{\sigma}} \frac{dx}{x}$$

for $p \geqq q$,

$$\Lambda_{p,q,\sigma} = \Lambda^*_{p,q,\sigma} \text{ for } p \geqq q \text{ and } \Lambda_{p,q,\sigma} = \Lambda^*_{q,p,\sigma} \text{ for } p \leqq q.$$

If $\omega = 1$, I define:

$$\varrho^{(\omega)}_{p,q,\sigma} = \varrho^{(1)}_{p,q,\sigma} = 2^{p+q} \left\{ \frac{\left\{\Gamma\left(1+\frac{1}{\sigma}\right)\right\}^{p+q}}{\Gamma\left(1+\frac{p}{\sigma}\right)\Gamma\left(1+\frac{q}{\sigma}\right)} \cdot \Delta_{p,q,\sigma} \right\}^{\sigma}$$

and

$$\gamma^{(\omega)}_{p,q} = \gamma^{(1)}_{p,q} = 2^{p+q} \cdot \Delta_{p,q,1}.$$

If $\omega = i\sqrt{m}$:

$$\varrho^{(\omega)}_{p,q,\sigma} = \varrho^{(i\sqrt{m})}_{p,q,\sigma} = \left(\frac{\pi\lambda}{4\sqrt{m}}\right)^{\frac{\sigma}{2}(p+q)} \cdot 2^{p+q} \cdot \left\{ \frac{\left\{\Gamma\left(1+\frac{2}{\sigma}\right)\right\}^{p+q}}{\Gamma\left(1+\frac{2p}{\sigma}\right)\Gamma\left(1+\frac{2q}{\sigma}\right)} \cdot \Delta_{p,q,\frac{\sigma}{2}} \right\}^{\frac{\sigma}{2}}$$

and

$$\gamma^{(\omega)}_{p,q} = \gamma^{(i\sqrt{m})}_{p,q} = \left(\frac{\pi\lambda}{4\sqrt{m}}\right)^{\frac{p+q}{2}} \cdot 2^{p+q} \cdot (\Delta_{p,q,1})^{\frac{1}{2}}$$

with $\lambda = 1$ if $m \not\equiv 3 \bmod 4$ and $\lambda = 2$ if $m \equiv 3 \bmod 4$.

I finally call $(z_1, \ldots, z_n)$ a "*lattice-point in* $R^{(\omega)}_n$" if $z_1, \ldots, z_n$ are integers of $K(\omega)$.

Now I have demonstrated the following two theorems:

Theorem 1. *If* $\sigma \geqq 1$, *then*

α. to every given $t > 0$ *there exists at least one lattice-point* $(z_1, \ldots, z_n) \neq (0, \ldots, 0)$ *in* $R^{(\omega)}_n$ *satisfying the inequalities*

$$Q \leqq \frac{2}{t}, \quad P \leqq 2\left\{\frac{t^q\,|\,\Delta\,|^{\sigma}}{\varrho^{(\omega)}_{p,q,\sigma}}\right\}^{\frac{1}{p}} \quad \text{and} \quad P^p \cdot Q^q \leqq \frac{|\,\Delta\,|^{\sigma}}{\varrho^{(\omega)}_{p,q,\sigma}}$$

with

$$P = |L_1|^{\sigma} + \ldots + |L_p|^{\sigma} \quad \text{and} \quad Q = |L_{p+1}|^{\sigma} + \ldots + |L_{p+q}|^{\sigma};$$

β. to every given $\varepsilon > 0$ *there exist an infinity of lattice-points* $(z_1, \ldots, z_n)$ *in* $R^{(\omega)}_n$ *satisfying*

$$P^p \cdot Q^q \leqq \frac{|\,\Delta\,|^{\sigma}}{\varrho^{(\omega)}_{p,q,\sigma}} \quad \text{and} \quad Q < \varepsilon.$$

Theorem 2.

α. To every given $t > 0$ *there exists at least one lattice-point* $(z_1, \ldots, z_n) \neq (0, \ldots, 0)$ *in* $R^{(\omega)}_n$ *satisfying the inequalities*

$$Q \leqq \frac{2}{t}, \quad P \leqq 2\left\{\frac{t^q\,|\,\Delta\,|}{\gamma^{(\omega)}_{p,q}}\right\}^{\frac{1}{p}} \quad \text{and} \quad P^p \cdot Q^q \leqq \frac{|\,\Delta\,|}{\gamma^{(\omega)}_{p,q}}$$

with

$$P = max(|L_1|, \ldots, |L_p|) \text{ and } Q = max(|L_{p+1}|, \ldots, |L_{p+q}|).$$

β. To every given $\varepsilon > 0$ *there exist an infinity of lattice-points* $(z_1, \ldots, z_n)$ *in* $R_n^{(\omega)}$ *satisfying*

$$P^p \cdot Q^q \leqq \frac{|\Delta|}{\gamma_{p,q}^{(\omega)}} \text{ and } Q < \varepsilon.$$

Remarks.

1. In both theorems β is a corollary of α.
2. If we replace $\Lambda_{p,q,\sigma}$ by

$$\overline{\Lambda}_{p,q,\sigma} = \frac{(p+q)^{\frac{p+q}{\sigma}}}{(2p)^{\frac{p}{\sigma}}\,(2q)^{\frac{q}{\sigma}}} \cdot \frac{\Gamma\left(1+\frac{p}{\sigma}\right)\Gamma\left(1+\frac{q}{\sigma}\right)}{\Gamma\left(1+\frac{p+q}{\sigma}\right)}$$

we find the results that would have been obtained by using the non-generalized theorem of MINKOWSKI.

3. If $A_n^{(\omega)}$ indicates the upper limit of all allowed values of A in the inequality (3), then theorem 1 implies

$$A_n^{(\omega)} \geqq p^p\, q^q\, \varrho_{p,q,1}^{(\omega)}. \quad \ldots\ldots\ldots \quad (5)$$

This result is best if $p = n-1$:

$$A_n^{(1)} \geqq (n-1)^{n-1}\, \varrho_{n-1,1,1}^{(1)} > \frac{n^n}{n!}\left\{1+\left(\frac{n-2}{n}\right)^{n+1} \cdot \frac{n^3}{(n-1)(n+1)(n+2)}\right\} \quad (6)$$

and

$$A_n^{IV(m)} \geqq (n-1)^{n-1}\, \varrho_{n-1,1,1}^{(IVm)} > \left.\right.$$

$$> \left[\left(\frac{\pi\lambda}{4\sqrt{m}}\right)^n \cdot \frac{2^n\, n^{2n}}{(2n)!}\left\{1+\left(\frac{n-2}{n}\right)^{2n+1} \cdot \left(\frac{2n-1}{n-1}\right)^2 \cdot \frac{n^2}{(n+1)(2n+1)}\right\}\right]^{\frac{1}{4}}. \quad (7)$$

4. Theorem 1 with $\sigma = 1$ had been proved already for $\omega = 1$ (the real case) by KOKSMA and MEULENBELD.

5. It is possible to deduce from theorem 2 by a special choice of the coefficients of $L_1, \ldots, L_n$:

There exist an infinity of lattice-points $(x_1, \ldots, x_p, y_1, \ldots, y_q)$ *in* $R_{p+q}^{(\omega)}$ *satisfying the inequalities*

$$X = max(|x_1|, \ldots, |x_p|) \geqq 1$$

and

$$|a_{j1}\, x_1 + \ldots + a_{jp}\, x_p - y_j| < \frac{1}{C X^{\frac{p}{q}}} \qquad (j = 1, \ldots, q)$$

with arbitrarily given coefficients a and

$$C = (\gamma_{p,q}^{(\omega)})^{\frac{1}{q}}.$$

If we speak of allowed values of C in the same way as we have done above with reference to the constants in the inequality (3), and if we denote the upper limit of those allowed values by $C_{p,q}^{(\omega)}$, we can also say

$$C_{p,q}^{(\omega)} \geqq (\gamma_{p,q}^{(\omega)})^{\frac{1}{q}} \ldots\ldots\ldots\ldots \quad (8)$$

This includes the results of BLICHFELDT and KOKSMA–MEULENBELD about the simultaneous approximation by rational *) fractions ($p = 1$) as well as the results of the last two about the approximation of zero by a linear form ($q = 1$).

6. In an article, published a few months ago, a very interesting study of the method of BLICHFELDT was given by MORDELL. He stated a few simple theorems in which the fundamental idea of BLICHFELDT was incorporated. In applying these theorems to different inequalities he not only proved more general results than KOKSMA and MEULENBELD, but also other results, e.g. concerning the inequalities (4) with $0 < \sigma < 1$ and

$$\sum_{(r)}^{n} |L_1 \ldots L_r| \leqq k\,|\Delta|^{\frac{r}{n}},$$

where the summation refers to the combinations of $L_1, \ldots, L_n$ taken r at a time. The results of MORDELL however, though more general than the results of KOKSMA and MEULENBELD, provided no improvement of these. MORDELL only considered the real case.

§ 3. I suppose there are now among the forms $L_1, \ldots, L_n$ s pairs with conjugate complex coefficients and $r = n - 2s$ with real coefficients. Let $B_{n,s,\sigma}$ be the upper limit of the values of B in the inequality (4) allowed in this case. Then VAN DER CORPUT and SCHAAKE proved:

Theorem 3.

$$B_{n,s,\sigma} \geqq \frac{2}{\left(1+\frac{n}{\sigma}\right)^{\frac{\sigma}{n}}} \left\{ \frac{\left\{\Gamma\left(1+\frac{1}{\sigma}\right)\right\}^{r} \cdot \left\{\Gamma\left(1+\frac{2}{\sigma}\right)\right\}^{s}}{2^{\frac{2s}{\sigma}} \cdot \left(\frac{2}{\pi}\right)^{s} \cdot \Gamma\left(1+\frac{n}{\sigma}\right)} \right\}^{\frac{\sigma}{n}} \quad \text{for } \sigma \geqq 2.$$

I myself have treated the case in which all the forms $L_1, \ldots, L_n$ may have arbitrarily complex coefficients, in which case the variables are to be integers of $K(i\sqrt{m})$. Let $B_{n,\sigma}^{(i\sqrt{m})}$ indicate the upper limit of the values of B in the inequality (4) allowed in that case. Then I have shown

*) Rational relative to $K(\omega)$.

Theorem 4.

$$B_{n,\sigma}^{(i\sqrt{m})} \geqq \frac{2}{\left(1+\frac{2n}{\sigma}\right)^{\frac{\sigma}{2n}}} \left(\frac{\pi\lambda}{4\sqrt{m}}\right)^{\frac{\sigma}{2}} \left\{ \frac{\left\{\Gamma\left(1+\frac{2}{\sigma}\right)\right\}^n}{\Gamma\left(1+\frac{2n}{\sigma}\right)} \right\}^{\frac{\sigma}{2n}} \qquad \text{for } \sigma \geqq 2.$$

Remarks.

7. Theorem 3 is for $2 < \sigma \leqq n$ an amelioration of the result of MINKOWSKI:

$$B_{n,s,\sigma} \geqq \left\{ \frac{\left\{\Gamma\left(1+\frac{1}{\sigma}\right)\right\}^r \cdot \left\{\Gamma\left(1+\frac{2}{\sigma}\right)\right\}^s}{2^{\frac{2s}{\sigma}} \cdot \left(\frac{2}{\pi}\right)^s \cdot \Gamma\left(1+\frac{n}{\sigma}\right)} \right\}^{\frac{\sigma}{n}} \qquad \text{for } \sigma \geqq 1. \quad . \quad . \quad (9)$$

The complex analogy of this result of MINKOWSKI is

$$B_{n,\sigma}^{(i\sqrt{m})} \geqq \left(\frac{\pi\lambda}{4\sqrt{m}}\right)^{\frac{\sigma}{2}} \left\{ \frac{\left\{\Gamma\left(1+\frac{2}{\sigma}\right)\right\}^n}{\Gamma\left(1+\frac{2n}{\sigma}\right)} \right\}^{\frac{\sigma}{2n}} \qquad \text{for } \sigma \geqq 1. \quad . \quad . \quad (10)$$

For $2 < \sigma < 2n$ the limit given in theorem 4 is higher than that given here.

8. From theorem 3 may be deduced

$$A_n^{(1)} \geqq \frac{\left(\frac{\pi n}{2}\right)^{\frac{n}{2}}}{\Gamma\left(2+\frac{n}{2}\right)} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and in the same way from theorem 4

$$A_n^{(i\sqrt{m})} \geqq \left(\frac{\pi\lambda}{4\sqrt{m}}\right)^{\frac{n}{2}} \left\{ \frac{(2n)^n}{(n+1)!} \right\}^4 . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

For $n \geqq 39$ (11) gives a higher limit than (6) and for $n \geqq 3$ (12) a higher limit than (7).

9. It is also possible to deduce from the theorems 3 and 4 a partial amelioration of theorem 1:

If $\sigma \geqq 2$. one may replace $\Lambda_{p,q,\sigma}$ in theorem 1 by

$$\overline{\overline{\Lambda}}_{p,q,\sigma} = \frac{(p+q)^{\frac{p+q}{\sigma}}}{p^{\frac{p}{\sigma}}\, q^{\frac{q}{\sigma}}} \cdot \frac{\Gamma\left(1+\frac{p}{\sigma}\right)\Gamma\left(1+\frac{q}{\sigma}\right)}{\Gamma\left(2+\frac{p+q}{\sigma}\right)}$$

or

$$\overline{\overline{\Lambda}}_{p,q,\sigma} = \frac{2^{\frac{p+q}{\sigma}}}{1+\frac{p+q}{\sigma}}\,\overline{\Lambda}_{p,q,\sigma}.$$

This result is only better than that of theorem 1 if $p+q$ is sufficiently greater than σ.

10. From the theorems mentioned in which the parameter σ occurs it is easy to deduce results for the case $0<\sigma<1$ or $0<\sigma<2$ by using the inequality

$$(G_1+\ldots+G_l)^\tau \leqq l^{\tau-1}(G_1^\tau+\ldots+G_l^\tau)$$

with non-negative $G_1, \ldots, G_l$ and $\tau \geqq 1$. These results are in some cases better than the results already found.

Second Part (Non-allowed values).

§ 4. In my dissertation I have introduced the following system of inequalities

$$X = max(|x_1|, \ldots, |x_p|) \geqq 1$$

and

$$|a_{j1}x_1+\ldots+a_{jp}x_p-y_j| < \frac{1}{C X^{\frac{p}{q}}} \qquad (j=1,\ldots,q)$$

with arbitrarily given coefficients a. As I have remarked already (§ 2) this system includes the simultaneous approximation by rational fractions ($p=1$) as well as the approximation of zero by a single linear form ($q=1$). The question is now to determine non-allowed values of the constant C.

FURTWÄNGLER [7]) demonstrated

$$C_{1,q}^{(1)} \leqq (D_{q+1})^{\frac{1}{2q}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

in which D_{q+1} indicates the minimum of the absolute values of the discriminants of all real algebraic fields of degree $q+1$ relative to $K(1)$.

HOFREITER [8]) proved a complex analogy which only holds good for simple *) fields $K(i\sqrt{m})$:

$$C_{1,q}^{(i\sqrt{m})} \leqq (D_{q+1}^{(i\sqrt{m})})^{\frac{1}{2q}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

in which $D_{q+1}^{(i\sqrt{m})}$ indicates the minimum of the absolute values of the discriminants of all algebraic fields of degree $q+1$ relative to $K(i\sqrt{m})$.

*) A simple field is a field in which factorization in prime numbers is unambignuous.

These results concern only the case $p = 1$, which is the simultaneous approximation by rational fractions. A general result concerning the approximation by a single linear form ($q = 1$) has not yet been found. Here I restrict myself to the case $q = 1$, $p = 2$. I will demonstrate

$$C_{2,1}^{(1)} \leqq (D_3)^{\frac{1}{3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and in the case of a simple field $K(i\sqrt{m})$

$$C_{2,1}^{(i\sqrt{m})} \leqq (D_3^{(i\sqrt{m})})^{\frac{1}{3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Since according to FURTWÄNGLER $D_3 = 23$, this implies $C_{2,1}^{(1)} \leqq 23$.

§ 5. I suppose $n = 3$, so I consider the form

$$L_k = a_{k1} z_1 + a_{k2} z_2 + a_{k3} z_3 \qquad (k = 1, 2, 3) \quad . \quad . \quad . \quad (17)$$

with coefficients a satisfying the condition

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

and the inequality

$$|L_1 L_2 L_3| \leqq \frac{|\Delta|}{A} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

I suppose that in the real case L_1 has real coefficients and L_2 and L_3 are either real or form a conjugate complex pair. Then the following general theorem is true:

Theorem 5. *) *If C is not an allowed value of A in the inequality* (19), *then*

$$C_{2,1}^{(\omega)} \leqq C.$$

First I prove that this theorem implies (15) and (16).

Let a_{11}, a_{12}, a_{13} in the real case be the base of a real algebraic field of degree 3 relative to $K(1)$ and a_{21}, a_{22}, a_{23} and a_{31}, a_{32}, a_{33} the conjugate bases of the conjugate fields. Then

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \sqrt{D} \neq 0,$$

where D is the discriminant of the field.

If z_1, z_2 and z_3 are integers of $K(1)$ not all zero, then

$$a_{11} z_1 + a_{12} z_2 + a_{13} z_3$$

*) This theorem holds good in the complex case for any field $K(i\sqrt{m})$ whether simple or not.

is an integer of the field and

$$|\,norm\,(a_{11}\,z_1 + a_{12}\,z_2 + a_{13}\,z_3)\,| \geqq 1.$$

That means here: For any lattice-point $(z_1, z_2, z_3) \neq (0,0,0)$ of $R_3^{(1)}$ the following inequality is true

$$|\,L_1\,L_2\,L_3\,| \geqq \frac{|\,\Delta\,|}{|\sqrt{D}|}.$$

This implies $\sqrt{D}$ is not an allowed value of A and so according to theorem 5 : $C_{2,1}^{(1)} \leqq D$. Since the field is chosen arbitrarily, this implies (15).

According to HOFREITER an algebraic field of degree n relative to $K(i\sqrt{m})$ always has a base if $K(i\sqrt{m})$ is simple. That means, there are in the field numbers $\omega_1, \ldots, \omega_n$ such that all numbers of the field are represented by

$$\alpha = u_1\,\omega_1 + \ldots + u_n\,\omega_n$$

with numbers $u_1, \ldots, u_n$ of $K(i\sqrt{m})$, α being an integer of the field if $u_1, \ldots, u_n$ are all integers of $K(i\sqrt{m})$. HOFREITER has further proved that the norm of an integer of the field relative to $K(i\sqrt{m})$ is an integer of $K(i\sqrt{m})$ and consequently in absolute value at least 1. So the same argument as in the real case is valid here and theorem 5 also implies (16).

§ 6. *Proof of theorem 5.* I may restrict myself to the real case, because in the complex case exactly the same argument can be used as in the real one if in the real case all coefficients a are taken real too.

a. I suppose C is not an allowed value of A in (19). That means

$$|\,L_1\,L_2\,L_3\,| \leqq \frac{|\,\Delta\,|}{C} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

has only a finite number of solutions. Then it is clear that none of the coefficients a can be zero. I put

$$\frac{a_{11}}{a_{13}} = a_1 \quad \text{and} \quad \frac{a_{12}}{a_{13}} = a_2.$$

If $C' < C_{2,1}^{(1)}$, then there exist an infinity of solutions in integers z_1, z_2 and z_3 of $K(1)$ *) of the inequality

$$a_1\,z_1 + a_2\,z_2 + z_3 < \frac{1}{C'\,X^2}$$

with

$$X = max\,(|\,z_1\,|, |\,z_2\,|) \geqq 1$$

Among those solutions there is certainly one with arbitrarily large X which is not a solution of (20). Let z_1, z_2, z_3 be such a solution, then I put

$$a_1\,z_1 + a_2\,z_2 + z_3 = \frac{\vartheta}{C'\,X^2}$$

*) $K(i\sqrt{m})$ in the complex case.

with $-1 < \vartheta < 1$. Now

$$z_3 = \frac{\vartheta}{C' X^2} - \frac{a_{11} z_1 + a_{12} z_2}{a_{13}}.$$

So

$$L_1 = a_{11} z_1 + a_{12} z_2 + a_{13} z_3 = \frac{\vartheta a_{13}}{C' X^2},$$

$$L_2 = a_{21} z_1 + a_{22} z_2 + a_{23} z_3 = \left(a_{21} - a_{23} \frac{a_{11}}{a_{13}}\right) z_1 + \left(a_{22} - a_{23} \frac{a_{12}}{a_{13}}\right) z_2 + \frac{\vartheta a_{23}}{C' X^2},$$

$$L_3 = a_{31} z_1 + a_{32} z_2 + a_{33} z_3 = \left(a_{31} - a_{33} \frac{a_{11}}{a_{13}}\right) z_1 + \left(a_{32} - a_{33} \frac{a_{12}}{a_{13}}\right) z_2 + \frac{\vartheta a_{33}}{C' X^2}.$$

Let further be $z_1 = \vartheta_1 X$ and $z_2 = \vartheta_2 X$ with $|\vartheta_1| \leqq 1$ and $|\vartheta_2| \leqq 1$ and

$$a_{21} a_{13} - a_{23} a_{11} = b_{21} \quad , \quad a_{22} a_{13} - a_{23} a_{12} = b_{22},$$
$$a_{31} a_{13} - a_{33} a_{11} = b_{31} \quad , \quad a_{32} a_{13} - a_{33} a_{12} = b_{32}.$$

Then

$$L_2 = \frac{(\vartheta_1 b_{21} + \vartheta_2 b_{22}) X}{a_{13}} + \frac{\vartheta a_{23}}{C' X^2} \text{ and } L_3 = \frac{(\vartheta_1 b_{31} + \vartheta_2 b_{32}) X}{a_{13}} + \frac{\vartheta a_{33}}{C' X^2}.$$

Since z_1, z_2, z_3 do not form a solution of (20) and X is arbitrarily large,

$$\frac{|\Delta|}{C} \leqq |L_1 L_2 L_3| < \frac{|\vartheta_1 b_{21} + \vartheta_2 b_{22}| \cdot |\vartheta_1 b_{31} + \vartheta_2 b_{32}|}{|a_{13}| \cdot C'} + \varepsilon$$

with arbitrarily small positive ε.

The number of solutions of (20) does not change if one column of the determinant (18) multiplied by an integer of $K(1)$ *) is added to another. I now assert that

in this way it is possible to transform the determinant so as to make valid the inequality

$$\frac{|\vartheta_1 b_{21} + \vartheta_2 b_{22}| \cdot |\vartheta_1 b_{31} + \vartheta_2 b_{32}|}{|a_{13}|} < |\Delta| (1 + \varepsilon')$$

with arbitrarily small positive ε', wathever the values of ϑ_1 and ϑ_2 may be.

After that transformation the argument remains true. So

$$\frac{|\Delta|}{C} < \frac{|\Delta|}{C'} (1 + \varepsilon') + \varepsilon$$

*) $K(i\sqrt{m})$ in the complex case.

or $C' \leq C$. This holds good for any $C' < C_{2,1}^{(1)}$. Consequently also $C_{2,1}^{(1)} \leq C$.

β. Now the above stated assertion remains to be proved. I put

$$a_{21}\,a_{12} - a_{22}\,a_{11} = b_2 \text{ and } a_{31}\,a_{12} - a_{32}\,a_{11} = b_3$$

and I transform the determinant (18) in the following way

$$a'_{j1} = a_{j1} + k\,a_{j2}\ ,\ a'_{j2} = a_{j2}\ ,\ a'_{j3} = a_{j3} + l\,a_{j2} \qquad (j = 1, 2, 3)$$

where k and l are integers of $K(1)$. Then $(j = 2, 3)$

$$\begin{aligned} b'_{j1} &= a'_{j1}\,a'_{13} - a'_{j3}\,a'_{11} = \\ &= (a_{j1} + k\,a_{j2})\,(a_{13} + l\,a_{12}) - (a_{j3} + l\,a_{j2})\,(a_{11} + k\,a_{12}) = \\ &= b_{j1} + k\,b_{j2} + l\,b_j. \\ b'_{j2} &= a'_{j2}\,a'_{13} - a'_{j3}\,a'_{12} = \\ &= a_{j2}\,(a_{13} + l\,a_{12}) - (a_{j3} + l\,a_{j2})\,a_{12} = \\ &= b_{j2} \\ b'_j &= a'_{j1}\,a'_{12} - a'_{j2}\,a''_{11} = \\ &= (a_{j1} + k\,a_{j2})\,a_{12} - a_{j2}\,(a_{11} + k\,a_{12}) = \\ &= b_j. \end{aligned}$$

First Case *). (*All coefficients a are real.*)

As will be known, it is possible to find an infinity of integers k and l of $K(1)$, for which

$$|k\,b_{32} + l\,b_3| \ldots\ldots\ldots\ldots (21)$$

is arbitrarily small. If

$$|k\,b_{22} + l\,b_2| \ldots\ldots\ldots\ldots (22)$$

for those values of k and l (29) always were to be smaller than a given number, k and l would also have to be limited, since

$$\begin{vmatrix} b_{32} & b_3 \\ b_{22} & b_2 \end{vmatrix} = a_{12} \cdot \Delta \neq 0. \ldots\ldots\ldots\ldots (23)$$

That is not possible. In other words k and l may be chosen so as to make (22) arbitrarily large and (21) arbitrarily small. Then $|b'_{21}|$ will also get arbitrarily large whereas the difference between b'_{31} and b_{31} will become arbitrarily small.

In the same manner the determinant may be transformed by a second transformation

$$a''_{j1} = a'_{j1}\ ,\ a''_{j2} = a'_{j2} + m\,a'_{j1}\ ,\ a''_{j3} = a''_{j3} + n\,a'_{j1} \qquad (j = 1, 2, 3)$$

so as to make

$$|b''_{32}| = |b'_{32} + m\,b'_{31} + n\,b'_3|$$

---

*) This case corresponds with the complex case.

arbitrarily large and the difference between b''_{22} and $b_{22} = b'_{22}$, which is

$$|b''_{22} - b'_{22}| = |m b'_{21} + n b'_2|$$

arbitrarily small. Since the second transformation leaves b'_{21} and b'_{31} unchanged, this implies

$$min(|b''_{21}|, b''_{32}|) > A \cdot max(|b''_{22}|, b''_{31}|)$$

with arbitrarily large positive A. Consequently

$$|\vartheta_1 b''_{21} + \vartheta_2 b''_{22}| \cdot |\vartheta_1 b''_{31} + \vartheta_2 b''_{32}| < |b''_{21}| \cdot |b''_{32}| \cdot \left(1 + \frac{1}{A}\right)^2 <$$

$$< |b''_{21} b''_{32} - b''_{22} b''_{31}| \frac{\left(1 + \frac{1}{A}\right)^2}{1 - \frac{1}{A^2}} <$$

$$< |b''_{21} b''_{32} - b''_{22} b''_{31}| (1 + \varepsilon''),$$

with arbitrarily small positive ε''. Since

$$b''_{21} b''_{32} - b''_{22} b''_{31} = a''_{13} \Delta,$$

this implies the assertion.

Second Case. (*Not all coefficients a are real.*)

In this case I take as second transformation:

$$a''_{j1} = a'_{j1} \ , \ a''_{j2} = a'_{j2} \ , \ a''_{j3} = a'_{j3} + m a'_{j1} \qquad (j = 1, 2, 3)$$

where m is an integer of $K(1)$. Then $(j = 2, 3)$

$$b''_{j1} = b'_{j1} = b_{j1} + k b_{j2} + l b_j \ , \ b''_{j2} = b'_{j2} + m b'_j = b_{j2} + m b_j.$$

Since a_{2j} and a_{3j} $(j = 1, 2, 3)$ are conjugate complex numbers and a_{1j} real, also b_{2j} and b_{3j} $(j = 1, 2)$ and b_2 and b_3 are conjugate complex numbers. This remains the case after the transformation.

Suppose

$$b''_{21} = p_1 + i q_1 \quad \text{and} \quad b''_{22} = p_2 + i q_2 \ . \ . \ . \ . \ . \qquad (24)$$

with real p_1, p_2, q_1 and q_2. Then

$$b''_{31} = p_1 - i q_1 \quad \text{and} \quad b''_{32} = p_2 - i q_2. \ . \ . \ . \ . \ . \qquad (25)$$

Now

$$|\vartheta_1 b''_{21} + \vartheta_2 b''_{22}| \cdot |\vartheta_1 b''_{31} + \vartheta_2 b''_{32}| =$$
$$= (\vartheta_1 p_1 + \vartheta_2 p_2)^2 + (\vartheta_1 q_1 + \vartheta_2 q_2)^2 =$$
$$= \{\vartheta_1 p_1 - \vartheta_2 q_1 + \vartheta_2 (p_2 + q_1)\}^2 + \{\vartheta_1 q_1 + \vartheta_2 p_1 - \vartheta_2 (p_1 - q_2)\}^2$$

and

$$|a''_{13}\,\Delta| = |b''_{21}\,b''_{32} - b''_{22}\,b''_{31}| =$$
$$= 2\,|p_1\,q_2 - p_2\,q_1| =$$
$$= 2\,|p_1^2 + q_1^2 - p_1\,(p_1 - q_2) - q_1\,(p_2 + q_1)|.$$

The assertion is true if it proves to be possible to chose k, l and m so as to make p_1, q_1 or p_1 and q_1 arbitrarily large whereas $p_2 + q_1$ and $p_1 - q_2$ remain limited.

From (24) and (25) we get

$$2\,p_1 = (b_{21} + b_{31}) + k\,(b_{22} + b_{32}) + l\,(b_2 + b_3)$$
$$2\,q_1 = i\,(b_{31} - b_{21}) + k\,i\,(b_{32} - b_{22}) + l\,i\,(b_3 - b_2)$$
$$2\,(p_1 - q_2) = (b_{21} + b_{31}) + i\,(b_{22} - b_{32}) + k\,(b_{22} + b_{32}) + l\,(b_2 + b_3) + m\,i\,(b_2 - b_3)$$
$$2\,(p_2 + q_1) = (b_{22} + b_{32}) + i\,(b_{31} - b_{21}) + k\,i\,(b_{32} - b_{22}) + l\,i\,(b_3 - b_2) + m\,(b_2 + b_3)$$

There are an infinity of integers k, l, m, for which both forms

$$|k\,(b_{22} + b_{32}) + l\,(b_2 + b_3) + m\,i\,(b_2 - b_3)|,$$
$$|k\,i\,(b_{32} - b_{22}) + l\,i\,(b_3 - b_2) + m\,(b_2 + b_3)|$$

are arbitrarily small, as both forms have real coefficients. If both forms

$$|k\,(b_{22} + b_{32}) + l\,(b_2 + b_3)|$$
$$|k\,i\,(b_{32} - b_{22}) + l\,i\,(b_3 - b_2)|$$

for those values of k, l and m always were to be smaller than a given number, k, l and m would also have to be limited. For either

$$\begin{vmatrix} (b_{22} + b_{32}) & (b_2 + b_3) & i\,(b_2 - b_3) \\ i\,(b_{32} - b_{22}) & i\,(b_3 - b_2) & (b_2 + b_3) \\ (b_{22} + b_{32}) & (b_2 + b_3) & 0 \end{vmatrix} = 2\,(b_3 - b_2)\,a_{12}\,\Delta \neq 0$$

or

$$\begin{vmatrix} (b_{22} + b_{32}) & (b_2 + b_3) & i\,(b_2 - b_3) \\ i\,(b_{32} - b_{22}) & i\,(b_3 - b_2) & (b_2 + b_3) \\ i\,(b_{32} - b_{22}) & i\,(b_3 - b_2) & 0 \end{vmatrix} = 2\,i\,(b_2 + b_3)\,a_{12}\,\Delta \neq 0,$$

since $b_3 - b_2 = b_3 + b_2 = 0$ implies $b_2 = b_3 = 0$ contrary to (23).

LITERATURE.

1. Toepassing van de meetkunde der getallen op ongelijkheden in $K(1)$ en $K(i\sqrt{m})$, Dissertation, Amsterdam (1945).

2, 3, 4, 5. For references to litearture see my dissertation.

6. Lattice points in some n-dimensional non-convex regions. Proc. Kon. Ned. Akad. v. Wetensch., **49**, 773—792 (1946).

7. Über die simultane Approximation von Irrationalzahlen. Math. Ann. **96**, 169—175 (1927), **99**, 71—83 (1928).

8. Über die Approximation von komplexen Zahlen. Mh. f. Math. u. Ph. **42**, 401—416 (1935).

Mathematics. — *On the elastic stability of sandwich plates.* II. By P. P. BIJLAARD. (Communicated by Prof. F. K. TH. VAN ITERSON.)

(Communicated at the meeting of January 25, 1947.)

As this case is very important in the design of aircrafts, we will examine here also a sandwich plate, submitted to the action of shearing forces $S = h\tau$, uniformly distributed along the edges $y = \pm c$ (fig. 1), c being half of the breadth b of the plate. The plate is assumed to be infinitely long in X-direction. We still consider antimetric buckling.

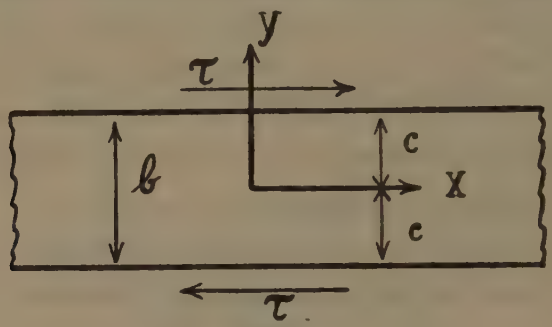

Fig. 1.

According to our preceding publications [1] [2], the critical shearing force per unit length for sandwich plates will be given by the equation [3])

$$S_{cr} = S_0 + (S_m^{-1} + S_s^{-1})^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which S_0 is the joint buckling stress of the outer plates (the faces), S_m is the buckling force if the modulus of rigidity G of the intermediate layer (the core) — and also of the faces — with respect to shearing stresses τ_{zx} and τ_{zy} is assumed to be infinite and S_s is obtained by assuming only the modulus of rigidity G of the core with respect to τ_{zx} and τ_{zy} to have a finite value. Again in the cases S_m and S_s the proper rigidity to bending of the faces is neglected; therefore its influence is added separately as S_0.

In order to compute S_m and S_0 we dispose of the solution as given for the same case for isotropic plates by SOUTHWELL and SKAN [4]), which we will have to consider, however, in some detail, in order to check the similarity of the deflection surfaces in the cases S_m and S_s. The differential equation for isotropic plates being [4])

$$D\left\{\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4}\right\} - 2S_m\frac{\partial^2 w}{\partial x\,\partial y} = 0 \quad . \quad . \quad . \quad (2)$$

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[1]) BIJLAARD. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, Nr. 10 (1946).
[2]) BIJLAARD. The same. Nr. 1 (1947).
[3]) Comp. eq. (10), lit. footnote 2.
[4]) SOUTHWELL and SKAN. Proc. Royal Soc. London, series A, vol. 105 (1924).

SOUTHWELL and SKAN assume

$$w = Y e^{i\alpha x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Y being a function of y only, so that the half wave length of the deflection surface in X-direction is $L = \pi/a$. After substitution of eq. (3) in eq. (2) they insert in the thus obtained ordinary differential equation

$$Y = e^{i\beta y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and consequently get

$$Y = C_1 e^{i\beta_1 y} + C_2 e^{i\beta_2 y} + C_3 e^{i\beta_3 y} + C_4 e^{i\beta_4 y}. \quad . \quad . \quad . \quad (5)$$

This expression is substituted in the boundary conditions at $y = \pm c$, yielding, on eliminating C_1, C_2, C_3 and C_4, a determinantal equation, which represents the buckling condition and in which β_1, β_2, β_3 and β_4 are functions of the shearing force S. The combination of equations, obtained in this way, will not be repeated here, as the authors calculated already all the data we need for the present problem. For, besides the critical shearing force, they communicated also the shearing forces, belonging to other wavelengths than that, yielding the minimum shearing force, for simply supported as well as for clamped edges $y = \pm c$. As with sandwich plates, owing to the deformation by shearing of the core, the minimum resistance is obtained with shorter wave lengths than with isotropic plates, we can use these results directly for our problem. If we write

$$S_m = k_s D/b^2 \quad \text{or} \quad S_m = k_c D/b^2 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

for simply supported and clamped edges $y = \pm c$ respectively, the results of SOUTHWELL and SKAN are given in fig. 2 by the curves for k_s and k_c.

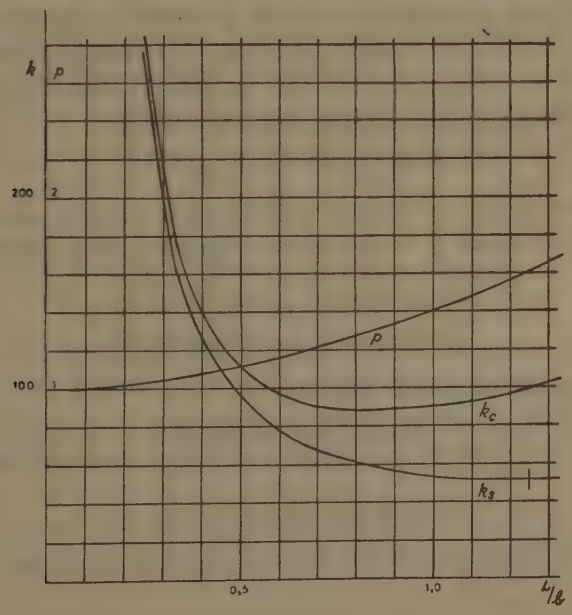

Fig. 2.

In eqs. (6) value D is given by eq. (2) of our preceding paper [2]). Value S_0 is obtained by equaling D to $E'_1 h^3/6$.

In our case S_s we know [5]), that the forces S_s, acting on an element $Hdxdy$, have a component $2 S_s \partial^2 w/\partial x \partial y$ in Z-direction, so that, according to figs. 1 and 2 of our preceding paper [2]), the equilibrium in Z-direction requires

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + 2 S_s \frac{\partial^2 w}{\partial x \partial y} = 0$$

whence, with eqs. (6) and (7) of that paper, we get the differential equation for case S_s, being

$$\frac{(t+h)^2}{t} G \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + 2 S_s \frac{\partial^2 w}{\partial x \partial y} = 0 \quad . \quad . \quad . \quad . \quad (7)$$

Inserting here also eq. (3), we obtain

$$\frac{(t+h)^2}{t} G (Y'' - \alpha^2 Y) + 2 i \alpha S_s Y' = 0 \quad . \quad . \quad . \quad . \quad . \quad (8)$$

primes denoting differentiation with respect to y. Assuming furthermore Y to be given by eq. (4), eq. (8) yields

$$\beta^2 + \frac{2 S_s t}{(t+h)^2 G} \alpha \beta + \alpha^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

whence

$$\beta_{5.6} = \alpha \left[- \frac{t}{(t+h)^2} \frac{S_s}{G} \pm \sqrt{\left(\frac{t}{(t+h)^2} \frac{S_s}{G} \right)^2 - 1} \right] . \quad . \quad . \quad (10)$$

Owing to the constraints along the boundaries $y = \pm b/2$ as well as by the principal tensile stresses, the critical principal compressive stress will be higher than the critical compressive stress for a strip, so that, according to eq. (33) of our first paper on this subject [1]), also S_s is higher than $G(t+h)^2/t$. Hence β_5 and β_6 are both real, whilst

$$Y = C_5 e^{i \beta_5 y} + C_6 e^{i \beta_6 y} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

With simply supported as well as with clamped edges $y = \pm b/2 = \pm c$, the boundary conditions $Y = 0$ at these edges may be expressed as follows

$$\left. \begin{aligned} C_5 \cos \beta_5 c + C_6 \cos \beta_6 c = 0 \\ C_5 \sin \beta_5 c + C_6 \sin \beta_6 c = 0 \end{aligned} \right\} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

whence we get the buckling condition

$$\tan \beta_6 b/2 = \tan \beta_5 b/2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

or

$$\beta_6 b/2 = \beta_5 b/2 + n \pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

n being integral. With $n = 0$ eqs. (12) would yield $C_6 = - C_5$, by which,

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[5]) TIMOSHENKO. Theory of elastic stability, p. 304.

according to eq. (11) $w = 0$, so that no buckling occurs. If n differs from zero, we obtain from eqs. (10) and (14)

$$S_s = \pm \frac{(t+h)^2}{t} G \sqrt{1 + \frac{n^2 \pi^2}{a^2 b^2}}$$

being as small as possible with $n^2 = 1$. Expressing a in the half wave length $L = \pi/a$ in X-direction, we find the buckling force per unit length

$$S_s = \pm \frac{(t+h)^2}{t} G \sqrt{1 + \frac{L^2}{b^2}} = \pm p \frac{(t+h)^2}{t} G. \quad . \quad . \quad . \quad (15)$$

In fig. 2 value p is plotted too as a function of L/b, the scale being one hundred times larger than that for values k.

With eqs. (1), (6) and (15) and fig. 2 the critical stress S_{cr} may be easily computed. Let us, for example, examine a sandwich plate with thicknesses $h = 0$, 1 cm and $t = 1$ cm of faces and core respectively, and supposing the moduli of elasticity of the plywooden faces and the onazote core to be $E_1 = 110.000$ kg/cm² and $E = 304$ kg/cm² respectively, with $\nu_1 = \nu = 0{,}25$. The breadth b of the plate is 50 cm and it is supposed to be simply supported at the edges $y = \pm b/2$. With the first equation (6) and eq. (15) we find, computing value D in eq. (6) according to eq. (2) of our preceding paper [2]) in order to obtain S_m and equaling D to $E'_1 h^3/6$ for S_0

$$S_m = 2{,}851\, k_s \quad , \quad S_0 = 0{,}0078\, k_s \quad , \quad S_s = 147\, p$$

all expressed in kg/cm, so that

$$S_{cr} = \left(0{,}0078\, k_s + \frac{1}{1/2{,}851\, k_s + 1/147\, p} \right) \text{kg/cm} \quad . \quad . \quad . \quad (16)$$

With the curves in fig. 2 we can calculate S_{cr} for various ratios L/b, yielding with slide rule accuracy, in kg/cm

L/b	1,25	1,00	0,95	0,90	0,67
S_o	0,41	0,43	0,44	0,45	0,54
S_m	150,3	155,5	158,1	162,5	198,8
S_s	235	208	202,8	197	176,5
S_{cr}	92,0	89,4	89,4	89,45	94,0

whence $S_{cr} = 89{,}4$ kg/cm and, the entire shear supposed to be taken by the faces only, $\tau_{cr} = 447$ kg/cm². The half wave length of the deflection surface in X-direction is about $0{,}95\, b$, whilst according to SEYDEL [6]) with isotropic plates the minimum critical shear occurs with a half wave length $L = 1{,}25\, b$.

The fact that the deflection surfaces for cases S_m and S_s have not the same form, causes S_{cr} to be smaller than the critical shearing force

[6]) SEYDEL. Ingenieur Archiv. 1933, p. 169.

belonging to the joint deflection surface, provided of course, that in both cases the same assumptions are made, as for example the neglection of the influence of stresses σ_z.

Also for the cases as treated in our preceding paper [2]), it will save work if curves for k and p are plotted as functions of the ratio $L/b = a/mb$, from which $P_{xm} = kD/b^2$ and $P_{xs} = p(t+h)^2G/t$ may be found, whence P_x is computed with eq. (10) of that paper, in the same way as shown here for S_{cr}.

The splitting off of the cases P_{xo} in the latter equation and S_o in eq. (1) of this paper, which device was originally applied in deriving our formula for built up columns as given in footnote 7 of our first paper [1]), enabled us to split up the remaining system in two parts. All these problems could strictly not have been solved with our original method [7]), expressed by $P = (P_1^{-1} + P_2^{-1})^{-1}$ or by the same one, derived later on other grounds by BUCKENS [8]), as a direct splitting up is not possible, because the part, assumed to be rigid, would prevent the deflection by deformation of the elastic one [9]).

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[7]) BIJLAARD. De Ingenieur in Ned. Indië. 1939, Nr. 3.

[8]) BUCKENS. Publications Int. Association for Bridge and Structural Engineering. 7th Volume. 1943/1944. He derived the equation $P = (\Sigma P_k^{-1})^{-1}$, but, as a splitting up is seldom possible, could only solve rather simple problems of bars and frames, using only the equation $P = (P_1^{-1} + P_2^{-1})^{-1}$. The formula $P = (\Sigma P_k^{-1})^{-1}$ follows also directly from our line of thought [7]) [1]). Let $w = \Sigma w_k$. Assume first only part 1 as elastic, the rest infinitely rigid. Then we obtain in the same way as before [1]) $P_1 w_1 = Pw$. Thinking successively only part 2, 3, etc. elastic, we get $P_2 w_2 = Pw$, $P_3 w_3 = Pw$, etc., from which it follows, as $w = \Sigma w_k$, that $w = P(\Sigma P_k^{-1})w$ or $P = (\Sigma P_k^{-1})^{-1}$. The formula $P = P_0 + (P_1^{-1} + P_2^{-1} + P_3^{-1})^{-1}$ underlies our formula for built up columns [1]).

[9]) The splitting off of the proper rigidities of the single struts in our formulae for built up columns has moreover the effect, that our formulae are more accurate than similarly built formulae, especially for very weak connections between the single struts. Our formula as given in footnote 7 of our first paper [1]) yields for example the right and exact value $\lambda_l = l/r_s$, being the slenderness ratio of the single struts, if the connections are infinitely weak, by which λ_c is infinite, whilst similar formulae yield an infinite λ_l, consequently $Pcr = 0$.

In the same way we computed the following formula for the critical thrust of latticed struts, consisting of a complete truss

$$P_{cr} = \frac{\pi^2}{l^2}\left(2B_s + \frac{l^2}{l^2 + \pi^2 B/S}B\right),$$

l being its free supported length, $B_s = TI_s$ the flexural rigidity of the single struts, $B = \frac{1}{2}TA_s h^2$ the flexural rigidity of the entire bar, the proper rigidities of the single struts left out of account. Values A_s and h are the sectional area of the single struts and the mutual distance of their axes respectively. Value S may be called the modulus of rigidity of the latticing, the angular distortion by a shearing force D being $\beta = D/S$. If the coupling is effected by diagonals and battens, the angle between diagonals and struts being α, we have

$$1/S = 1/EA_d \sin^2\alpha\cos\alpha + \tan\alpha/EA_v,$$

A_d and A_v being the total cross sectional areas of diagonals and battens in each panel

We will now derive the exact formula for symmetric buckling of the faces of a sandwich plate. Let us first consider a strip, subjected to axial

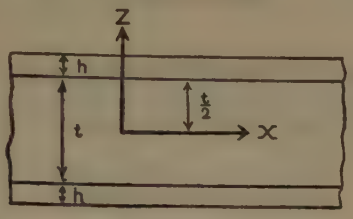

FIG. 3

compression. According to eqs. (7) and (12) of our first paper [1]) the AIRY stress function for this case is $\Phi = Z \cos \lambda x$, in which

$$Z = a\,(\cosh \zeta + C_1 \sinh \zeta + C_2\, \zeta \cosh \zeta + C_3\, \zeta \sinh \zeta)$$

whilst $\lambda = \pi/a$ and $\zeta = \lambda z$. The symmetry in the case on hand (fig. 3) requires C_1 and C_2 to be zero, yielding

$$Z = a\,(\cosh \zeta + C_3\, \zeta \sinh \zeta). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Furthermore eq. (5a') of our first paper, being

$$\frac{h}{2}\,\lambda^2 Z''' + \lambda Z'' - \left\{\frac{h}{2}\,(2+\nu)\,\lambda^2 - \frac{E}{R'}\right\} Z' + \nu \lambda Z = 0 \quad . \quad . \quad . \quad (18)$$

represents our boundary condition for $z = t/2$ or $\zeta = \lambda t/2 = \beta$, whilst according to eqs. (9), (10a) and (11) of the same paper we know that the critical thrust of an outer plate is given by

$$h\varrho_1 = B\lambda^2 + E'/2\lambda + \tfrac{1}{4}\psi h E' \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

in which

$$E' = 2E\left\{\frac{Z}{(2+\nu)\,Z' - Z'''}\right\}_{\zeta=\beta} \quad \text{and} \quad \psi = \left(\frac{Z'}{Z}\right)_{\zeta=\beta} \quad . \quad . \quad (20)$$

Substitution of eq. (17) in eq. (18) yields C_3, after which we find with eqs. (20)

$$E' = \frac{\{(1+\nu)\lambda^2 h - 2E/R'\}\,a - \{(1-\nu)\lambda^2 h + 2E/R'\}\sinh a - 4\lambda(\cosh a + 1)}{(1+\nu)^2 \lambda a - 2(\cosh a - 1)\,E/R' - (3-\nu)(1+\nu)\lambda \sinh a}\,E \quad (21)$$

$$\psi = 2\,\frac{(1+\nu)\lambda a - (1-\nu)\lambda \sinh a - \lambda^2 h(\cosh a - 1)}{\{(1+\nu)\lambda^2 h - 2E/R'\}\,a - \{(1-\nu)\lambda^2 h + 2E/R'\}\sinh a - 4\lambda(\cosh a + 1)} \quad (22)$$

in which $a = \lambda t = \pi t/a$. If t is infinite, these formulae are transformed in eqs. (15a) and (16a) of our first paper, valid for a plate, supported by a semi infinite substratum. According to eqs. (17) and (18) of that paper

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respectively. With crossed diagonals or diagonals forming a WARREN truss the term with A_v vanishes, so that $S = EA_d \sin^2 a \cos a$. Also in this formula the case $A_d = 0$ yields the exact value $P_{cr} = 2\pi^2 B_s/l^2$, whilst similarly built formulae yield the wrong value $P_{cr} = 0$.

it is sufficiently accurate to equal the half wave length $L = a = \pi/\lambda$ of the corrugations to

$$a = \pi\,(4B/E')^{1/3}$$

by which

$$h\varrho_1 = \tfrac{3}{4}\,(4BE'^2)^{1/3} + \tfrac{1}{4}\,\psi h E'$$

and to equal E/R' to zero. With $B = E'_1 h^3/12$ and $\psi E'/4 = E''$ these equations then yield

$$L = a = \pi h\,(\tfrac{1}{3}\,E'_1/E')^{1/3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

$$\varrho_1 = \tfrac{1}{4}\,(9E'_1\,E'^2)^{1/3} + E'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

in which

$$E'_1 = E_1/(1-\nu_1^2)$$

$$\left.\begin{aligned} E' &= \frac{\{(1+\nu)\,\alpha - (1-\nu)\sinh\alpha\}\,h\lambda - 4\,(\cosh\alpha + 1)}{(1+\nu)\,\{(1+\nu)\,\alpha - (3-\nu)\sinh\alpha\}}\,E \\ E'' &= \frac{(1+\nu)\,\alpha - (1-\nu)\sinh\alpha - (\cosh\alpha - 1)\,h\lambda}{2\,(1+\nu)\,\{(1+\nu)\,\alpha - (3-\nu)\sinh\alpha\}}\,E \end{aligned}\right\} \quad . \quad . \quad (25)$$

with $\alpha = \pi t/L$.

In many cases ϱ_1 will be much higher than the critical stress with antimetric buckling, so that it will suffice to check this fact with the safe values for infinite value t

$$E' = \frac{4 + (1-\nu)\,h\lambda}{(1+\nu)\,(3-\nu)}\,E \quad \text{and} \quad E'' = \frac{1 - \nu + h\lambda}{2\,(1+\nu)\,(3-\nu)}\,E \quad . \quad . \quad (26)$$

or with the still simpler formulae obtained from eqs. (26) by equaling $h\lambda = \pi h/L$ to zero.

In the example of a sandwich strip, subjected to compression, as given at the end of our first paper [1]), the critical stress for antimetric buckling was 431 kg/cm². With symmetric buckling eqs. (24) and (26), with $h\lambda = 0$, yield for this combination plywood-onazote values $E'_1 = 117.300$ kg/cm², $E' = 354$ kg/cm², $E'' = 33$ kg/cm², giving $\varrho_1 = 1300$ kg/cm², so that it is unnecessary to use the more accurate formulae. With the quoted values we find according to eq. (23) a half wave length $L = 15{,}1\ h$ cm of the corrugations.

As the wave length of the corrugations with symmetric buckling is very small, the influence of further constraints will be insignificant. Therefore it will be sufficiently accurate to equal the critical principal compressive stress for plates with arbitrary boundary conditions to that for a plate with free sides, subjected to simple compression. In this case we have to use eqs. (5) and (10) instead of (5a) and (10a) of our first paper, by which we find, that values E' and E'' in eqs. (23) and (24) are now given by

$$\left.\begin{aligned} E' &= \frac{\{\alpha - (1-2\nu)\sinh\alpha\}\,h\lambda - 4\,(1-\nu)\,(\cosh\alpha + 1)}{(1+\nu)\,\{\alpha - (3-4\nu)\sinh\alpha\}}\,E \\ E'' &= \frac{\alpha - (1-2\nu)\sinh\alpha - (1-\nu)\,(\cosh\alpha - 1)\,h\lambda}{2\,(1+\nu)\,\{\alpha - (3-4\nu)\sinh\alpha\}}\,E \end{aligned}\right\} \quad . \quad (27)$$

yielding for infinite value t

$$E' = \frac{4(1-\nu)+(1-2\nu)h\lambda}{(1+\nu)(3-4\nu)}E \quad \text{and} \quad E'' = \frac{1-2\nu+(1-\nu)h\lambda}{2(1+\nu)(3-4\nu)}E. \quad (28)$$

So eq. (24), combined with eqs. (27) or (28), gives also the value of the critical shearing stress τ_{cr} with symmetric buckling, and for the example given above here, with $\tau_{cr} = 447$ kg/cm² for antimetric buckling, it yields, with E' and E'' computed according to eqs. (28), with $h\lambda = 0$, the value 1330 kg/cm².

In all these investigations we have assumed the faces and the core to be isotropic. An extension to orthotropic behaviour or to plastic buckling, involving also an anisotropic deportment [10]), may however be given along the same lines.

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[10]) BIJLAARD. Proc. Kon. Akad. v. Wetensch., Amsterdam, Nrs 5 and 7 (1938). BIJLAARD. Publications Int. Association for Bridge and Structural Engineering. 6th Volume. 1940/1941.

Paleontology. — *Protohistoric Mammals from the Linderbeek, province of Overijssel, the Netherlands.* By D. A. HOOIJER. (Communicated by Prof. H. BOSCHMA.)

(Communicated at the meeting of January 25, 1947.)

Recent excavations at the Linderbeek, between Den Ham and Vroomshoop, in the province of Overijssel, have led to the discovery of mammalian remains. The collection belongs to the Rijksmuseum Twenthe at Enschede and was kindly put at my disposal for identification by Dr. C. C. W. J. HIJSZELER. According to a pollen analysis of Mr. F. FLORSCHÜTZ the age of the fauna is transition Subboreal — Subatlanticum, 650 B.C. If not worked or utilized, the bones [1]) are generally in a good state of preservation. In some cases, e.g., that of the dog of the *palustris*-type, the roe, and the horse, we have a number of bones of the extremities, often both of the right and of the left side, which must have belonged to one and the same individual. The most interesting forms in the fauna are the domestic cat, protohistoric remains of which are exceedingly scarce, and the dog of the St. Bernard's-type, which was not known until now in the subfossil state. The annotated list of the species is given below.

Castor fiber L. A right femur, a left tibia with part of the fibula, and a left os coxae are indistinguishable from the corresponding bones of the recent beaver. There is also an immature right femur, without epiphyses.

Felis catus L. The domestic cat is represented in the collection by a right humerus (pl. II fig. 1) a right tibia (pl. II fig. 5) and a left os coxae, with the medial boundary of the obturator foramen missing (pl. II fig. 4).

The European wild cat, *Felis silvestris* Schreber, has been included in table I to show its greater dimensions.

Canis familiaris L. subsp. The dog is represented in our collection by a complete skull, the principal bones of the extremities, vertebrae, and ribs, all of one and the same individual. The calvarium is nearly perfect, it lacks only the left I^1 and I^3, the right I^2 and I^3, and the P^1 and P^2 on both sides. In the lower jaw the I, C, P_1 and M_3 are missing.

The skull is adult and of moderate size. The anterior narial orifice is wider than high, the nasals gradually decrease in width and are concave from before backward. The frontals are relatively narrow and are slightly depressed in the median line. The orbito-frontal angle is 51°. The post-orbital constriction is not pronounced. The braincase is well arched, the sagittal crest has weakly developed and runs almost horizontally. The occiput hardly projects posteriorly beyond the condyles.

As to the measurements I followed in part WAGNER (1929); of the indices I selected only those which are of racial importance. WAGNER

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[1]) One bone indicates the presence of a bird in the fauna. Dr. G. C. A. JUNGE identified it as a right humerus of *Anser* spec. It is damaged on both ends, and identical in form with that of *Anser anser* (L.), but about 2 cm longer than the specimen compared in the Leiden Museum.

TABLE I.

	Felis catus L.						*Felis silvestris* Schreber			
		Leiden Museum		DEGERBØL, 1933, pp. 417—420			Leiden Museum no. 383	FREUDENBERG 1914, pp. 200/01	DEGERBØL, 1933, pp. 416—420	
		cat. *a*	cat. *b*							
Humerus, greatest length	97.0	94.0	90.5	95.0	91.0	97.0	121	127	121.6	124.5
Proximal antero-posterior diameter	21.0	18.8	19.0	20.0	19.7	20.5	27.0	—	25.5	26.0
Proximal width	16.5	16.4	15.6	16.7	15.3	16.5	21.3	—	20.1	20.5
Middle width	7.0	6.6	6.3	7.3	6.5	7.0	8.7	8	8.3	8.7
Distal width	18.7	18.2	17.0	20.1	17.5	17.7	24.6	23	22.0	22.6
Tibia, greatest length	103.8	ca. 108	ca. 101	108.5	—	111.8	145	—	139.8	130.4
Proximal width	18.0	19.0	16.4	19.3	—	18.6	24.0	—	22.6	20.5
Middle width	6.8	7.4	7.2	7.0	—	7.0	9.1	—	8.3	7.5
Distal width	12.2	—	ca. 12	14.3	—	14.1	19.5	—	16.6	15.0
Os coxae, greatest length	73.0	76.5	73.3				96	—		
From acetabulum to anterior border of ilium	40.1	42.7	39.6				52.5	—		
Greatest width of ilium	13.0	14.5	13.8				21.4	—		
Greatest width of shaft of ilium	9.5	10.0	10.0				15.1	13		
Greatest width of shaft of ischium	7.6	9.2	8.7				12.3	12		

distinguishes two groups of races among recent dogs, viz., a "normal" and a short-faced group. In some of his diagrams the points representing the different breeds fall into two distinct bands, and the wolf stands about at the intersection of them.

The rostrum-cranial cavity length index plotted against the length of the cranial cavity (WAGNER, l.c., p. 40/41) characterizes our dog as "normal" (type *a*). The correlation of the palate length-width index (l.c., p. 45) and that of the rostrum-mandible length index (l.c., p. 48/49) give the same result. In size and proportions our dog agrees with the "Schnauzer", the Norwegian greyhound or Dunker, the Lapland dog and the large whippet. As the anterior premolars are missing I cannot use the diagrams in which the sum of the lengths of the upper and that of the lower teeth is correlated with the length of the cranial cavity. They give no different values for the two groups, however (l.c., pp. 59—61).

The lower jaw of our subfossil animal is somewhat higher and more massive than in the four recent races to which it resembles so closely in the proportions of its calvarium. This is shown by the three indices given in table III.

Besides in its slightly superior size, the calvarium has a great resemblance to that of *Canis familiaris palustris* Rütimeyer, as described and figured by RÜTIMEYER (1862), STUDER (1901) and DEGERBØL (1939) from the Swiss lake dwellings and the Danish kitchen middens. The range of variation of the measurements given by these authors is given in the sixth column of table II.

The type of *Canis familiaris intermedius* Woldrich, from the Bronze Age of Austria (table II, seventh column, after WOLDRICH, 1877, pp. 74—78, and STUDER, 1901, p. 88) is very near in dimensions, and differs from our skull only in the slightly greater diastemata between the anterior premolars, which makes the tooth series longer than in our specimen, though the carnassial and the molars are shorter. With a view to the variability observed in other races this difference seems to be of no value.

STUDER (1901, p. 86) records skulls from the Swiss lake dwellings which form a transition between the two races, and believe *intermedius* to be an offshoot of *palustris*. WINGE (cf. BRINKMANN, 1925, p. 28) regards *intermedius* as a large variety of *palustris*.

The lower jaw agrees in general size with that of *intermedius* (table III), but the ascending ramus is a trifle higher (57 mm from the angular process in our specimen, against 53 in *intermedius*), the P_4 is shorter and the M_1 longer. It is indistinguishable from some specimens from Ølgaarde, Denmark's Earliest Stone Age (Azilian or Azilio-Tardenoisian in age: DEGERBØL, 1933, p. 620/21), which are referred to *Canis familiaris inostranzewi* Anutschin (DEGERBØL, 1943, p. 176/77, nos. 3—7, last column in my table III). The basal length of the type skull of *inostranzewi*, however, is 177 mm (STUDER, 1901, p. 49), and the estimated length of the Danish skull ca. 180 mm (BRINKMANN, 1925, p. 7). Those of the skulls to

TABLE II.

		WAGNER, 1929, tables 7 and 10				*Canis f. palustris*	*Canis f. intermedius*
		"Schnauzer"	Norwegian greyhound	Lapland dog	Large whippet		
Length of cranial cavity	81	80— 85	73— 84	78— 86	77— 85	—	—
Total length of calvarium	181	162—187	148—193	159—197	162—174	147—171	184
Condylobasal length	172	—	—	—	—	136—161	—
Basal length	163	149—167	138—171	141—166	143—159	130—154	164
Basicranial length	45	42— 45	36— 49	41— 43	42— 45	35— 44	45
Basifacial length	118	108—123	103—123	101—123	102—114	92—109	119
Basion-middle of frontal	90	85— 89	75— 93	84— 93	82— 85	—	—
Middle of frontal-prosthion	104	93—111	82—112	93—113	95—106	84— 99	—
Anterior border of orbit-prosthion	77	69— 85	65— 85	67— 84	69— 80	62— 73	82
Length of palate	90	80— 92	78— 96	76— 90	79— 89	71— 86	92
Length of horizontal portion of palatine	30	25— 30	24— 32	24— 29	24— 32	—	—
Height of braincase	54	47— 52	49— 55	51— 57	45— 51	41— 50	55
Width of braincase at parieto-frontal suture	55	57— 67	51— 60	52— 60	52— 58	46— 56	55
Width of occiput	62	56— 63	53— 64	52— 65	54— 57	52.5— 62	63
Least width between orbits	34	33— 35	30— 38	31— 39	31— 36	26.5—34.5	36
Postorbital width	49	51— 52	43— 56	45— 56	46— 54	37— 48	51
Postorbital constriction	38	34— 38	36— 40	35— 39	36— 40	29—36.5	37
Zygomatic width	104	95—140 [1]	89—103	90—105	89— 98	82—100	—
Greatest palatal width	64	59— 65	58— 66	54— 62	52— 59	51— 60	64
Smallest palatal width	34	30— 34	30— 36	24— 33	24— 30	29— 31	35
Width across canines	37	31— 39	32— 38	29— 38	27— 35	30— 33	38
Length P^1—M^2	63	57— 66	57— 69	57— 68	55— 62	52— 60	65.5
Length of P^2	(10)	10.0— 12.0	7.0— 11.0	9.0—11.0	6.0—10.0	—	—
Length of P^3	12.3	11.0— 14.0	11.0— 13.5	12.0—13.0	9.5—12.0	—	—
Length of P^4	19.2	17.0— 19.5	17.0— 19.0	17.0—19.0	15.0—18.0	14.7—18.3	18.5
Width of P^4	10.0	—	—	—	—	6.2— 9	10
Length of M^1	13.8	13.0— 14.0	12.0— 15.0	12.0—14.0	10.0—13.0	—	13
Length of M^2	7.5	7.0— 8.0	7.0— 8.0	7.0— 8.0	7.0— 7.0	—	7
Rostrum-cranial cavity length index	95.1	85.5—100.0	87.8—103.7	85.9—97.7	84.2—94.1	—	—
Palate width index	53.1	48.4— 52.3	50.0— 55.6	44.4—53.3	44.1—52.6	—	54.7
Palate length-width index	71.1	70.7— 73.8	68.1— 74.4	68.9—75.0	64.6—72.0	—	69.8
Rostrum-mandible length index	57.5	55.0— 60.7	54.8— 60.7	55.7—59.6	56.7—61.1	—	—

[1]) Apparently a misprint for 104.

TABLE III.

		WAGNER, 1929, tables 7 and 10				*Canis f. intermedius*	*Canis f. inostranzewi*
		"Schnauzer"	Norwegian greyhound	Lapland dog	Large whippet		
Length of mandible (condyle to front)	134	124—140	114—142	119—141	119—131	—	136
Length from processus angularis	135	123—138	115—141	118—142	118—129	135	135
Height of ascending ramus	54.5	45— 49	44— 55	43— 53	41— 49	—	—
Height of ramus behind M_1	23.5	19— 22	20— 23	18— 22	17— 21	—	21.5—25.7
Antero-posterior diameter of ascending ramus	32.5	26— 33	28— 35	27— 33	27— 31	—	—
Width across processus angularis	ca. 75	60— 76	60— 77	73— 75	59— 69	—	—
Greatest width of horizontal ramus	13	9— 12	10— 12	9— 12	9— 10	12	11.0—12.0
Length P_1—M_3	73	66— 71	63— 76	64— 72	62— 70	73	69.5—72.5
Length P_1—P_4	38	36— 41	38— 42	35— 39	33— 40	—	37—38
Length M_1—M_3	35	32— 36	29— 38	31— 35	30— 33	—	34.5—37.6
Length of P_2	9.7	9.0—10.5	5.0— 9.5	8.0—10.0	7.0— 9.0	—	9.5
Length of P_3	10.6	10.0—11.5	10.0—11.5	11.0—11.0	9.0—11.0	—	10.0
Length of P_4	11.8	11.5—13.0	11.0—13.0	12.0—12.5	9.0—12.0	12.5	11.5—12
Length of M_1	22.5	20.5—24.0	18.5—23.0	20.0—22.0	18.0—21.0	21.5	21.4—23.5
Width of M_1	9.2	—	—	—	—	—	8.8— 8.9
Length of M_2	9.4	9.0—10.0	8.5—10.5	8.0—10.0	7.5—10.0	9.5	—
Ascending ramus height-length index	40.7	34.9—38.6	35.4—40.0	36.1—39.2	33.9—37.4	—	—
Horizontal ramus height-length index	17.5	15.3—15.7	14.8—17.7	15.1—16.0	14.0—16.5	—	—
Horizontal ramus width-length index	9.7	7.0— 8.6	7.4— 9.1	7.6— 8.8	6.9— 8.4	—	—

which the mandibles from Øgaarde belong are calculated (after BRINKMANN) as 159—168 mm (DEGERBØL, 1943, p. 176), which would agree well with our specimen (163 mm).

The limb bones associated with the skull in our collection are longer than those of *palustris* (RÜTIMEYER, 1862, p. 119) and slightly shorter than those of *intermedius* (WOLDRICH, 1877, p. 80). They are almost exactly of the size of those of a complete skeleton from Ertebølle in Denmark (Campignian) which is considered by BRINKMANN (1925, p. 14) to have belonged to a cross between *palustris* and *inostranzewi* (table IV, after DEGERBØL, 1927, p. 71).

TABLE IV.

		Canis f. palustris	*Canis f. intermedius*	*Canis f.* subsp.
Humerus greatest length	150	127—144	153	150
Proximal width	28	—	—	26
Middle width	12	—	12.5	11
Distal width	30	—	30	28.5
Femur, greatest length	166	127—144	173	162
Proximal width	34.5	—	32	36
Middle width	13	—	12	12
Distal width	30	—	29	28
Tibia, greatest length	165	144	170	163
Proximal width	32	—	31	30
Middle width	12	—	13	12
Distal width	20.5	—	20	21

I do not venture to ascribe the present dog to any of the subspecies cited above. Even recent races cannot always be distinguished osteologically, and interbreeding will have occurred in former times as well as at the present day.

In the Linderbeek collection there is moreover the calvarium of a rather large dog, three views of which will be found on plate I. It is of the size of a wolf. That it, however, represents a domestic animal is evident from the orbito-frontal angle which is not less than 60° [2]). The size of the teeth, moreover, is distinctly less than in *Canis lupus* L.

The nasal process of the right premaxilla has broken off, and the right postorbital process is injured. The I^1, I^2, P^1 and the right P^3 are missing, but their alveoli are present. The pterygoids are incomplete and the right bulla is damaged, but otherwise the calvarium is perfect. It differs from that of the smaller dog described above, besides in its greater size, in its

[2]) STUDER (1901, p. 13) regards as belonging to wolves skulls in which the angle between the plane of the orbit and that of the brow is 40—45°, and refers to dogs those in which the angle is greater than 45°. REYNOLDS (1909, p. 24) has shown that the species overlap to a certain extent (between 42° and 48°) so that the character cannot be relied upon in all cases. Nevertheless he regards it as probably the most important character to distinguish between a wolf skull and a dog skull.

longer muzzle, more expanded frontals, less swollen braincase, and better developed sagittal crest which has a backward slope. The teeth are relatively weaker; the premolars are separated by intervals of 2—4 mm.

The rostrum-cranial cavity length index plotted against the length of the cranial cavity in WAGNER's diagram (WAGNER, 1929, p. 40/41) shows our larger dog to belong to the short-faced group of races, very near to the St. Bernard, the Newfoundland dog and the Great Dane. The palate length-width index points to the same (l.c., p. 45). The comparative measurements are given in table V.

It will be seen that our larger dog falls completely within the range of variation of the measurements of the three large recent breeds, and that it differs from the wolf (fifth column) in its wider frontal and smaller teeth.

TABLE V.

		WAGNER, 1929, tables 1, 4 and 5			
		St. Bernard	Newfound-land dog	Great Dane	Wolf
Length of cranial cavity	110	101—115	97—109	91—117	102—115
Total length of calvarium	236	240—276	223—257	197—277	235—283
Condylobasal length	231	—	—	—	—
Basal length	218	213—244	199—223	173—239	206—246
Basicranial length	65	56—71	56—64	50—72	55—68
Basifacial length	153	156—172	143—165	115—175	153—175
Basion — middle of frontal	121	116—135	108—125	97—135	108—127
Middle of frontal — prosthion	138	135—151	122—146	105—155	140—161
Anterior border of orbit-prosthion	103	101—115	94—111	75—117	105—126
Length of palate	115	119—130	108—121	96—130	113—134
Length of horizontal portion of palatine	42	40—49	36—43	35—49	38—44
Height of braincase	71	68—82	65—77	53—80	60—75
Width of braincase at parieto-frontal suture	62	67—70	62—69	60—73	62—72
Width of occiput	83	81—95	73—87	66—93	75—85
Least width between orbits	52	45—62	44—55	39—57	37—54
Postorbital width	ca. 80	69—83	60—79	53—80	49—75
Postorbital constriction	41	40—47	36—47	37—55	37—48
Zygomatic width	131	131—148	120—136	106—151	118—160
Greatest palatal width	78	81—90	74—85	64—90	73—91
Smallest palatal width	41	46—54	39—47	35—53	37—53
Width across canines	43	50—61	43—52	38—61	42—46
Length P^1—M^2	76.5	72—83	70—80	64—82	83—97
Length of P^2	11.2	9.0—13.0	11.0—13.0	8.5—15.0	11.0—17.5
Length of P^3	13.5	14.5—16	13.5—16.0	12.0—16.0	16.0—19.0
Length of P^4	19.7	21.0—23.0	20.0—22.0	17.0—24.0	23.0—29.0
Width of P^4	10.8	—	—	—	—
Length of M^1	14.2	15.5—17.0	14.0—16.0	13.0—17.0	16.0—20.0
Length of M^2	8.6	9.0—10.0	9.0—13.5	7.5—10.0	9.0—11.0
Rostrum-cranial cavity length index	93.6	100.0—103.9	90.4—106.9	82.4—113.0	100.9—111.7
Palate width index	52.6	54.1—60.0	51.2—57.9	48.0—64.1	47.6—58.2
Palate length-width index	67.6	66.9—72.3	65.0—71.9	59.1—79.2	64.0—72.0

No prehistoric dog known up to now presents dimensions as great as the present calvarium. As remarked already above, WAGNER (1929) has found that in some correlation diagrams the bands representing the two groups of domestic dogs intersect approximately at the point representing the wolf. This is regarded by him as supporting the theory that the wild ancestral type of dog should correspond to the wolf in size and skull proportions (l.c., p. 82/83). According to STUDER (1901, p. 127) the small size of the prehistoric dogs relative to the wolf militates against this theory. Captivity of wolves, however, brings about a decrease in size of the skull (cf. WOLFGRAMM, 1894). The large calvarium described above is by no means more wolf-like than that of the recent forms; on the contrary the orbito-frontal angle is even greater than in any of the Newfoundland dogs and St. Bernards of which STUDER (1901, pp. 57—66) has given the measurements. It is in so far important as it appears to be indistinguishable from these large recent breeds and consequently shows that the domestication of the type of the St. Bernard, Newfoundland dog and Great Dane must date back into protohistoric times.

Finally the collection from the Linderbeek contains a pair of humeri and a right femur which I must refer to the dog. The bones are not especially large and seem to indicate the presence of a third type of dog in the fauna, intermediate in size between the two described above. The size is just below that in *Canis familiaris matris optimae* Jeitteles, one of the largest of the prehistoric races (table VI).

TABLE VI.

		Canis f. matris optimae (DEGERBØL, 1927)	St. Bernard (WAGNER, 1929)	Great Dane (BRINKMANN, 1921)
Humerus, greatest length	163	166—179.5	220—222	223—247
Proximal width	26	—	56—58	—
Middle width	11.5	12.5—15	—	22—23
Distal width	28.5	34—36	51	—
Femur, greatest length	192	193—202	245—247	245—278
Proximal width	41	42—43	57	—
Middle width	13	14—15	21—22	19—22.5
Distal width	33	33—34	47	—

Ursus arctos L. Two scapulae, one of the right and the other of the left side, betray the presence of the brown bear. The thin supra- and infraspinous fossae, as well as the upper border of the right (pl. II fig. 2) and almost the whole of the anterior border of the left specimen, are missing. The recent brown bear varies in size to a great extent, and in prehistoric times there were particularly powerful specimens (DEGERBØL, 1933, pp. 628—633). The present bones are larger than the corresponding recent in the Leiden Museum, but a specimen in the British Museum, of which REYNOLDS (1906, p. 19) has given the measurements, shows that they are still within the limits of the recent form (table VII).

TABLE VII.

Ursus arctos L.

			Leiden Museum,		REYNOLDS, 1906, p. 19
			cat. *a*	cat. *b*	
Length from acromion to upper border at the spine	ca. 290	ca. 290	270	230	319
Greatest antero-posterior diameter of infra-spinous fossa	ca. 140	145	137	130	—
Antero-posterior diameter of glenoid cavity	66	64	60	55	61
Transverse diameter of the latter	43	37	35	34.5	—
Height from top of acromion to costal border of glenoid cavity	96	81	86	78	109

Sus scrofa scrofa L. The anterior portion of a lower jaw presents dimensions which are the maximum found in the male wild boar by REITSMA (1935). A large broken left lower canine likewise belongs to the present form.

TABLE VIII.

		Sus scrofa scrofa REITSMA, 1935, table 26		*Sus s. palustris* DEGERBØL, 1939, p. 136/37
		♂	♀	
Total length of symphysis	117	—	—	66—79
Idem, without anterior prominence	110	87—110	72—81	55—70
Greater diameter of canine alveolus	30	24— 30	14—17	12—20

Sus scrofa palustris Rütimeyer. The present subspecies is represented by an incomplete left horizontal ramus of the mandible. It agrees in size with the lower jaws from Bundsø, Denmark's Late Stone Age (Robenhausenian, DEGERBØL, 1933, p. 621); a locality which furnished also remains of the wild boar. DEGERBØL (1939, p. 139) states that there is no transition in size between the two forms, but the recent wild boar may be smaller than the subfossil, though it remains larger than the present specimen, as shown by the subjoined measurements.

TABLE IX.

	Sus scrofa palustris		*Sus scrofa scrofa*	
		DEGERBØL, 1939, p. 136/37		REITSMA, 1935, table 26
Length M_1—M_3	ca. 68	63—78	78—89	72—84
Length of M_3	ca. 33	30—41	41.5—47	36—45

A right horizontal ramus of the mandible with $pd_4 — M_1$, as well as the left half of a young mandible with $pd_3 — pd_4$ are not sufficiently characteristic to determine whether they belong to the wild or to the domestic form.

Capreolus capreolus (L.). The right half of a mandible and some limb bones agree with the recent roe in all essential structural characters, and

added to that they are not larger. BOULE (1910, p. 200) records mandibles, which are in general of greater dimensions than those of the recent roe, from the Grimaldi caves.

Cervus elaphus L. subsp. The red deer is relatively abundant in the collection: three incomplete skulls, two of which with part of the antlers, about forty fragments of antlers, parts of lower jaws, and limb bones. In one skull the greater parts of both antlers are preserved. The tines are broken; at the left side the bez tine springs off immediately above the brow tine, but in the right antler it arises 60 mm above the latter. This variation is also seen in the series of more or less complete antlers. In one of them the bez tine is quite short and springs off about at one-third of the distance between the brow and the trez above the former, while in another it is reduced to a mere knob, close to the brow.

The brow tine, if preserved, has the normal position, viz., just above the burr; one left antler, broken off at the trez tine, is remarkable for having a malformed brow tine which is shorter than the bez (pl. II fig. 3). The only complete antler, which is of the left side, measures 840 mm along the curve. It has three terminal tines; its measurements are given in the fifth column of table X.

TABLE X.

Cervus elaphus L. subsp.

	Skull				
	r.	l.			
Circumference of burr	210	205	215	175	210
Circumference of beam above bez tine	150	135	140	105	125
Length of brow tine	—	—	310	125	255
Length of bez tine	—	—	60	205	120
Length of trez tine	—	—	245	—	225
Distance between brow tine and trez tine	—	310	260	—	250
Height of bez tine above brow tine	60	—	90	95	55

Alces alces (L.). This species must have been rare in comparison to the red deer. I found only the base of a young left antler with a small portion of the frontal referable to the Moose.

Capra hircus L. or *Ovis aries* L. Goat and sheep are commonly found together. An incomplete right ramus of the mandible and a left radius do not yield specific characters. I have also a pair of metacarpals and a pair of metatarsals which are sheep-like in being longer and more slender than those in the goat (cf. DUERST, 1908, p. 381). In one sheep skeleton (cat. b) these bones hardly differ in proportions from *Capra hircus* L., however.

Bos primigenius Bojanus. The Urus is represented by a right horn-core with a basal girth of 225 mm. The tip has broken off, and the core measures 440 mm along the outer curve. This specimen must have belonged to a female individual. There are also a right half of the mandible (incomplete anteriorly), an epistropheus, and some limb bones. Their measurements are given in table XII.

TABLE XI.

		Ovis aries L. Leiden Museum			*Capra hircus* L. Leiden Museum	
		cat. *a*	cat. *b*	cat. *c*	cat. *c*	cat. *n*
Radius, greatest length	153	182	ca. 175	ca. 145	162	165
Proximal width	31.5	33	ca. 40	ca. 33	31	ca. 28
Middle width	15.0	15.5	20	16	15.5	19.5
Distal width	28	29.5	36.5	—	ca. 29	ca. 29
Metacarpal, greatest length	134	150	ca. 128	ca. 117	115	122
Proximal width	22.7	23	ca. 30	ca. 25	ca. 24	ca. 25
Middle width	11.6	12.5	16.5	12.5	15	17.2
Distal width	24	23.5	30	26	29	ca. 27
Metatarsal, greatest length	145	158	ca. 143	ca. 125	125	ca. 128
Proximal width	20.8	20	ca. 25	—	ca. 21	ca. 22
Middle width	12.0	11.5	14.5	10.5	12	14
Distal width	23.6	23	28	24.5	27	26

TABLE XII.

	Bos primigenius Bojanus	*Bos taurus* L. subsp.
Length of mandible from mental foramen to condyle	430	ca. 305
From mental foramen to P_2	84	ca. 61
Length P_2—M_3	166	142
Length M_1—M_3	103	ca. 87
Height of ramus behind P_4	62	43

		Bos primigenius Bojanus (RÜTIMEYER.1862, TSCHERSKI, 1892)
Epistropheus, ventral length	155	151—168
Idem, without dens	128	124—136
Greatest anterior width	124	124—143
Height of anterior articular processes	81	78—99
Humerus, distal width	105	92—109
Radius, greatest length	350	ca. 370—394.5
Proximal width	109	ca. 103—129.5
Ulna, total length	460	ca. 407—481
Radio-ulna, distal width	94	105—111
Tibia, greatest length	400	—
Distal width	71	76—82

Bos taurus brachyceros Owen. Whether this is the only domestic form of *Bos* in the collection I do not know. It is represented by a left horn-core (circumference at base 115 mm, length along outer curve 95 mm). We have further the right half of a mandible (table XII), skull fragments and fragmentary limb bones.

Equus caballus L. subsp. The hinder portion of a calvarium measures

198 mm over the zygomatic arches; the distance from foramen magnum to vomer is 127 mm. The limb bones indicate animals of different size; one radius has a length of 377 mm, while another measures only 309 mm. The dimensions of four metacarpals are included in table XIII.

TABLE XIII.

Equus caballus L. subsp.

Metacarpal III, length	219	229	234	248
Proximal width	48	46	ca. 57	52
Least width of shaft	33	32.5	34.5	36
Distal width	49	47.5	52	55
Length-proximal width index	22	20	24	21

The age of the Linderbeek fauna can be determined by the mammals only between certain limits; the lower limit is given by the domestic animals and the upper by the wild. I shall discuss the former first.

The horse, domestic ox and sheep or goat came to light at Øgaarde, a locality from Denmark's Earliest Stone Age, but they were found in the upper layer which is regarded as Late Stone Age or even younger (DEGERBØL, 1943). In the Early Stone Age (Campignian: DEGERBØL, 1933, p. 621) the occurrence of the domestic ox is not definitely settled [3]), and sheep or goat are absent (DEGERBØL, 1942). In the Late Stone Age of Denmark (Robenhausenian: DEGERBØL, 1933, p. 621) we have the cow, sheep, goat, pig and dog (Bundsø, DEGERBØL, 1939) side by side with the wild animals mentioned in the present paper. Except for a doubtful record from the Early Stone Age, however, the tame horse is not known before the refuse heaps of the Bronze Age (DEGERBØL, 1933, p. 621/22). Our Linderbeek fauna is of that age at the most; the only species which might point to a younger age is *Felis catus* L. Though in Egypt, the country in which the animal was most probably originally domesticated, it dates from the 16th century B.C., I do not know of protohistoric records of the domestic cat in Europe except in the artificial mounds of refuge or "terpen" of the northern provinces of our country. They date from 400 B.C. to 800 of the Christian era and the assemblage of mammals found in them is still the same as that of the Linderbeek. As the Moose and the brown bear have vanished from our country since that time, the Linderbeek fauna is at the least as old as the terpen fauna.

The mammalian remains indicate that the Linderbeek fauna must be placed somewhere between the Neolithic and the beginning of our era, and most probably closer to the latter than to the former. This is well in accord with the palaeobotanical evidence.

The presence of a domestic dog which belongs to the St. Bernard, Newfoundland dog and Great Dane group gives a special cachet to the Linder-

[3]) In our country, however, the *brachyceros*-cow has been recorded from the transition Boreal-Atlanticum (VAN DER VLERK, 1942, p. 5).

beek fauna, since these large breeds were not known until now before late historic times.

LITERATURE.

BOULE, M., 1910. Les Grottes de Grimaldi (Baoussé-Roussé), vol. 1, fasc. 3, Géologie et Paléontologie (Suite). Monaco, pp. 157—236, figs. 19—33, ps. XIV—XXIX.

BRINKMANN, A., 1921. Canidenstudien I—III. Vidensk. Medd. Dansk. nat. Foren., vol. 72, pp. 1—43, pls. I—III, 6 figs., 3 tables.

———, 1925. Canidenstudien V—VI. Bergens Museums Aarbok 1923—24, Naturv. raekke no. 7, 57 pp., 4 pls., 3 figs., 4 tables.

DEGERBØL, M., 1927. Über prähistorische, dänische Hunde. Vidensk. Medd. Dansk. nat. Foren., vol. 84, pp. 17—71, pls. I—IV.

———, 1933. Danmarks Pattedyr i Fortiden i Sammenligning med recente Former. I. Ibid., vol. 96, pp. 357—641, pls. XII—XXIV, 21 figs., 12 tables.

———, 1939. Bundsø. En Yngre Stenalders Boplads paa Als. IV. Dyreknogler. Aardb. nord. Oldk. og Hist., pp. 85—198 and IX—XVIII, 3 pls., 36 figs.

———, 1942. Et Knoglemateriale fra Dyrholm-bopladsen, en Aeldre Stenalder-Køkkenmødding. Kong. Danske Vidensk. Selsk. Ark.-Kunsthist. Skr., vol. 1, no. 1, pp. 79—135, 25 figs.

———, 1943. Om Dyrelivet i Aamosen ved Undløse paa Sjaelland i Stenalderen. Nord. Fortidsm., vol. 3, fasc. 3, pp. 165—206 and 221—226, figs. 84—103.

DUERST, J. U., 1908. Animal Remains from the Excavations in Anau, and the Horse of Anau in its Relation to the Races of Domestic Horses, in: Explorations in Turkestan. Expedition of 1904. Prehistoric Civilizations of Anau. Origins, Growth, and Influence of Environment. Edited by Raphael Pumpelly. Vol. 2. Carnegie Inst. of Wash., Publ. no. 73, pp. 339—442, pls. 71—91, figs. 490—492.

FREUDENBERG, W., 1914. Die Säugetiere des älteren Quartärs von Mitteleuropa mit besonderer Berücksichtigung der Fauna von Hundsheim und Deutschaltenburg in Niederösterreich nebst Bemerkungen über verwandte Formen anderer Fundorte. Geol. Pal. Abh., n.s., vol. 12, fasc. 4/5, 219 pp., 20 pls., 69 figs.

REITSMA, G. G., 1935. Zoölogisch Onderzoek der Nederlandsche Terpen. II. Het Varken. Wageningen, 58 pp., 93 figs., tables, diagrams.

REYNOLDS, S. H., 1906. A Monograph of the British Pleistocene Mammalia, vol. 2, part 2, The Bears. Palaeontographical Soc. London, 35 pp., 8 pls.

———, 1909. Ibid., vol. 2, part 3, The Canidae. 28 pp., 6 pls.

RÜTIMEYER, L., 1862. Die Fauna der Pfahlbauten der Schweiz. Neue Denkschr. Allg. Schweiz. Ges. ges. Naturw., vol. 19, pp. 1—248, pls. I—VI, 11 figs.

STUDER, TH., 1901. Die praehistorischen Hunde in ihrer Beziehung zu den gegenwärtig lebenden Rassen. Abh. schweiz. pal. Ges., vol. 28, pp. 1—137, pls. I—IX, 16 figs.

TSCHERSKI, J. D., 1892. Beschreibung der Sammlung posttertiärer Säugethiere. Mém. Acad. Imp. Sci. St.-Pétersbourg, ser. 7, vol. 40, no. 1, pp. 1—511, pls. I—VI.

VLERK, I. M. VAN DER, 1942. Kwartaire Bovidae van Nederland. De schedels en hoornpitten, welke zich bevinden in het Rijksmuseum van Geologie te Leiden. Leidsche Geol. Med., vol. 13, fasc. 1, pp. 1—28, pls. I—XI, 7 tables.

WAGNER, K., 1929. Rezente Hunderassen. Eine osteologische Untersuchung. Skr. Norske Vidensk. Akad. Oslo, I. Mat. Nat. Kl., 1929, no. 9, 157 pp., pls. I—XII, 36 figs.

WOLDRICH, J. N., 1877. Ueber einen neuen Haushund der Bronzezeit (Canis familiaris intermedius). Mitt. anthrop. Ges. Wien, vol. 7, pp. 61—85, pls. I—V.

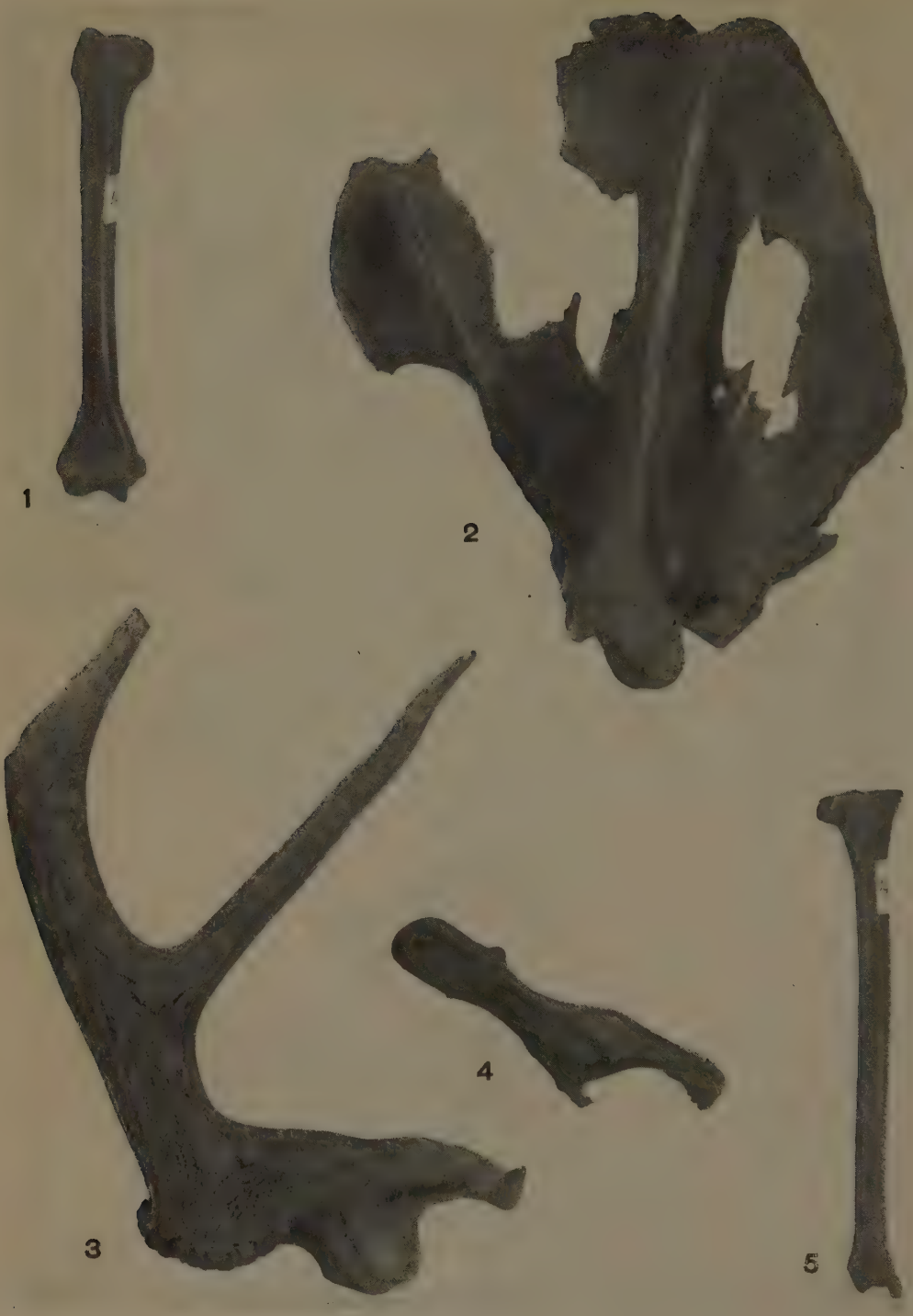
WOLFGRAMM, A., 1894. Die Einwirkung der Gefangenschaft auf die Gestaltung des Wolfsschädels. Zool. Jahrb., Abt. Syst., vol. 7, pp. 773—822, pls. 24—26, 4 tables.

Rijksmuseum van Natuurlijke Historie, Leiden.

PLATE I.

Canis familiaris L. subsp., calvarium; upper, left, and lower view, About $\frac{1}{2}$ nat. size.

PLATE II.



Figs. 1, 4 and 5, *Felis catus* L.; fig. 1, humerus dext., anterior view; fig. 4, os coxae sin., left view; fig. 5, tibia dext., anterior view.

Fig. 2, *Ursus arctos* L., scapula dext., right view.

Fig. 3, *Cervus elaphus* L. subsp., left antler, front view.

Figs. 1, 4 and 5, about 7/10 nat. size; figs. 2—3, about 1/3 nat. size.

Medicine. — *On contrasts in X-ray pictures: A comparison of the influence of the wave-length with that of the size of the representing beam.* By R. H. DE WAARD. (X-ray department of the Medical University Clinic, Utrecht.) (Communicated by Prof. H. R. KRUYT.)

(Communicated at the meeting of January 25, 1947.)

1. *Statement of the problem.*

Fig. 1 gives a schematic picture of the simplest arrangement applied in medical radiography. A beam of X-rays coming from the focus F of a tube is bounded by a diaphragm D, traverses an object Ob, and gives a picture of this object on the film f. We will now consider two small areas p and q

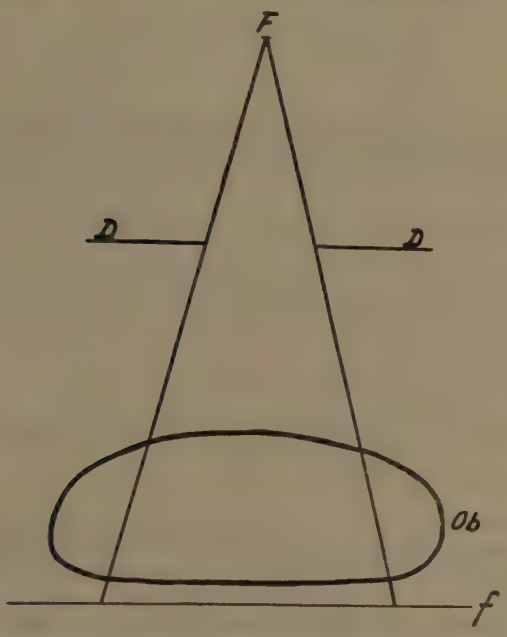


Fig. 1. Simplest arrangement applied in medical radiography: an object Ob is represented on a film f by a beam, originating in the focus F of an X-ray tube and bounded by a diaphragm D.

of the picture and denote the densities in these areas by z_p and z_q. If $z_q > z_p$ the contrast is then $z_q - z_p$ and its magnitude is partly determined by the technique of exposure. We can, for instance, increase it by making use of softer rays, or by reducing the size of the representing beam. Now let us suppose that a definite increase of the softness of the rays and a definite reduction of the size of the beam give rise to the same increase of the contrast $z_q - z_p$. The question arises whether these two changes will then have equal influence on other contrasts as well. If this is the case we can consider the two changes as equivalent; it is, however, also possible that the two changes giving rise to the same increase of the contrast $z_q - z_p$ affect other contrasts very differently. In the present paper we will consider this matter in the discussion of some simple schematic cases.

2. *A formula for small contrasts with respect to a background of uniform density.*

We first consider the case represented in fig. 2. A homogeneous circular beam of vertical X-rays falls on the surface of a horizontal water sheet, and we will assume that all the rays have the same wave-length λ. In the water is a small and thin horizontal sheet of bone of thickness d, and this

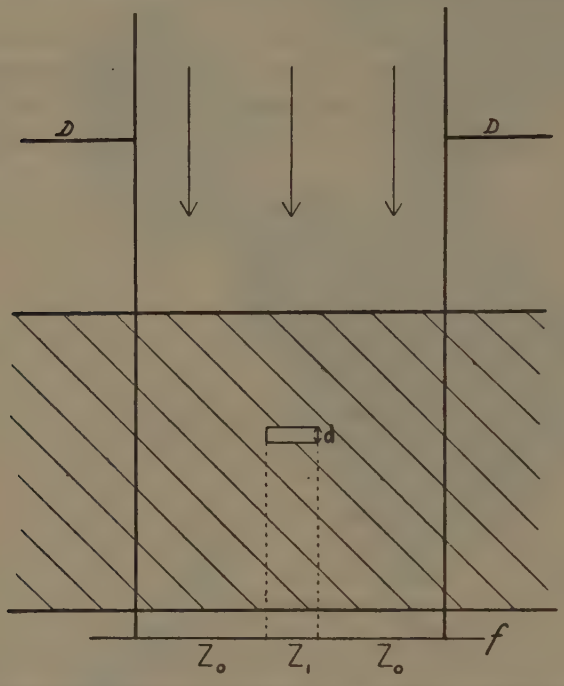


Fig. 2. A bone sheet of thickness d enclosed in a sheet of water is represented on a film f by a beam of vertical X-rays bounded by a circular diaphragm D.

bonesheet is represented on a film f which is placed under the bottom of the watersheet in immediate contact with it. The density-curve of the film has the form represented in fig. 3; on the axis of abscissac is plotted the

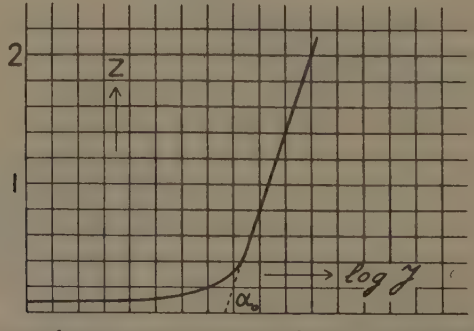


Fig. 3. Schematic X-ray density-curve corresponding to some definite time of exposure: the density of the film is plotted against the logarithm of the intensity J of the incident radiation. Gradation $\gamma_0 = tg\ \alpha_0$.

logarithm of the total intensity J with which the film is reached by the radiation falling on it from different directions. The slope of the curve is characterized by the quantity

$$\gamma = \frac{dz}{d(\log J)}$$

the value γ_0 assumed by it in the steep part of the curve is called the gradation of the film.

We will now denote the density effected in the shadow of the bone sheet by z_1, and that is the surrounding area by z_0; the contrast $z_0 - z_1$ is then given by the formula

$$z_0 - z_1 = 0.434\,\gamma\,d\,\frac{a\,\lambda^3 + a'}{1 + \frac{S}{D}} \quad \ldots\ldots\ldots \quad (1)$$

where the meaning of the symbols S and D is as follows:

S is the average intensity with which scattered radiation reaches the exposed part of the film.

D is the average intensity with which direct radiation reaches the exposed part of the film.

As to the meaning of the symbols a and a' we observe that the coefficient of enfreeblement μ of X-rays propagating in water is given by the formula

$$\mu_w = 2.5\,\lambda^3 + 0.18$$

and for X-rays propagating in bone by

$$\mu_b = 34\,\lambda^3 + 0.34$$

when λ is expressed in Ångström-units.

Now the definition of a and a' follows from the formula

$$\mu_b - \mu_w = a\,\lambda^3 + a'$$

and so we have

$$a = 34 - 2.5 = 31.5 \text{ and } a' = 0.34 - 0.18 = 0.16$$

3. *Proof of the formula.*

In order to prove the formula we can proceed as follows. We first observe that the intensities with which the areas 0 and 1 of the film are reached by scattered radiation will not be sensibly different. Therefore, if D_0 and J_0 are the intensities with which direct and total radiation reach the area 0 we shall have

$$J_0 = D_0 + S$$

and for the area 1 we have in the same way

$$J_1 = D_1 + S.$$

Hence

$$\left.\begin{aligned} z_0 - z_1 &= \gamma\,(\log J_0 - \log J_1) = 0.434\,\gamma\,(\log \text{nat}\, J_0 - \log \text{nat}\, J_1) = \\ &= 0.434\,\gamma\,\Delta\,(\log \text{nat}\, J) = 0.434\,\gamma\,\frac{\Delta J}{J} = 0.434\,\gamma\,\frac{\Delta D}{D_0 + S} = \\ &= 0.434\,\gamma\,\frac{1}{1+\frac{S}{D}}\,\frac{\Delta D}{D_0} \end{aligned}\right\} \quad (a)$$

where

$$\Delta\,(\log \text{nat}\, J) = \log \text{nat}\, J_0 - \log \text{nat}\, J_1$$
$$\Delta J = J_0 - J_1$$
$$\Delta D = D_0 - D_1.$$

Now, if D^* is the intensity with which direct radiation falls on the watersheet and h the thickness of this sheet, we shall have

$$D_0 = D^*\, e^{-(2.5\lambda^3 + 0.18)h}$$

whilst

$$D_1 = D^*\, e^{-(2.5\lambda^3+0.18)(h-d)}\, e^{-(34\lambda^3+0.34)d} =$$
$$= D^*\, e^{-(2.5\lambda^3+0.18)h}\, e^{-(a\lambda^3+a')d} = D_0\, e^{-(a\lambda^3+a')d}.$$

Hence

$$\frac{\Delta D}{D_0} = \frac{D_0(1 - e^{-(a\lambda^3+a')d})}{D_0} = a\lambda^3 + a' \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

and from this result and (a) formula (1) follows at once.

4. *Application of the formula.*

The properties of the film find expression in formula (1) by the factor γ, the thickness h of the watersheet and the size of the representing beam by the quotient $\frac{S}{D}$, and the wavelength λ both by this quotient and by the factor $a\lambda^3 + a'$. The application of the formula is considerably facilitated by figs. 4*a* and 4*b* in which $\frac{S}{D}$, λ-curves are given corresponding to different values of h and of the radius R of the circular cross section of the representing beam. These curves were obtained by means of a system of curves and formulae given by the author in another paper [1]).

In figs. 5*a* and 5*b* two curves derived from formula (1) are represented. These curves refer to a 20 cm thick watersheet ($h = 20$); fig. 5*a* gives the relation between $\frac{z_0 - z_1}{\gamma d}$ and λ for $R = 10$ cm, and fig. 5*b* the relation

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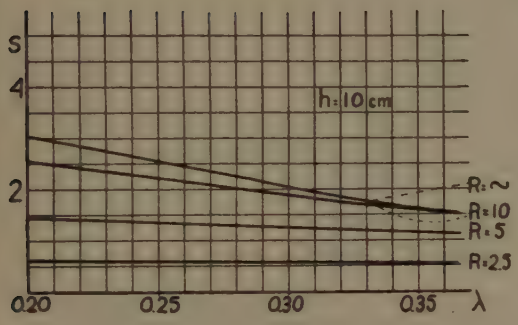
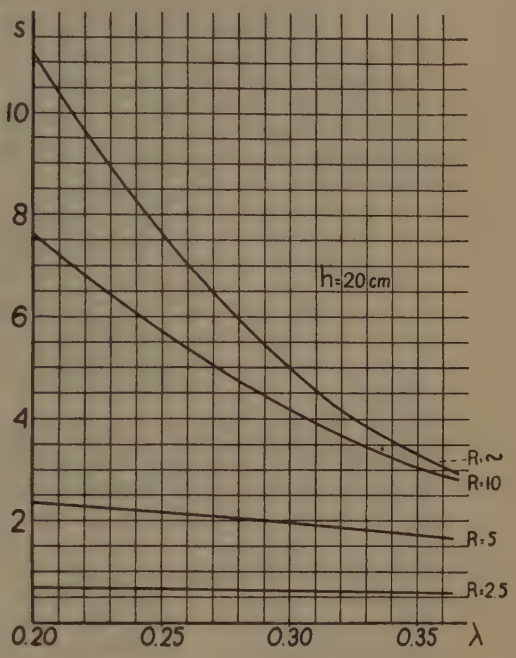
[1]) R. H. DE WAARD, The intensity of scattered X-radiation in medical radiography. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **49**, 955 and 1016 (1946).

between $\frac{z_0 - z_1}{\gamma d}$ and R for $\lambda = 0.2$ Å. With the help of these curves we can determine the increase of λ giving rise to the same increase of the contrast $z_0 - z_1$ as a given reduction of R. Fig. 6 gives a series of results obtained in this way.





Figs. 4a, b. Relations between $s = \frac{S}{D}$ and λ corresponding to incident beams with four different radii R; in fig. 4a for a water sheet of thickness $h = 10$ cm., in fig. 4b for one of thickness $h = 20$ cm.

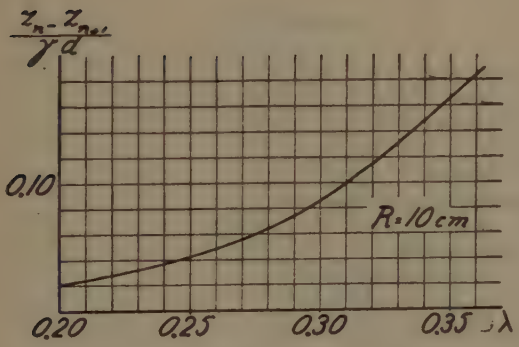


Fig. 5a. Relation between $\frac{z_0 - z_1}{\gamma d}$ and λ corresponding to a beam of radius $R = 10$ cm. falling on a water sheet of thickness $h = 20$ cm.

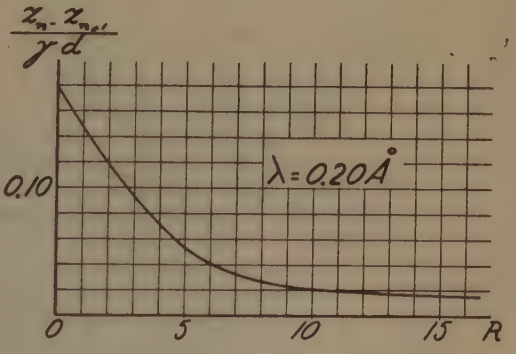


Fig. 5b. Relation between $\frac{z_0 - z_1}{\gamma d}$ and R corresponding to a beam of wavelength $\lambda = 0{,}20$ Å falling on a watersheet of thickness $h = 20$ cm.

5. *Deduction and application of a more general formula for small contrasts.*

We will now consider the more complicated case to which fig. 7 refers. Here the bone sheet of the preceding case is replaced by a bone ladder the

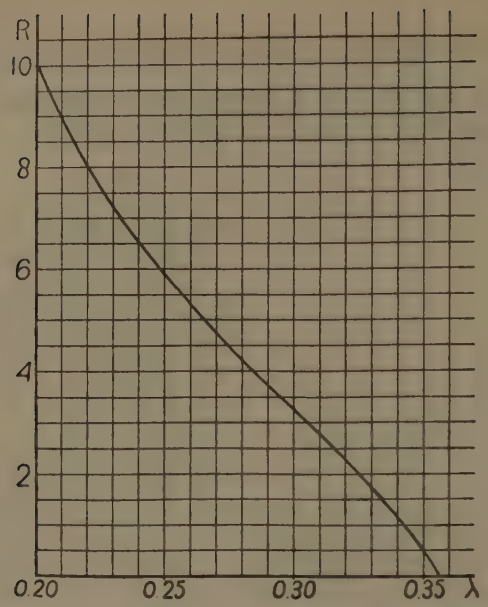


Fig. 6. Curve showing the radii R giving rise to equal contrasts when combined with a wavelength of 0.20 Å, as a given wavelength λ when combined with a radius of 10 cm. Thickness of water sheet: $h = 20$ cm.

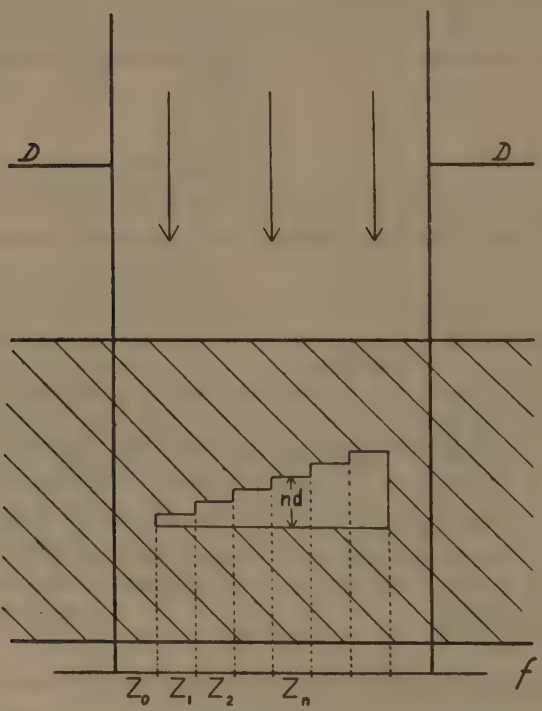


Fig. 7. Variation of the arrangement shown in fig. 2: the bone sheet of thickness d is replaced by a bone ladder with steps of height d.

steps of which have all the same height d. The contrast of the images of the steps n and $n+1$ is then

$$z_n - z_{n+1} = 0.434\,\gamma\,\frac{\Delta J_n}{J_n} = 0.434\,\gamma\,\frac{\Delta D_n}{D_n + S_n}$$

where

$$\Delta D_n = D_n - D_{n+1} \quad \text{and} \quad \Delta J_n = J_n - J_{n+1} = \Delta (D_n + S).$$

Now it is easy to see that

$$D_n = D^* e^{-(2.5\lambda^3+0.18)(h-nd)}\, e^{-(34\lambda^3+0.34)\,nd} =$$

$$= D^* e^{-(2.5\lambda^3+0.18)h}\, e^{-(a\lambda^3+a')\,nd} = D_0\, e^{-(a\lambda^3+a')\,nd}.$$

Hence

$$\Delta D_n = D_n\,(a\lambda^3 + a')\,d$$

and consequently

$$z_n - z_{n+1} = 0.434\,\gamma\,d\,\frac{a\lambda^3 + a'}{1 + \frac{S}{D}\,e^{(a\lambda^3+a')\,nd}}. \quad \ldots\ldots \quad (2)$$

With the help of this formula it is possible to answer the question raised in the first section of this paper. We will illustrate this by a study of the case $h = 20$ cm, $\lambda = 0.2$ Å, $R = 10$ cm. Curve I of fig. 8 shows the corresponding relation between $\frac{z_n - z_{n+1}}{\gamma d}$ and nd. Now it follows from fig. 6 that an increase of λ by 0.115 Å and a decrease of R by $7\frac{1}{2}$ cm give rise to the same increase of $z_0 - z_1$, and it is for these two cases

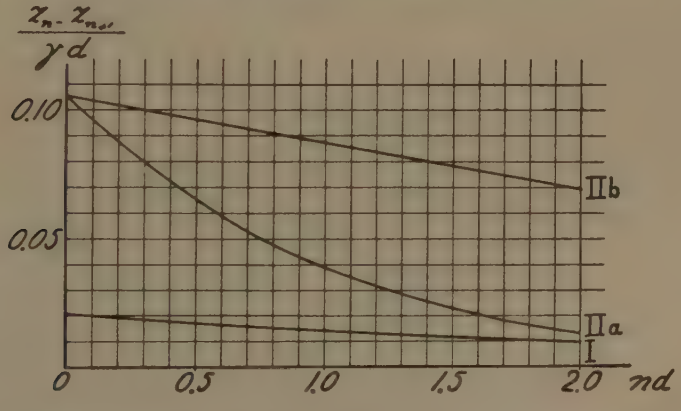


Fig. 8. Relations between $\frac{z_n - z_{n+1}}{\gamma d}$ and bone thickness nd:

I for $h = 20$ cm., $\lambda = 0.20$ Å, $R = 10$ cm.
IIa for $h = 20$ cm., $\lambda = 0.315$ Å, $R = 10$ cm.
IIb for $h = 20$ cm., $\lambda = 0.20$ Å, $R = 2.5$ cm.

The data of the curves IIa and IIb have been chosen in such a way that the contrasts corresponding to $n = 0$ are equal.

($h = 20$ cm, $\lambda = 0.315$ Å, $R = 10$ cm and $h = 20$ cm, $\lambda = 0.2$ Å, $R = 2.5$ cm) that the curves II*a* and II*b* of fig. 8 give the relations between $\frac{z_n - z_{n+1}}{\gamma d}$ and nd resulting from formula (2).

For the case in discussion the curves give the answer to the question raised in the first section of this paper. They show that in the lighter parts of the image (which correspond to the greater bone thicknesses nd traversed) the increase of contrasts brought about by a decrease of the diameter R of the representing beam is far more important than that resulting from an increase of the wavelength λ. In case of a decrease of R the increase of contrasts in the lighter parts is even comparatively greater than in the denser parts of the image.

Summary.

In this paper are derived two formulae by which contrasts in X-ray pictures can be calculated. With the help of these formulae a comparison is made of two measures by which contrasts in X-ray pictures can be increased, viz. 1. an increase of the wavelength, and 2. a reduction of the size of the representing beam. These two measures are found to be by no means equivalent: in a special case considered an increase of wavelength is shown to affect chiefly the contrasts in the denser parts of the image whereas a reduction of the size of the representing beam has comparatively more effect in the lighter parts.

Geology. — *The chromite deposits of the Guleman-concession (Vilayet Elaziz, Turkey).* By P. DE WIJKERSLOOTH. (Communicated by Prof. H. A. BROUWER.)

(Communicated at the meeting of January 25, 1947.)

Introduction.

In our paper "Die Chromerzprovinzen der Türkei" (1942) we had occasion to give a general summary of the chromite-bearing areas of Turkey. In that article the chromite deposits of the Guleman-concession, which will be treated here, were said to belong to the "southern or tauridic chrome ore-province of alpine origin". Though the map, added to the earlier paper, shows in general the geographical and tectonic-magmatic distribution of the turkish chromite deposits, it does not give any complete information about the occurrence of ophiolite and chrome ore in eastern Turkey. The sketch, which we furnish now, should in this connection fill the gaps. It shows clearly, that the southern or tauridic ophiolite-province in the territory of eastern Turkey splits into two belts. Between both lies the "kratogen land" of the Van-lake region. Further to the SE the two belts join again and form one single trace, which continues in the direction of the mountaineous region of Kermanshah in Iran.

SKETCH OF THE CHROMITE-BEARING AREAS OF EASTERN TURKEY

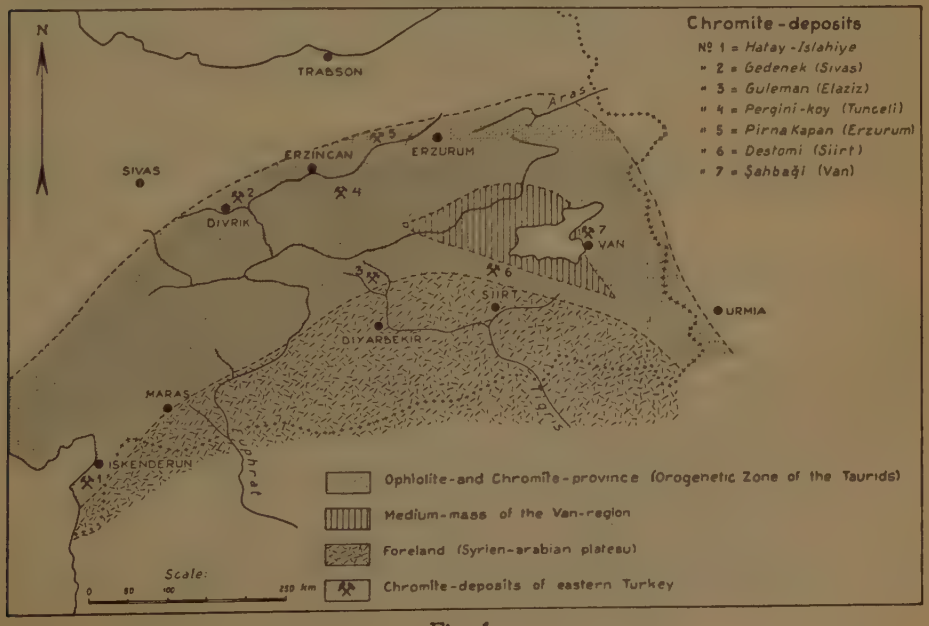


Fig. 1.

About the chrome ore richness of the two belts little is known, as no systematical exploring for chrome ore has been done here. Only some incidentally discovered finding places are known. The northern belt contains for instance the chromite deposits of Pirna Kapan (Vil. Erzurum) and of Şahbağı (Vil. Van), while from the southern belt only one chrome ore-deposit near Destomi (Vil. Siirt) became known. Notwithstanding this scanty knowledge it is evident, that both belts are chromite-bearing and that the southern or tauridic chrome ore-province continues to the east.

The chrome ore region of Guleman is located near the above mentioned virgation point of the tauridic orogenic zone, lying west of the Van-lake.

Location and transportation-possibilities — historical notes.

The Guleman concession is situated about 20 km E. of Ergani-Maden, between the two tributaries of the Tigris, namely the Bahru-çay in N. and the Maden-suyu in S. It occurs at an altitude of appr. 1300 meters in a mountaineous region, belonging to the Armenian Taurus.

A good highway connects the mine with Ergani-Maden and the railroad-station of Maden. The ore is transported by an aerial cableway of some 20 km length to a bunker and a loading-installation near the railroad. From there the ore is shipped by railroad via Malatya to the port of Mersin for export (a railroad-route over 600 km long).

The management of the nearby copper-mine of Ergani knew since long of the existence of the chromite deposits of Guleman. From this source R. Pilz (1917) evidently obtained the news, mentioned as a side-issue in his paper on Ergani-Maden.

In the year of 1936 the Eti-bank has started to mine this rich chrome ore region. Untill the end of 1938 the chrome ore was transported by trucks. It was then that the aerial cableway-installation was first put in operation.

During the last years the annual output has varied widely according to the very changing economical (and political) conditions. In 1940 it has reached the maximum of 90.000 metric tons.

Earlier published litterature.

The chromite deposits of Guleman have not been treated until now in the main reference books on ore-deposits. They are not mentioned, for instance, in the "Lehrbuch der Erzlagerstaettenkunde" (1941) by H. Schneiderhöhn.

A. Helke (1938) has given the first short description of the chrome ores of Guleman and the surrounding area. A sketch, accompanying his article gives a good survey of the great number of places where chrome ores are found in this region, named by the author "the chrome ore-province of eastern Turkey". In the time of his investigations the opening and the mining of the Guleman-concession was just starting and therefore the

author was obliged to confine himself to a few general remarks on the nature and the size of the ore-bodies.

G. ROSIER (1942) in a short paper has treated essentially the stratigraphy and the tectonic structure of the Guleman-concession and the surrounding areas. In the field we could always verify and reestablish the accuracy of his geological studies. On the nature of the ore deposits themself he gave little account.

In the same year V. KOVENKO published a study on the chromite deposits of Guleman, which contains many petrografical data and genetic considerations.

Although the above mentioned works exist, there is not yet any detailed description of the very important chrome ore-deposits. The aim of this paper is therefore to give somewhat more detailed information about them, as, because of their size and geological position, they merit to be more generally known.

Stratigraphy of the Guleman region.

Cristalline schists and marbles.

The oldest members of the rocks that appear here, are highly cristalline mica schists, which often contain quarzite intercalations, and white marbles. About their age we know very little. We can only establish that they are older than uppercretaceous (see below).

The ophiolitic rocks.

The principal components of the ophiolitic rocks are serpentine, forming the main body, pyroxenite and gabbro.

The serpentine contains sporadic large individuals of diallag. It looks just like its homologue in the Ergani-district, which is clearly derived from peridotite. Here, however, the serpentinization has made much more progress and left behind scarcely any relics of the original rock, from which it is evolved.

The pyroxenite consist of monoclinal pyroxene. It is found mostly near the chrome ore-bodies as around Gölalan (see the map). It forms irregular streaks ("schlieren") in the serpentine.

The gabbro is medium-grained and even macroscopically can be seen that it is built up from plagioclase and pyroxene. It appears in the form of large solid masses inside the more plastic serpentine. Mostly its borders are tectonic and more or less polished by differential movements. As was already remarked by G. ROSIER (see the references), the gabbro seems to be of younger age than the serpentine and is represented by tectonically heavily deformed bodies, which are intrusive in the latter.

All the members of this ophiolitic series are covered transgressively by the Lower-eocene, whose basal parts are developed as conglomerates and breccias. In the clastic elements of this formation we find products of erosion of ophiolitic rocks and even of chrome ore. According to this we

conclude that the basic rocks and the chrome ores of the Guleman-district will be of pre-lowereocene age. Moreover G. ROSIER informs us that outside of the Guleman-concession near the village of Kelhasi (on the bank of the Bahru-river) uppercretaceous flysch lies in transgression on the serpentine. From this it may be referred that the ophiolitic rocks should even be of pre-upper-cretaceous origin.

To fix the lower age limit there is no supporting point at our disposal. But if we take into consideration that the basic rocks of this district belong to the tauridic ophiolitic province (see "Die Chromerzprovinzen der Türkei"), it can be admitted that probably they are not older than the mesozoic period.

The cretaceous.

The cretaceous consist of shaly marls, sandstones and light phyllitic slates which as a whole show a typical flysch-facies. In general the formation has a grey-yellow color, but fresh samples, taken from unweathered parts, exhibit mostly a color of grey-green. Its layers are generally poor in fossils. In spite of this G. ROSIER has found here Jereminella Pfenderae Lugeon, which indicates the presence of uppercretaceous. On the other hand A. HELKE mentions that he has found in this series and particularly in its bituminous components, Ponopaea cf. gurgites Brongn., Pecten cf. orbicularis Sow. and Ostrea vescularis Lam. As finding places he mentions the villages Heridan, Pertek-Pirajman and Deri, all situated in the neighbourhood of the Guleman-concession. These fossils prove that locally in the flysch-formation the middle- and even the lower-cretaceous are represented.

The lower-eocene.

Exactly as in the Ergani-district a transgressive formation of red-violet schists and limestones is found in the Guleman-concession. Although no characteristic fossils were met here, we consider it to be of lowereocene age, in virtue of the excellent analogy with the above mentioned neighbouring district. As described in our paper on Ergani-Maden (see references) the lowereocene of Ergani is rich in schmall nummulites, orthofragminae and alviolinae. An other analogy lies in the fact, that in both districts the lowereocene contains many and mostly thick intercalations of melaphyre.

The basal parts of the lowereocene are conglomeratic (see above). Frequently only a slight rounding up of the coarse clastic elements has taken place, so that it gives the impression that we have to do with slope breccias, which have no components of foreign character. Mainly products of erosion from the basic rocks are met, accompanied by sporadic pieces of cristalline schist and marble.

The middle-eocene.

Without any sharp boundary the lower-eocene is locally covered by

limestone layers which contain large-sized nummulites of the middle-eocene (as at Saritepe). These small limestone-occurrences represent scanty relics of a formerly widely spread middle-eocene formation.

Tectonic of the Guleman-district.

As the accompanying map indicates, the chromite deposits of Guleman are situated near the south-border of the Taurids. This has its special significance.

In this connection we like to point out that L. KOBER (1915) has established around Maras that the frontal units of the Taurids had been pushed as a uniform mass upon the syrien-arabian plateau during a late orogenic phase and he reports that the miocene of the foreland of the Taurids was overrun.

We believe that we have observed something similar to this in Hatay, between Kirkhan and Hassa. Here the serpentine (together with cristalline elements), representing the frontal mass of the Taurids, seems to have been pushed forwards upon the syrien-arabian foreland, characterized by a frequency of basaltic flows.

An excellent analogy with the orogenic structure of the Dinarids is offered here. A. PILGER (1941) in his description of the Yugoslavian overthrust-sheets (nappes) reports that the Merdita-nappe, with its predominent serpentine, was pushed upon the folded foreland as a relatively solid mass, causing the serpentine to slide over on top, due to its ability to glide on other rocks.

Besides these results of great horizontal movements, which we meet now as widespread overthrust-masses, the frontal elements of the orogen were mostly affected by a strong digitation, dissecting the overthrust-mass into smaller units.

Probably on a smaller scale both structures seem to exist in the Guleman-district. An overthrust-mass of serpentine (locally with cristalline schists and marbles), lying on an uppercretaceous base, and a manifold breaking up of this unit can be distinguished in form of a generally existing slice-structure. G. ROSIER (1942) mentions in his valuable paper three slices, namely one of Soridağ, one of Kundikan and one of Guleman. It must be emphasized, however, that these represent only the most evident three members of a much more complicated slice-structure. They are secondary phenomena which were of later origin and infringed on a structure of much more importance for the understanding of the local orogeny.

As it can be gathered from our map of the Guleman-concession, the chromite deposits that exist here, are all found in the thrust-over serpentine, which shows a small thickness inside the concession-district. At more than one place the overridden uppercretaceous base comes out in form of inliers.

Therefore we should not wonder that the chromite deposits which, together with the serpentine, have been submitted to a great horizontal

transportation, exhibit a strong tectonically affected picture. This is one of the reasons why we have reserved so much space for the treatment of the tectonic problems.

This appearance is specially striking for all the chief ore-bodies of Guleman, as Saysin, Tosin and Gölalan, as well as for the neighbouring masses. They all form flat lenses, bordered by gliding surfaces which gives the impression that they are torn up shreds of an earlier, more or less coherent chrome ore mass. Also the internal structure of the ore reveals the effect of a strong tectonical deformation (see below).

The chromite deposits of Guleman are therefore "rootless" ore-bodies that swimm in a foreign serpentine mass. This is a very important knowledge for the true estimation of the ore reserves and for the further success of the total opening up, which has not yet ended.

The chromite deposits of the Guleman-concession.

The chromite ore-bodies are found on an appr. 1700 m long line, which runs NWW–SEE (see the map). Three main groups can be distinguished, namely the Saysin-, the Tosin- and the Gölalan-group. Each group shows a more or less farreaching partitioning in small and large ore-lenses. It must be emphasized, however, that possibly there may be many more ore-lenses which, concealed under a cover of serpentine, have not been discovered yet.

The ore-group of Saysin which on account of its being situated in the neighbourhood of the installation of the aerial cable-way was mined first, has delivered an amount of chrome ore of about 50.000 tons and seemed to be entirely exhausted. Recently, however, a new ore-lense was discovered, seperated from the earlier mined members of this group by a veinlike shaped mass of serpentine. Its ore content will be about 20.000 tons. There is a possibility, that in the near future even more lenses, belonging to this group, will be found.

The ore-group of Tosin consists of a main body and a few smaller sur-

NNW-SSE SECTION OF THE CHROMITE-DEPOSIT OF TOSIN

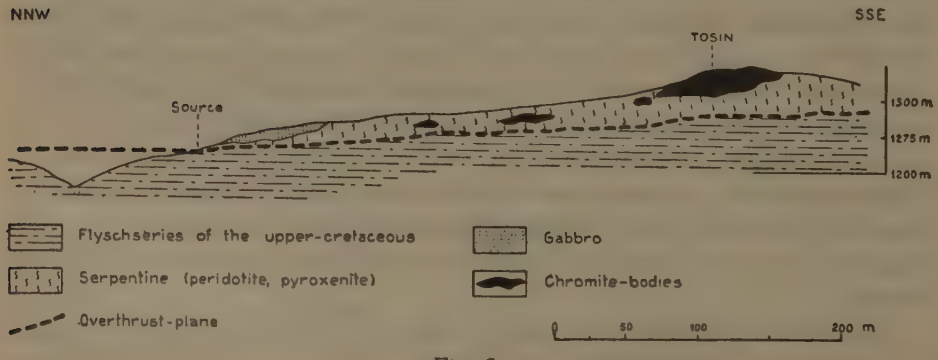


Fig. 2.

rounding lenses (see the section). The main body became sufficiently known through the opening works. It forms a flat ore mass, up to 20 m thick, which in horizontal projection shows very irregular borders and hooklike apophyses. Its contact with the serpentine has a wavy course and is sharp-lined. The whole morphology of the ore-body indicates that strong squeezing out forces were active here. At many places the ore-body was cracked and so we find many fissures, now filled up by diapyric intruded serpentine. All learns that, if the ore-body had been displaced over a still greater distance, a total splitting up would have resulted and the unity of the main ore-body of Tosin completely lost.

The ore content of this group is estimated to be at least 130.000 tons. To this amount must be added the possible reserves of the neighbouring small ore lenses, about which our knowledge is very restricted.

The ore-group of Gölalan (in which we include also the ore-lenses of Sayver) is not known so well as the two above-mentioned groups, since the opening works that have to carry away the covering serpentine, are only partially advanced. To get information about the deeper parts of the chrome ore a few shafts and drillings were made. On the base of these few informations and the natural outcrop we assume the content of the main ore-body to be at least 300.000—400.000 tons.

By exact observation it can be established that here also, quite as we showed in Tosin, a cutting up of the main body has taken place. Many fractures, filled up by serpentine, are crossing the chrome ore, dividing the main body in different smaller parts. The surfaces of these single parts are polished and show magnificent mirrors.

The total ore-content of the Guleman-concession is, according to the data mentioned above, about 500.000—600.000 metric tons. It should be pointed out, however, that the exploring of the concession-district must be continued and that it may be possible, that still important discoveries can be made, which might change considerably the above mentioned estimation.

The chrome ore of the Guleman-concession is almost entirely high-grade and massive ("Derberz"). It is not necessary that the ore is washed and it can be delivered as it comes out of the open pit. The annual output shows an average content of chrome oxyde ranging from 50 to over 50,5 per cent. The ores of the main body of Tosin have an average content of 49—50 per cent, while those of the main body of Gölalan are somewhat richer and show an average of 51—52 per cent. The small ore lenses N of the main body of Tosin (called Aşağa Tosin) contain ores with an average Cr_2O_3 value of 55,5 per cent, being the richest type found in the concession.

The following table gives a few analytical values about the ore of Tosin:

Cr_2O_3	49.91	50.17	50.31 %
FeO	16.11	16.78	19.83 %
SiO_2	6.09	5.83	5.60 %

From the mineralogical point of view the ore of Guleman is quite interesting. As already mentioned above, it is, almost without exception, very pure and massive (Derberz). Only in the group of Saysin it shows locally a strong intergroth with magmatic pyroxene.

Striking is the considerable grain size of the chromite. It is not rare that the crystals (subsequently deformed — see below) have a diameter of 2.5—3.0 cm. In no other ore deposit in Anatolia have we found such a pronounced macro-crystallinity of the chromite. It must be admitted that here a very slow and continuous crystallization of chromite from the peridotitic magma has taken place.

A second remarkable feature of the chromite of Guleman is its widespread foliation. In this connection we want to refer to our paper on the chrome ore of Çatak in Vilayet Bursa (see reference No. 12) in which a similar phenomenon was described. We showed there that chromite under the influence of tectonic movements can be crossed by a system of parallel orientated ruptures. It has been pointed out that the direction of these ruptures is conditioned only by the tectonic action and that there is no connection with the internal crystal structure of the affected chromite. In this way the chromite of Guleman is broken up in a complex of laminae in a large degree. As a result of the release of stresses sometimes differential movements have taken place between the laminae and so the chromite individuals lost their original form and have been stretched in the direction of the tectonical action. At the same time some short transverse fractures cracked the laminae which, in the extreme cases, lost hereby their unity. Finally a stratified mylonite of finest chromite particles resulted. But generally the chromite laminae, although somewhat broken up, preserved their characteristic forms.

This shearing deformation of the chromite has made itself noticeable throughout all the chrome ore masses of Guleman. A pronounced foliation exists everywhere. A. HELKE has interpreted this phenomenon as being a consequence of a "plastic" deformation of chromite and therefore he described the chromite laminae as "breitgedrückte Chromitkristalle" (widened out chromite crystals). We have accepted this interpretation in our article on the chrome ore provinces of Turkey (see reference) and used the expression "flattened ore grains". Our later investigations, however, have proved that here no plastic, but a shearing deformation has taken place.

It could be established by accurate observation that the shear surfaces between the chromite laminae are covered with a coat of Kaemmererite. This mineral and also the uwarovite which is found specially in the main ore-body of Gölalan, are of secondary origin. Generally we find them in fissurès and on faultplanes which in great number are cutting the ore-bodies. The uwarovite appears often in well-shaped crystal-aggregates which indicate a clear idiomorphy of this mineral (pentagondodecahedron).

Microscopically, the chromite shows still other mineralassociations, such as olivine and rutile. We meet them only as very fine sized enclosures in the chromite. The olivine represents an "old foreign guest" (see SCHNEIDERHÖHN "Die Mineraleinschlüsse in Erzmineralien", reference No. 7) which as a foreign substance was included in the chromite individuals during their growth. On the other hand the rutile seems to be a "family guest", which was formed by unmixing of the constituent elements of the chromite *). This conclusion may be based on its symmetrical orientation in the chromite crystals. Thus we observed that the rutile is grouped in several systems of parallel individuals. All indicates that there is a general orientation parallel to the (111) directions of the host crystals. The rutile appears in the form of very fine and long needles, having sharp ends. This crystal habit seems a justification for giving to these individuals the name "sagenite".

For completeness the "reconversion enclosures" (Umbildungseinschlüsse) must be added here. First we should mention the ferritchromite (see our article "Mikroskopische Beobachtungen an anatolischen Chromerzen"). It is formed preferably in the fissure cracks and on the grain borders of the chromite. Less general is the internal segregation of the iron oxyde which we can find in the weakly magnetic parts of the Guleman ore. As we have described already in 1942 (see reference) this alteration product is found along fissure cracks, as extremely fine and mostly tabular (in polished section needle-like) aggregates, whose structure can only be observed by the use of a very strong enlargement and oil immersion. In the formation of these iron-oxyde aggregates the haematite occupies an important place. Magnetite could not surely be identified which must be attributed to the hard identification of these microcrystalline intergrowth of iron-oxyde.

Summary.

The deposits of the Guleman-concession which contain important reserves of chromite, consist of very coarse grained and pure ore. The macrocrystallinity of the ore shows that the segregation of the chromite took place in a very large reservoir of peridotitic magma, which allowed a slow crystallization and a quiet growth of the chromite-individuals.

The chromite deposits occur all together in an overthrust-mass of serpentine which was pushed from the north to the south over a great distance over its upper-cretaceous base. The chrome-ores which perhaps originally have formed a single mass, are during the overthrust movement broken up in several bodies of different size. They are found in the form of lenses, shreds and rounded blocks, all rolled out and bordered by sliding surfaces. Everywhere the chromite exhibit a strong foliation.

The main ore-groups of Saysin, Tosin and Gölalan have together a

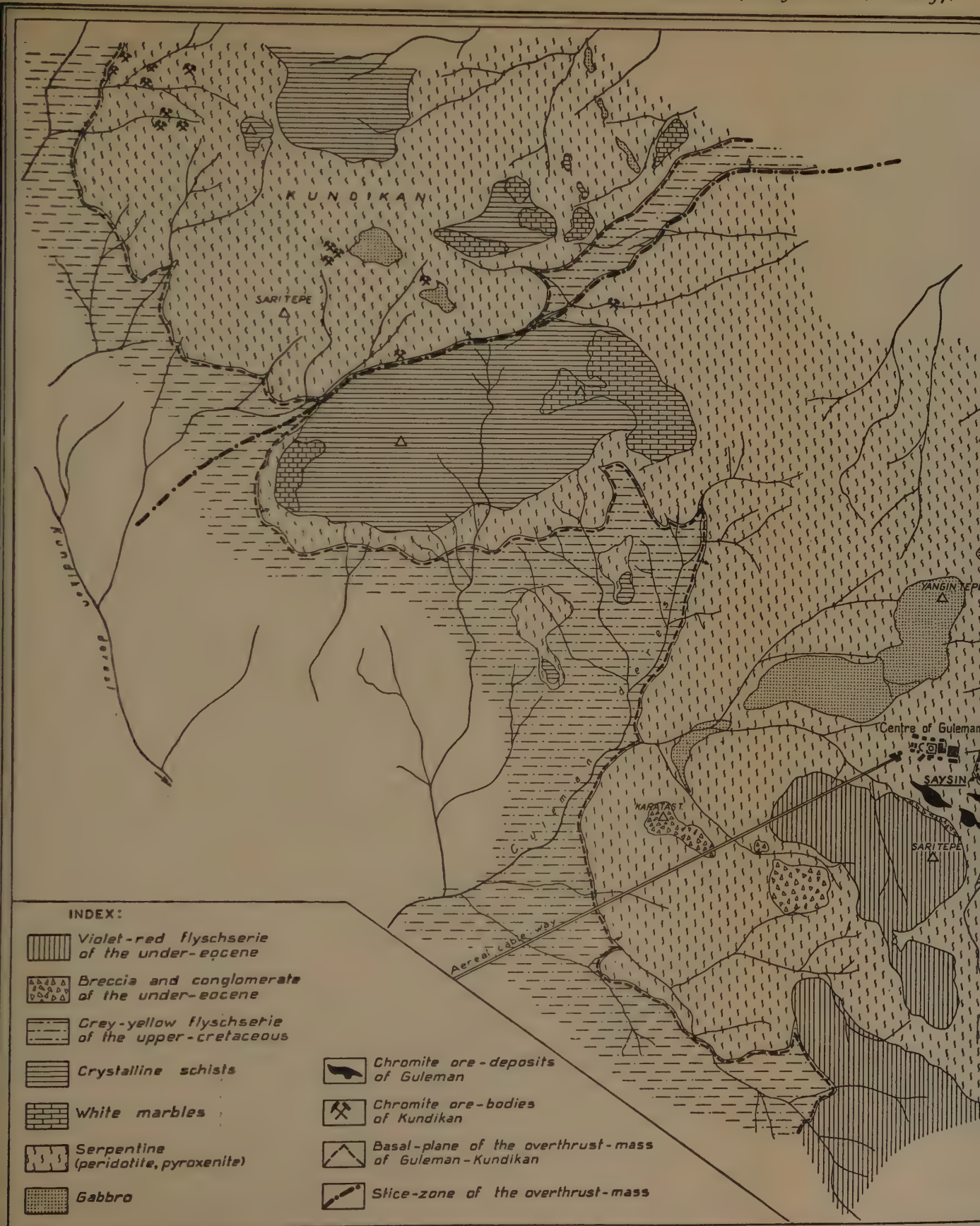
*) In an earlier publication ("Mikroskopische Beobachtungen an anatolischen Chromerzen") it was described by mistake as an "older foreign guest".

reserve of about 500.000—600.000 tons. Further search of the ore-masses, now covered by serpentine, promises the discovery of other chrome ore. The exact knowledge of the overthrust base and the other local tectonic features is necessary for the success of the difficult exploration.

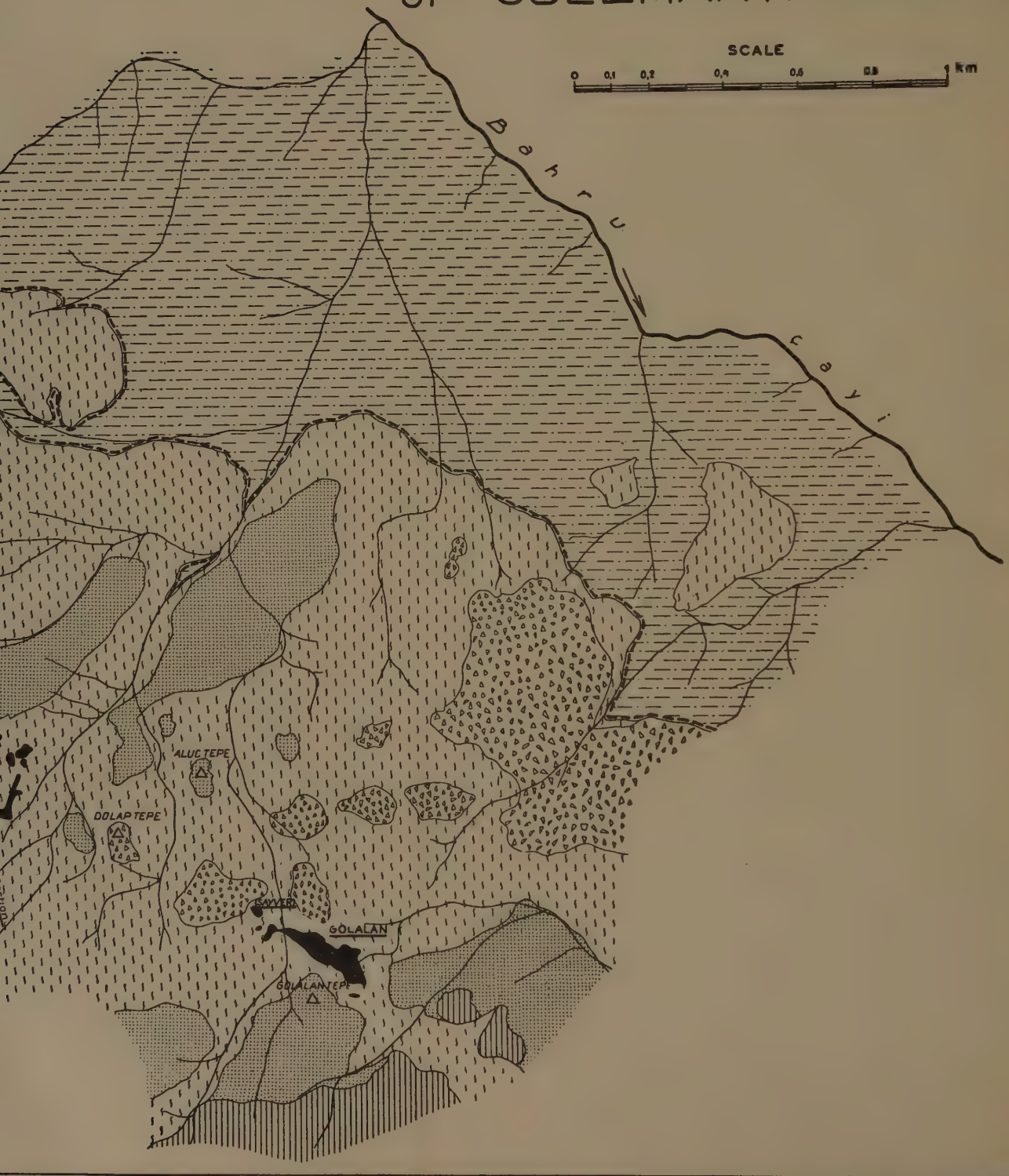
Local geological observations showed us that the formation of the chrome ore as well as the genesis of the ophiolitic mother-rocks is of pre-eocene age. According to the general historic development of the Tauridic orogeny, the chrome ore of Guleman should be assigned a mesozoic age.

REFERENCES.

1. A. HELKE: Die Chromerzlagerstaetten des Vilayets Elâzig, M. T. A. mecmuasi, N°. 3, 1938.
2. L. KOBER: Geologische Forschungen in Vorderasien, Denkschr. d. Math.-Naturw. Kl. d. kais. Ak. Wiss., Bd. 91, Wien, 1915.
3. V. KOVENKO: Les gîtes de chromite de Guleman (Turquie), M. T. A. mecmuasi, N°. 3, 1942.
4. A. PILGER: Zur Entstehung der jugoslavischen Decken, Abh. d. pr. Ak. Wiss., Math-Naturw. Kl., N°. 3, Berlin, 1941.
5. R. PILZ: Beitrag zur Kenntnis der Kupfererzlagerstaetten in der Gegend von Arghana-Maden, Ztschr. f. pr. Geol., 1917.
6. G. ROSIER: Sur la géologie et les gisements de chromite de la région de Guleman (Vil. d'Elâzig), C. R. Soc. de Phys. et d'Hist. nat., Genève, 1942.
7. H. SCHNEIDERHÖHN: Mineraleinschlüsse in Erzmineralien, N. J. f. Min. Geol. u. Pal., Abt. A., Heft 1, 1943.
8. P. DE WIJKERSLOOTH: Die Chromerzprovinzen der Türkei und des Balkans und ihr Verhalten zur Grosstektonik dieser Laender, M. T. A. mecmuasi, N°. 1, 1942.
9. ————: Metamorphose des Chromspinelles in den türkischen Lagerstaetten, M. T.A. mecmuasi, N°. 2, 1942.
10. ————: Mikroskopische Beobachtungen an anatolischen Chromerzen, M. T. A. mecmuasi, N°. 2, 1943.
11. ————: Neuer Beitrag zur Kenntnis der Kupferlagerstaette „Ergani-Maden" in Vilayet Elâzig (Türkei), M. T. A. mecmuasi, N°. 1, 1945.
12. ————: Die hydrothermalen Umwandlungen des Chromites als Begleiterscheinung der Magnesitbildung im Westanatolien, M. T. A. mecmuasi, N°. 2, 1945.


KUNDIKAN
SARITEPE
YANGINTEPE
Centre of Guleman
SAYSIN
KARATAS
SARITEPE
Aereal Cable-way
INDEX:
Violet-red flyschserie of the under-eocene
Breccia and conglomerate of the under-eocene
Grey-yellow flyschserie of the upper-cretaceous
Crystalline schists
White marbles
Serpentine (peridotite, pyroxenite)
Gabbro
Chromite ore-deposits of Guleman
Chromite ore-bodies of Kundikan
Basal-plane of the overthrust-mass of Guleman-Kundikan
Slice-zone of the overthrust-mass

GEOLOGICAL MAP
OF THE CHROME ORE-REGION
OF GULEMAN (TURKEY)



L. W. v. VULPEN

Geology. — *Origin of the Dutch coast.* By J. H. F. UMBGROVE.

(Communicated at the meeting of February 22, 1947.)

1. According to prevailing opinions the Dutch coast originated as a large spit which grew from Sangatte near Calais towards the North and North-East under the influence of tidal currents passing from the Atlantic Ocean towards the North Sea by way of the English Channel.

Marine clay — the so-called old sea clay — accumulated in the bay behind the spit. Afterwards the bay became a fresh-water lake in which a few meters of peat accumulated.

In a still later stage the sea broke through the sand-bars. Its deposits are called young sea clay (fig. 1).

This hypothesis has been propagated by TESCH in several publications [1]). A synopsis of this point of view may be seen in the graphic representation, fig. 2. It shows the thickness of the respective strata, the supposed time of their origin and the situation of sea-level during the last 7000 years.

Before entering into a discussion of TESCH'S hypothesis yet another feature of the coast must be mentioned. The sand formation along the Dutch coast consists of two elements which are of different ages. The older element is a set of parallel low sand barriers. Part of the Westernmost

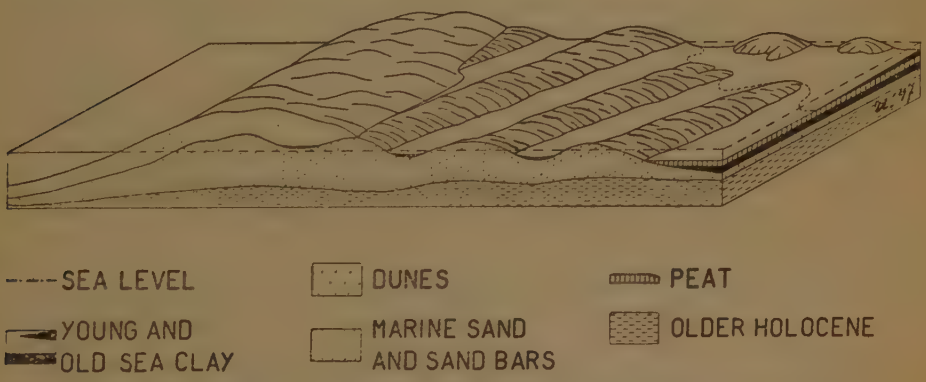


Fig. 1. Schematic block-diagram of part of the Dutch coast between The Hague and Leyden.

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[1]) P. TESCH, Duinstudie I—XIII (Tijdschr. Kon. Ned. Aardrijksk. Genootsch. **37—38**, 1920—1930).

P. TESCH, De vorming van de Nederlandsche duinkust. (Wolters, Groningen 1935.)

P. TESCH, Over de woorden Laagveen en Hoogveen. (Tijdschr. Kon. Aardrijksk. Genootsch. **57**, 1940.)

P, TESCH, De Noordzee (Mededeel. Rijks Geol. Dienst, A no. 9, 1942).

of these ridges is covered by the more recent and much higher dunes as illustrated in the schematic block-diagram fig. 1. The old sand ridges are considered as old dunes by TESCH. The supposed age of the formation of the two coastal elements is also indicated in the graph (fig. 2). Another remarkable feature is that the "old dunes" are intersected abruptly by the coast South of The Hague.

Originally it was supposed by EUGÈNE DUBOIS [2]) that the material of the dunes had derived from the rocks that once formed the connection between France and England. From this it followed that the time of origin of the Dutch coast coincided with the time of origin of the open sea-connection between the Atlantic and the North Sea, i.e. the opening of Dover straits.

From that moment on the spit would have grown by the accumulation of erosion products of the rocks that formerly connected England and France.

2. However, it later became apparent that the Dover straits dates from an earlier epoch than the time of origin of the Dutch coast.

A petrological investigation of bottom samples from the Nort Sea, carried out by BAAK [3]) yielded a quite different result. For it appeared that a great part of the dune sand derived from land areas in the South East, certainly not from the Dover straits and the English Channel.

Moreover VAN VEEN [4]) pointed out that the coast has more the character of a lido or sand barrier coast which means that the old sand barriers were more a product of forces working in a direction at right angles to the coast than of the predominant lateral growth of a spit.

The same author demonstrated in a convincing manner that there is no sand transport of any significance from the Dover straits towards the Dutch coast at present. Therefore, it remained a puzzle what caused the sudden accumulation of the dunes in the nineth century A.D. No satisfactory explanation has been given as yet.

According to TESCH the whole sequence of deposits formed during an uninterrupted rise of sea-level, the amount of which is shown by the inclination of the curve in the graph, fig. 2.

However, in the year 1929 Miss B. POLAK [5]) published the results of

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[2]) E. DUBOIS, Over het ontstaan en de geologische geschiedenis van Vennen, venen en zeeduinen. (Arch. du Musée Teyler, ser. 3, vol. 4, 1919.)

[3]) J. A. BAAK, Regional petrology of the Southern North Sea. (Acad. Thesis, Leiden 1936.) BAAK ascribed a Scandinavian origin to part of the sands (the so-called A-group). Possibly, however, these were transported also by the Rhine. (See J. I. S. ZONNEVELD, Beschouwingen naar aanleiding van de korrelgrootte der zware mineralen in zandige sedimenten. Geolog. en Mijnbouw, 8, 1946, p. 103.)

[4]) J. VAN VEEN, Onderzoekingen in de hoofden (Acad. Thesis, Leiden 1936).

[5]) B. POLAK, Een onderzoek naar de botanische samenstelling van het Hollandsche veen (Acad. thesis, Amsterdam, 1929).

a botanical examination of the peat deposits. She clearly demonstrated that the peat of the western provinces is no low level bog-peat that grew under freshwater conditions.

Instead, she proved that it belongs to the type called upland moor or

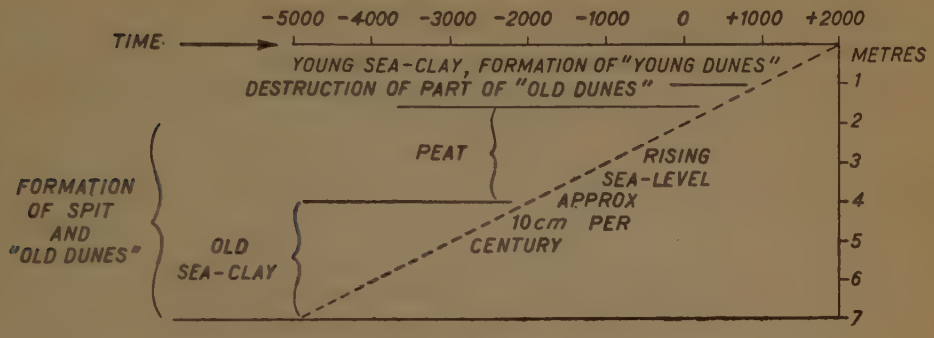

Fig. 2. Graphic representation of Dr. TESCH's hypothesis of a continuous rising sea-level.

Sphagnum peat, which means that it grew under terrestrial conditions. It is impossible to reconcile these results with the theory of a continuously rising sea-level. There could not be the slightest chance for the development of land conditions behind the sand barriers because these barriers had several gaps through which the sea had free passage. There must evidently have been gaps from the very beginning even if there were no other reason than the necessary outlet of the water from the great rivers. And we know that such a gap is widened and scoured out by the sea. In the case of a continuously rising sea-level, as supposed by TESCH, the area of the old sea-clay could not possibly become a freshwater lake. A fortiori it could not change to a land surface on which the upland moor or Sphagnum peat could start to grow.

We do not know how far the barriers extended Southward, but at any rate we need not look for a connection with a local spit near Calais.

3. The peat beds give convincing evidence that their formation had been made possible by a relative fall of sea-level. The graph has to be altered accordingly (fig. 3). Such a procedure implies the acceptance of a subsequent relative rise of the sea in more recent times. For without artificial human constructions present sea-level would extent twice as far Eastward as it did during the formation of the old sea clay. Moreover the lower surface of the peat now lies 4 metres below sea-level.

As a matter of fact such a retreat of the sea was claimed by LORIÉ [6]) as early as 1893, and afterwards by C. A. WEBER and J. VAN BAREN and

[6]) J. LORIÉ, Binnenduinen en bodembewegingen. Tijdschr. Kon. Ned. Aardr. Gen. **10**, 1893.

again recently by GODWIN [7]). But their opinion was rejected by TESCH who has propagated a formation of the peat layers under conditions of continued rise of sea-level [8]).

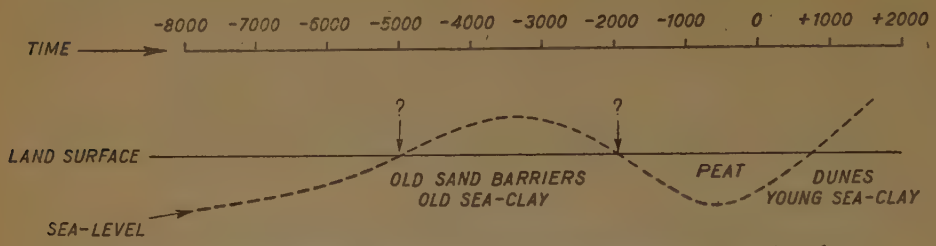

Fig. 3. Relative movement of sea-level when supposing a stable land surface.

4. Now, however, we may correct the term relative fall of sea-level and say that it was a true or so-called eustatic lowering of sea-level.

As early as 1919 DALY [9]) came to suspect a subrecent eustatic lowering of sea-level by a few metres that he placed 3000 to 4000 years ago. Wave-cut benches at approximately the same amount of some 6 metres above sea-level were found in the Pacific, the East-Indies, Africa, the Atlantic, America, in fact all over the world.

Of course emergence of a coastal tract need not be found in areas of a notable subsidence of the bottom like Holland. But Holland would look very different, if no subrecent lowering of sea-level had taken place; and certainly there would be no Sphagnum peat deposits in the area directly behind the coastal dunes.

The regression of the sea as well as the formation of the peat probably coincides approximately with the Sub-Boreal epoch which started about 4000 years ago [10]).

The amount of the emergence shown by more stable parts of the earth is of paramount importance for untangling the interwoven effects of sea-level movement and land movement in Holland. For even if sea-level dropped by an unknown amount, say 6 + x metres and rose subsequently, there always remains an eustatic drop of sea-level of 6 metres since the Sub-Boreal regression started. On the other hand an accurate dating of the peat deposits in Holland is at the same time of importance for an estimate of the age of the 6 metre terraces as far away as the central Pacific!

For the sake of simplicity an amount of 6 metres will be used in our

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[7]) H. GODWIN, Pollen analysis and Quarternary Geology (Proc. Geol. Assoc. LII, 1941, 1941, pp. 328—361); Coastal peatbeds of the North Sea Region, as indices of land- and sea-level changes (New Phytologist, 44, 1945). See also J. A. STEERS, The coastline of England and Wales, 1946, pp. 195 and 426—439.

[8]) TESCH, op. cit. 1922, p. 74; 1928, pp. 78, 79; 1935, p. 14.

[9]) R. A. DALY, The Changing world of the ice Age (Yale Univ. Press 1934).

[10]) Cf. TESCH, XIII, 1930, p. 173, and 1942, Table 1.

further considerations, although one need not expect exactly the same amount all over the world.

A first attempt at analysis of the interplay of the true events that occurred is shown in the curves of fig. 4. The full-drawn line shows the

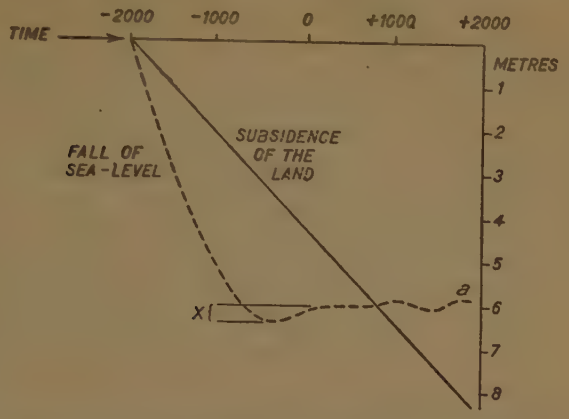

Fig. 4. Analysis of movements of sea-level and subsidence of the bottom, a first rough approximation.

supposed subsidence of the land at a constant rate of 21 cm per century during the last 4000 years. The dot-dash line represents the Sub-Boreal fall of sea-level by an amount of 6 metres. At the point where the two curves intersect, the transgression of the sea starts in Holland. The Sub-Boreal regression was eustatic and worldwide. It was, moreover, directly or indirectly, due to glacial changes. On the other hand the transgression that started about A.D. 850 was regional and caused by subsidence of the bottom which in Holland resulted in a relative rise of sea-level. According to the estimates given just now this relative rise may have amounted to approximately 2 meters since A.D. 850.

The dot-dash line shows a small undulation at the extreme right. GUTENBERG [11]) found from the records of 71 tide gauges over the whole world that since about 1885 sea-level has risen at a secular rate of about 12 cm. It is well known that the glaciers and ice-caps have shrunk all over the world in the course of the last century and this apparently is the cause of the eustatic rise of sea-level.

If this amount be added to the 21 cm subsidence of the land, tide-gauges in the Western coastal districts of Holland ought to show a relative rise of sea-level of 33 cm. Now 30 cm is the amount calculated by RAMAER, and VAN VEEN for the period between the years 1880 and 1920. A diagram in a publication by ESCHER [12]) illustrates this phenomenon very clearly. It

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11) B. GUTENBERG, Changes in sea-level, post-glacial uplift, and mobility of the earth's interior (Bull. Geol. Soc. America, 52, 1941).

12) B. G. ESCHER, Het vraagstuk van de daling van den bodem van Nederland (Geologie en Mijnbouw 1940, pp. 173—196).

shows moreover the increasing movement of later times which is obviously due to the additional eustatic rise.

5. Notwithstanding the remarkable agreement with the records of tide-gauges the graph of fig. 4 has to be regarded as a first and rough approximation. The construction of a more accurate curve, as tentatively shown by fig. 5, is a procedure which meets great difficulties. For there are several uncertain factors. (1) One is the preliminary character of the time scale, the starting point of the Sub-Boreal fall of sea-level being only approximately at 2000 B.C. (2) Another uncertain factor is the exact amount of the Sub-Boreal fall of sea-level. (3) Then, the subsidence of the land is represented by a straight line in fig. 4. It may be that it has to be replaced by an undulating line showing one or more accelerations and retardations as drawn in the curve of fig. 5. (4) Probably the movement

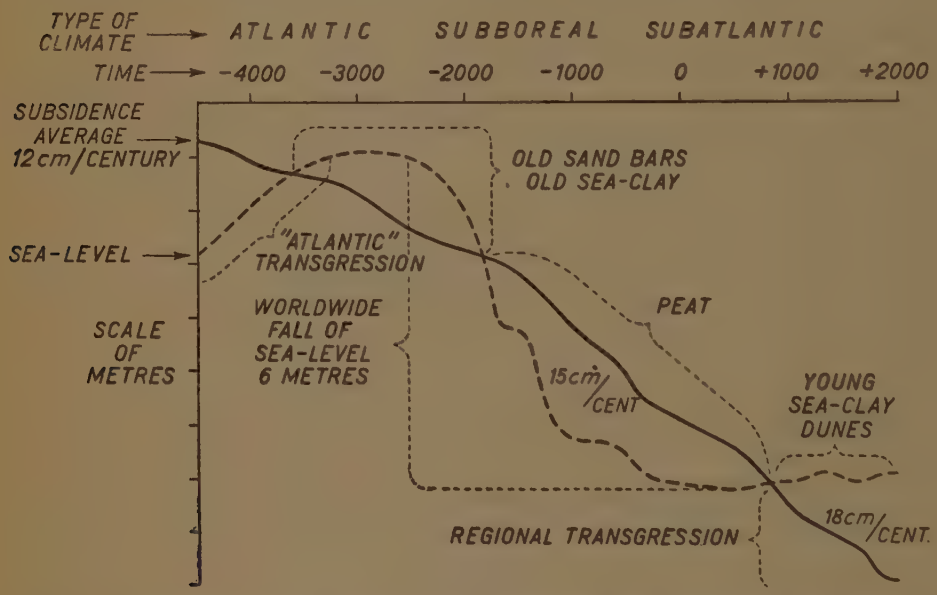

Fig. 5. Tentative curves for the movement of sea-level and subsidence of the bottom.

of sea-level was also intermittent [13]). However, the exact place of the retardations as well as the steepness of the sea-level curve are entirely arbitrary. (5) The subsidence line intersects the sea-level curve at A.D. 850. From historical data this seems the right point for the Western part of Holland. It is quite well possible that a differt epoch is more probable in other districts and then the graph has to be altered accordingly.

In Flanders, for example, the point of intersection seems to be at an earlier date, viz. approximately A.D. 200. (6) According to a recent paper by FLORSCHÜTZ [14]) the boundary between Boreal and Atlantic, which is

[13]) Cf. PH. H. KUENEN, Geology of coral Reefs. Snellius Exp. 1933.

[14]) F. FLORSCHÜTZ, Laagterras en veen op grootere diepte onder Velzen. Tijdschr. Kon. Ned. Aardr. Gen. 61, 1944.

assumed at about — 5000 by TESCH, ought to be placed somewhat later. In FLORSCHÜTZ's opinion deposition of the old sea clay started somewhere in the climatic stage called Atlantic by botanists. It remains to be seen what the correct chronological dating is of this climatic indication. Provisionally the beginning of the sedimentation of the old sea clay is inserted in the graph of fig. 5 somewhere between 4000 and 3000 B.C. It will have to be adjusted to future more accurate data. (7) But there are still other pittfalls. The thickness of the old sea clay can not be deduced from the curve. For although it depends on two factors which can be expressed by the curves (viz. rise of sea-level and subsidence) there are still others that cannot be included in the graph, such as the scouring and excavating effect excercised by sea-currents on the original land surface.

Another uncertain factor is the distance between sea-level and surface of sedimentation of the old sea clay.

Obviously sea-level had been lowered a certain amount before the surface of the old sea clay had become dry land and peat could start to grow.

Apparently the water was not deep. This may be deduced from the following comparison. At present large areas of Holland are about two metres below sea-level. The boundary of the area which is below sea-level is, however, twice as far landward as the Eastern boundary of the old sea clay.

(8) Probably the peat was subjected to a stronger compaction than the underlying sand and clay layers. This process was still accelerated since the water level was lowered artificially during the past ten centuries. Compaction of the sediments is one factor which finds expression in the subsidence of the bottom. Accordingly the land curve of fig. 5 corresponds with an average subsidence of 12 cm per century during the period from — 3500 to — 2000, against 15 cm secular from — 2000 to + 850 and 18 cm from + 850 to present times. Again the extra secular compaction of 3 cm before + 850 and of 6 cm after that year are figures that need further precision.

(9) Finally, the records from tide-gauges have to be taken cum grano salis. In a recent paper KUENEN [15]) has published a new table of recalculated tide-gauge records. According to his results the records show an uninterrupted relative rise of sea-level of some 20 cm per century since 1860. Assuming an eustatic rise of sea-level of 12 cm the bottom movement would have been in the order of 8 cm [16]). If these data have to be

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[15]) PH. H. KUENEN. De zeespiegelrijzing der laatste decennia. (Tijdschr. Kon. Nederl. Aardr. Genootschap 62, 1945.)

[16]) KUENEN gives an average subsidence of 5 cm per century, and he thinks this to be the right figure for longer times. Without the sub-boreal fall of sea-level an average of 5 cm would suffice to explain the present location of the upper surface of the young sea-clay. However, taking into account a fall of sea-level of 6 meters the subsidence must have been in the order of 21 cm secular. The same amount is needed to explain the situation of North- and South-Holland at an average of two meters below present

regarded as well established the result would point to a slowing down of the subsidence. This possibility is tentatively indicated at the utmost right of the graphs. However, I believe that all these data have a preliminary character and that an accurate calculation of these amounts have to be left until more trustworthy data will be available. All these remarks show the provisional character of fig. 5. The graph has no other meaning than a preliminary attempt at illustrating the events that co-operated in the formation of the coastal districts of the Western provinces of Holland.

6. A second problem is solved if the Sub-Boreal regression and the subsequent regional transgression be accepted. For now the destruction of the "old dunes" and the reasons for the formation of the new dunes are no longer a puzzle. The location of the coastline before the last transgression, i.e. before the Romans arrived is well illustrated in a map by VAN VEEN. Since then the coast has receded in many places, for example two kilometers near Den Helder between A.D. 1571 and 1866 [17]).

The shape of the coast adjusted itself to the relative rise of sea-level and the prevailing system of currents as has been fully described by VAN VEEN. The present shape of the coast dissects the trend of the former coast and the old sand barriers.

The sea received a great quantity of sand not only from the demolished "old dunes", but also from deeper parts due to the adjustment of the submarine profile to the new coastline (the stippled part of fig. 6 as well as the part that shows vertical shading). This material was heaped up in the younger dunes. As a result of the then and still prevailing strong Westerly (SW and NW) winds, the dunes are formations which are very different from the low and parallel sand barriers, which were predominantly modelled by the action of the sea.

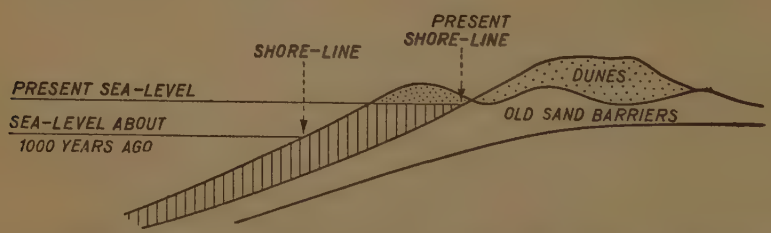

Fig. 6. Schematic section illustrating origin of dunes and changing submarine profile, due to a relative rise of sea-level.

The relative rise of sea-level caused many invasions of the sea (leaving the younger sea clay deposits behind). At this stage man was forced to occupy himself intensively with the situation. Gradually dikes and canals

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sea-level. KUENEN's estimate of 5 cm was an average for the whole coast. For North and South Holland, the provinces we have considered, he is inclined to assume about 10 cm (personal communication).

[17]) VAN VEEN, op. cit. p. 141, fig. 117.

were built; in short a system of polders came into existence in order to drain the land of the Western provinces which are actually a few metres below sea-level.

7. It is difficult to say what the cause is of the suggested downward movement of the land surface. Three factors may have been of special influence viz. (1) compaction of the sediments, (2) subsidence of the peripheral belt around the last Scandinavian ice-cap, (3) accelerated subsidence of the basement. The Netherlands form part of the so-called North Sea Basin. Subsidence of the floor of the basin started in the Permian and continues up to our days at an average rate of 0.4 cm per century [18]). But undoubtedly periods of accelerated and retarded subsidence alternated. Apparently we are living in a period of accelerated subsidence. No data are available on which a quantitative estimate of the influence of each factor can be based. As mentioned in section (5) the compaction of the subrecent sediments is estimated tentatively at 3 up to 6 cm secular since 2000 B.C. The combined effect of the three factors certainly surpasses the average subsidence taken over long geological periods. For even the left part of the subsidence curve of fig. 5 shows an average movement of 12 cm per century.

The structural boundary of the North Sea basin is formed by the massif of Brabant in the South and by the axis of Erkelenz in the East. During the long history of the basin its shape underwent some changes. The basin of London formed as a prolongation in its Southwestern corner in Tertiary times. A structural map of the pre-Pleistocene surface shows the isobases of —50 and —100 metres to run nearly at right angles to the axis of Erkelenz in the Northeastern part of the Netherlands [19]). In the same way a prolongation of the basin originated in a Southwestern direction thus forming a depression right across the former prolongation of the Brabant Massif.

This area covers part of the coastal districts of Belgium as far South as the neighbourhood of Calais. As a matter of fact the Holocene sequence of strata in Flanders strongly resembles that of Holland [20]). It seems, however, that the last relative rise of sea-level took place between A.D 200 and 500 in this area [21]). On the other hand a shore-line at about 5 metres above present sea-level was mentioned by several authors from the coasts of the Mediterranean and the Atlantic coast of Northern France

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[18]) Cf., UMBGROVE. Periodical events in the North Sea basin. Geolog. Magaz. 82, 1945, p. 238, Table A and The Pulse of the Earth, Sec. ed. 1947, pp. 312—313.

[19]) P. TESCH. Tektonische lijnen in Nederland. Geologie en Mijnbouw 1932, p. 233, kaart 2.

[20]) R. BLANCHARD. L'Origine des Moëres de la plaine maritime de Flandre. Bull. de la Soc. de Geographie 31 (1917) pp. 337—346.

[21]) Cf. TESCH. Duinstudies XIII, 1930, p. 174.

including the area of the English Channel. According to BRIQUET [22]) the 5 metre level in Northern France originated during the third or Riss glaciation. That sea-level was higher than it is now during the greatest Pleistocene glaciation would be an absurd supposition. At any rate BRIQUET asserts that the successive formation of the Flandrian clays, peat deposits and Dunkirk clays — which possibly correspond respectively with the old sea clay, peat, and young sea clay in the coastal district of Holland [23]) — took place after the formation of the 5 metre level. There is, however, no reliable basis for a correlation of the 5 m shore-lines in the area of the English Channel with the eustatic fall of sea-level of Sub-Boreal times. Tectonic and isostatic forces may have interfered. For the same reason there is still great confusion concerning the correlation of higher terraces [24]).

At any rate, however, the "Sub-Boreal" regression must be taken into account for the explanation of any coastal district if this phenomenon is considered of world-wide importance. The aim of the present paper is to show its importance for a good understanding of the origin of the Dutch coast.

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[22]) A. BRIQUET, Le littoral du Nord de la France et son évolution morphologique. (A. Colin, Paris 1930) p. 395.

[23]) G. DUBOIS, Classification du Quarternaire du Nord de la France et comparaison avec le Quarternaire danois. C. R. Ac. des Sci. 179, 1924.

G. DUBOIS, Reserches sur les terrains quarternaires du Nord de la France. Mém. Soc. géol. du Nord, tome VIII, no. 1, 1924, pp. 1—335.

[24]) BLANC gives, for example, an interpretation of coastal terraces in the Mediterranean which differs in many respects from the well-known scheme of DÉPÉRET. Cf. A. C. BLANC, Low levels of the Mediterranean sea during the Pleistocene Glaciation. (Quart. Journ. Geol. Soc. London, vol. 93, 1937, pp. 621—651).

Physics. — *Convection-currents in the Earth* *). By F. A. VENING MEINESZ.

(Communicated at the meeting of February 22, 1947.)

In a previous paper the writer has drawn attention to the fact that the hypothesis of convection-currents in the subcrustal layer under the eastern half of the Indian Archipelago may give a good explanation of the deep-focus and the intermediate earthquakes in this area. He emitted this hypothesis already in 1932 [1]) for explaining the sinking down of the deep basins which are clearly connected with the great folding processes in the neighbouring tectonic belts but which probably have originated with a great time-lag of many millions of years after the folding. In the Banda arc the last great folding period is put by UMBGROVE [2]) in tertiary *f2*, i.e. some twenty million years ago, while the sinking of the Banda basin, although difficult to determine exactly by direct evidence, is probably much more recent. Direct evidence that this area has provided erosion products to the surrounding tectonic belt and that it, therefore, must have been above sealevel, dates further back but, as MOLENGRAAFF already pointed out and as UMBGROVE also is inclined to assume, it seems likely that the principal part of the sinking of the basin is simultaneous with the rising of the adjacent tectonic belt which took place in the pleistocene, i.e. only some 1 or 2 million years ago. MOLENGRAAFF explained, for example, that in Timor, the miocene folding is intersected by the present coast-line of the deep basins and that it must, therefore, be anterior to the sinking. His opinion about the simultaneity of the rising and sinking movements is widely accepted. It agrees with the explanation of both movements by the hypothesis of a convection-current in the subcrustal layer which at the same time must bring about a rising above the rising current and a sinking above the sinking one.

This hypothesis may also explain the evident connection of these crustal movements with the folding phenomenon and the great time-lag between both. During the folding-period the Earth's crust may be supposed to have down-buckled along the tectonic belt, thus forming a considerable crustal bulge at the lower boundary of the crust, which according to the negative anomalies and the topography in this belt and assuming a density difference with the substratum of 0,6 must have a cross-section of 1500—2000 km^2. It must have pushed away the subcrustal material. As this last material is poorer in radio-active constituents than the crust, we may expect a slow heating up of this area by the excess of radio-active radiation caused by the concentration of crustal material along the belt.

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*) Lecture on December 21, 1946.

1) F. A. VENING MEINESZ, J. H. F. UMBGROVE, PH. H. KUENEN, Gravity Expeditions at Sea, Vol. II, Publ. Netherl. Geod. Comm., Waltman, Delft, p. 135.

2) Id. p. 140 e.s.

This heating up must have disturbed the equilibrium in the substratum. If we assume with most geophysicists that the Earth is cooling notwithstanding the amount of radio-active minerals present in the outer layers, the substratum must have shown a downward temperature-gradient tending to bring about instability as it causes layers of lower temperature to overlie higher temperature layers. The writer, however, agrees with JEFFREYS in supposing that these layers have some strength or, in other words, that below a certain limit, stresses only bring about elastical deformations; the stresses have, therefore, to exceed this limit before flow can take place. According to the smallness of the deviations from isostatic equilibrium of 10—15 mgal the writer supposes this limit to lie between 25 and 50 kg/cm^2. If only the vertical temperature-gradient caused by the cooling of the Earth is present, this strength must prevent any convection-current to originate and it has, therefore, a stabilising effect. If, however, at the same time, a temperature gradient in a horizontal sense is present, the normal density equilibrium in horizontal layers is disturbed. If this gradient is sufficient the resulting stresses will overcome the strength-limit of the substratum and a current must set in which, because of the vertical temperature-gradient, takes the character of a convection-current.

We must expect this heating of the subcrustal layer below the tectonic belt to a temperature sufficient to start the above phenomenon, to be a slow process and so it does not seem unlikely that we can thus explain the time-lag of many millions of years between the folding in the tectonic belt and the coming about of the convection-current. It is important to investigate this numerically to see whether this is possible.

We shall begin by assuming that the whole crustal root consists of granite. This leads to a density difference from the substratum of about 0.6 (peridotite $=$ 3.23, dunite $=$ 3.29) as we assumed above when mentioning the cross-section of 1500 — 2000 km^2. For our deductions we shall adopt a cross-section of 1700 km^2.

For the radio-active heat-production we shall use the figures for different types of rocks given by GUTENBERG in table 22 on page 155 of "Internal Constitution of the Earth" and the figures of Table 18—2 on page 270 of "Handbook of physical constants" by BIRCH, SCHAIRER and SPICER. The mean figure for granite given by GUTENBERG is 7.8×10^{-13} cal/cm^3/sec and by the second table 5.6×10^{-6} cal/gram/year which gives 4.8×10^{-13} cal/cm^3/sec. The mean figure is 6.3×10^{-13} cal/cm^3/sec. Taking the mean for peridotite and dunite in GUTENBERG's table we find 1.5×10^{-13} cal/cm^3/sec and the Handbook gives for ultra-basic rocks 0.9×10^{-6} cal/gram/year $= 0.9 \times 10^{-13}$ cal/cm^3/sec; the mean is 1.2×10^{-13} cal/cm^3/sec. The excess heat developed in the root is the difference and so we obtain 5.1×10^{-13} cal/cm^3/sec. For the whole root of 1700 km^2 cross-section this gives 9 cal/sec/cm (the cm dimension at right angles to the cross-section).

This figure has been derived for a root consisting entirely of granite.

This is, however, unlikely. If the root has been formed by the downward buckling of the crust, as it has been supposed, we must expect deep crustal layers to be present and it is usually assumed that these layers are not granitic but more basic. As, according to the assumed mode of originating of the root, these layers may be expected to form the outer shell of it and as we may suppose the root to have partially melted away because of the higher temperature these materials must have been subjected to when they were pushed downwards, it would of course be possible that a great part of this outer shell has disappeared. The explanation of the more acid type of volcanism in the nearby inner Banda-arc by the flowing off of the molten material of the root, as it has e.g. been advocated by ANDERSON [3]), would seem to point to the granitic central part of the root being now exposed to the melting and this would appear to confirm our last supposition.

We cannot, however, come to any certainty about this point and so it is no doubt possible that the root partially consists of deeper rocks than granite. It is simple to see that if for these rocks the difference of the heat-production from that of the sub-crustal material would be proportional to the difference of the density from that of this same subcrustal layer our result for the excess of the heat-production of the root would remain the same. The figures for the heat-production given by GUTENBERG and others, however, deviate from this proportionality in the sense that the heat-production in the crust diminishes probably quicker with depth. Resuming we must recognize that the above mentioned figure of 9 cal/sec/cm for the root may be too high and so we shall reduce it to 7.5 cal/sec/cm.

Adopting this figure the problem has to be solved what temperature distribution in the subcrustal material is caused by this source of heat after a lapse of time of some 18 million years. For this solution we have to apply the formula of heat-conduction for the two-dimensional case represented by our problem. This is given by the differential equation for the temperature θ

$$a \Delta \theta = \frac{\partial \theta}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1\,A)$$

where $\Delta\theta$ expressed in polar coordinates r, φ is

$$\Delta \theta = \frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \theta}{\partial \varphi^2} \quad . \quad . \quad . \quad . \quad . \quad (1\,B)$$

and

$$a = \frac{\lambda}{c\varrho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1\,C)$$

λ = coeff. of thermal conductivity,
c = heat-capacity,
ϱ = density.

The quantities λ, c and ϱ all refer to the subcrustal material.

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[3]) W. Q. KENNEDY and E. M. ANDERSON, Crustal layers and the origin of magmas, Bull. Volcan. S. II, Tome III, 1938.

We shall simplify our problem by the suppositions that the root has a circular cross-section with a radius r_0 and that the heat-production is concentrated in the centre. We shall furthermore assume that the phenomenon is cylindrically symmetric. As the fundamental law of heat-conduction given by 1 A is linear in θ our phenomenon is not affected by the normal cooling of the Earth and so from this point of view there is no objection to adopting this assumption.

The presence of the surface of the Earth at a distance of about 49 km (thickness of the crust = 30 km + distance of centre below the lower crustal boundary = ± 19 km) must, however, affect it and so we have to prove that this effect is negligible. It is simple to do this. The effect of the boundary can be taken account of by assuming the presence of a sink of heat of the same amount as the heat-source in the root and at the same distance of 49 km above the Earth's surface. This sink is, therefore, at a distance of 98 km from the source. We shall, however, find that the effect of the sink at this distance is negligible and so we can maintain our supposition of cylindrical symmetry.

According to this supposition formulas 1 A and 1 B become

$$a\left(\frac{\partial^2\theta}{\partial r^2}+\frac{1}{r}\frac{\partial\theta}{\partial r}\right)=\frac{\partial\theta}{\partial t}. \quad \ldots\ldots\ldots \quad (2)$$

The solution of this equation is an exponental integral, which we shall as usual indicate by the symbol Ei

$$\theta=\frac{q}{4\pi\lambda}\left[-Ei\left(-\frac{r^2}{4at}\right)\right] \qquad \left(Eix=\int_{\infty}^{-x}\frac{e^{-u}}{u}\,du\right) \quad . \quad . \quad (3)$$

where q is the heat-production in cal/cm³/sec.

We introduce

q = 7.5 cal/cm³/sec.
λ = 0.01 (see "Handbook" p. 254).
c = 0.20 (see "Handbook" p. 235 e.s.).
ϱ = 3.27.

This leads to

a = 0.015.

We compute θ for a time t = 18 million years = 5.67×10^{14} sec.

Using for $Ei\,(-x)$ the table in JAHNKE u. EMDE "Funktionentafeln" p. 21, 22, we find the following values for θ. For the radius r_0 of the root we introduce 23 km which corresponds to the supposed cross-section of the root of 1700 km².

r km	$x=\frac{r^2}{4at}$	$-Ei(-x)$	θ
23	0.1555	1.432	+ 85°.4
40	0.470	0.599	+ 35°.8
60	1.058	0.200	+ 12°.0
80	1.880	0.0506	+ 3°.0
100	2.940	0.0141	+ 0°.8

It is remarkable to see that a temperature of 85° at the surface of the root requires such a long lapse of time to come into being. It is also interesting to find that the heat after this long interval practically did not come beyond some 100 km. We see here our supposition confirmed that the effect of a heat-sink at this distance is negligible.

For making a rough estimate of the magnitude of the stresses caused by this temperature distribution we compute the rise of the surface of the substratum brought about by the expansion while neglecting the elastical deformations caused by the resulting stresses. For this rough estimate we adopt a rectangular cross-section for the root of a height of 37 km and a breadth of 46 km, and we assume in the column of the substratum bordering

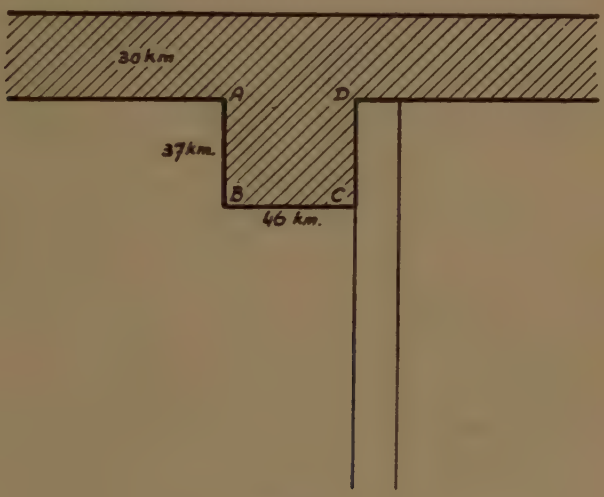

Fig. 1.

the root a temperature of 85°.4 over 37 km height and below this a falling off of the temperature as given by the above table i.e. at 17 km below the point C a temp. of 35°.8, at 37 km below it a temp. of 12°, etc. We adopt a volumetric thermal expansion of the substratum of 3×10^{-5} and we assume that the adjustment of the hydrostatic equilibrium of these layers leads to the entire expansion appearing at the surface. We then find a rise there of 146 meters which for a density of 3.27 represents an anomaly of 14 mgal and an excess pressure of 45 kg/cm^2. This is exactly the order of magnitude we may presume for the strength-limit of the subcrustal layers and so we see that the great time-lag of 18.000.000 years before the starting of the convection-current could well be explained. When the strength-limit is exceeded by further heating, the subcrustal matter would flow off from the neighbourhood of the root and this would disturb the equilibrium in the column below it; a rising current would set in here while a sinking one would originate in the area which had become loaded by this flow while the pressure difference in the deeper layer would

bring about a flow contrary to the one at the surface. The convection-current would thus be started and as it begins by bringing about an increase of the temperature-difference between the rising and sinking columns it would accelerate till after some time a maximum difference would be reached; this would probably occur after about a quarter of a complete turn. The speed will then decrease again till the current has about made half of a complete turn, i.e. when it has brought the higher temperature matter on top and the lower temperature matter below; the rising and sinking columns will then have assumed the same mean temperature and the equilibrium is restored. The heat conduction will more or less alter this picture and the amount of movement needed for restoring the equilibrium but it does not change the principle of the phenomenon.

If we assume that up to now about a quarter of a complete revolution has been made and supposing in accordance with the dimensions of the Banda basin and the depth of about 400 km of the deep earthquakes in this area that the current takes place inside a cross-section of about 500 km height and 500 km breadth, we find that in 2.000.000 years it must have travelled a distance of about 200 km and so the mean velocity must have been about 10 cm per year. During this interval which is only one ninth of the interval involved in the above problem, the cooling cannot have got much further than to a depth of about 40 km, i.e. to a fifth of the depth of the upper horizontal part of the current. It does not seem likely that this can have seriously affected the course of events.

For obtaining an estimate of the sinking of the surface above the sinking column we have to assume the temperature distribution in the subcrustal layer before the current started and outside the area heated by the root. Referring to the curve given by GUTENBERG in "Internal Constitution of the Earth" page 162, we think that fig. 2 gives an acceptable estimate. We may probably adopt the temperature at the lower boundary of the rigid crust at about 700° which would mean a gradient in the crust varying from 30°/km at the surface to 10°/km at the bottom.

If we assume that during the history of the Earth the subcrustal layer has been turned over several times by convection-currents, the temperature-curve in this layer has to show, as GUTENBERG remarks, a much smaller gradient of, for instance, 1°/km; from 100 km — 500 km depth he assumes a temperature of 1500 — 1800°. In the upper layer, however, where it touches the crust a cooling curve must have formed since the last convection-current took place, which might have occurred after the Eocene folding period in the tectonic belt. This has been represented by the curve of fig. 2.

From this curve we may estimate the mean temperature of the upper 200 km of the subcrustal layer at some 1200° and after a quarter revolution of the current this mean temperature must have reigned in the greatest part of the sinking column, increased, however, by a certain amount because of the heat-conduction from the surrounding matter. As a result we may

make the rough estimate that over the lower half of this column the mean temperature will be about 500° below the normal one. Adopting again a thermal expansion coefficient of 3×10^{-5} this amounts to a shortening of the column of 3.8 km.

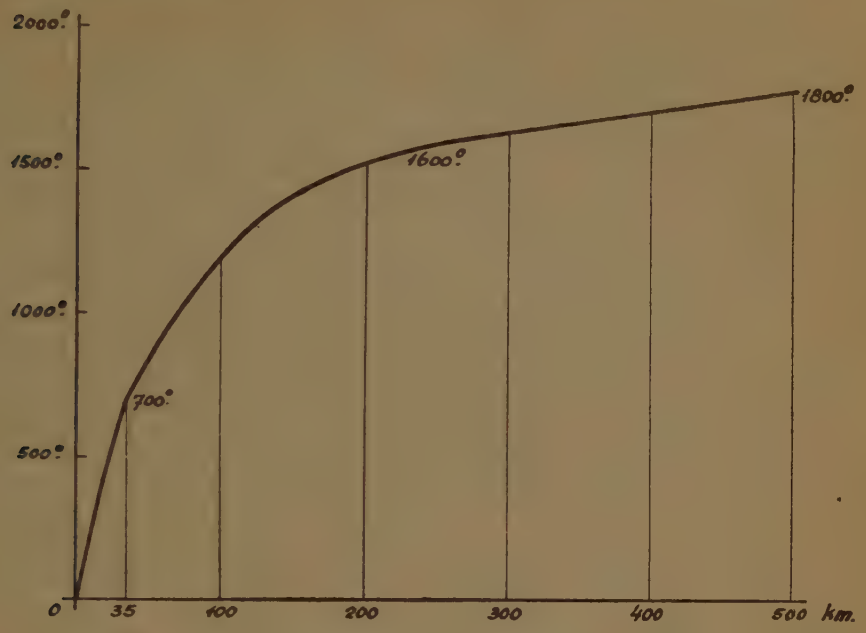

Fig. 2. Temperature curve in the upper 500 km.

We may assume that about half of this shortening will appear at the surface and the other half at the bottom of the column, thus causing the pressure gradient needed for the horizontal parts of the currents at the surface and at the bottom, the first being directed towards the sinking column and the second away from it. This rough assumption is in harmony with the formulas for a steady convection-current; in another paper the writer hopes to enlarge on these deductions. We come to the conclusion that we may expect a sinking of the surface over the sinking column of about 2 km.

The reverse must be true for the rising column, and so we may estimate the rising of the surface here at this same value. We neglect here the rising at the start caused by the radio-active effect of the root as derived above. The result would thus be a difference between the two areas of about 4 km. This difference is in good harmony with the actual topography of the Banda arc area and, in fact, of the whole eastern half of the archipelago but there is an obvious disagreement in the fact that the mean level of the whole region has not remained the same when the deep basins originated but that it has been lowered over 1.5 or 2 km. This point has already been raised by MAC GILLAVRY in 1930; he drew attention to the difficulty of explaining this lowering of the mean level, while we might

rather expect that the thickening of the sialic crust by the orogenic phenomena would bring about a rising of this mean level.

Our hypothesis of convection-currents in the subcrustal layer can give a simple explanation of this difficulty and it can at the same time make clear that this area as a whole must show an excess of gravity as the anomaly-field of the East Indies shows it to be the case. The explanation is that the presence of convection-currents brings about an increased cooling of the Earth in this area. This is clear as the sinking current brings matter of lower temperature downwards which absorbs heat from the surrounding matter while the excess heat brought to the surface by the rising current is for a large part lost to the Earth by an increased radiation at the surface. Admitting this excess of cooling up to a depth slightly larger than that to which the currents go, it follows that the density is correspondently larger than normal while the shrinking must lower the surface. If the whole area consisted of rising and sinking columns and if the excess heat of the rising column was entirely radiated into space we should thus obtain a mean lowering of the surface of half the shrinking of 3.8 km derived before for the sinking column, i.e. about 2 km. This figure may be too large but it is certainly possible that it would attain the amount of 1.5 km which is present. It is also possible that part of this amount has been caused by older current-systems. In that case this part of the lowering would, however, already have been present during the last folding in tertiary *f* 2.

The discussion of the gravity field may be postponed to a future paper. The writer wishes here only to add a short remark which follows from the last deductions. If we are right in attributing the low level of the mean topography in the eastern half of the East Indies and, in fact, in the whole area east of Asia where we find tectonic basins to an abnormal cooling here of the Earth up to a depth of some 600—800 km, this must mean a disturbance of the thermal equilibrium of the Earth of a much greater size than that caused directly by the presence of the crustal root as derived before. The question then arises whether this would not be likely to start a correspondently greater type of convection-current, made possible by the presence to a much greater depth of a slight temperature gradient of the order of 1°/km or less caused by the cooling.

This current-system rising under a broad borderzone of the continent and sinking under the adjoining oceanic zone would thus have continental dimensions and might even be supposed to involve the whole thickness of the mantle up to a depth of 2900 km. It might be imagined to bring about the major cycles of orogeny well known in the geological history of the Earth [4]). During each cycle the current would only make about one half of a revolution in the same way as it has been explained for the smaller type

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[4]) See also: DAVID GRIGGS, A Theory of Mountain-building. Amer. Journ. o. Sc. **237**, 611—650 (1939).

of current dealt with above. After each cycle the temperature gradient must have disappeared and it takes a long period of rest before the cooling has brought it again into being.

There is much to be said in favour of this explanation of the orogenic cycles. It explains their occurring more or less periodically with a variable period. It likewise explains the regressions in the beginning of an orogenic period and the slow change to transgressions in the end of this period and during the long period of rest between the cycles; these regressions are connected with the rise of the surface over the borderzones of the continents when a rising current originates there, while the transgressions come into being when the current is slowly dying out.

If these speculations may be allowed we should come to the conclusion that the initial disturbance of the thermal equilibrium needed for bringing about these large cycles would be caused by the smaller types of current-system here discussed which themselves are started as a direct consequence of the folding together of the Earth's crust. We thus would see here a chain of events of steadily increasing size started in the beginning by al small surface effect of the Earth and leading eventually to a big current-system having its main cause in the cooling of the Earth which provides it with the energy required.

Physics. — *Brittle rupture of plastic material.* By F. K. TH. VAN ITERSON.

(Communicated at the meeting of January 25, 1947.)

Rupture at sharp incisions.

All material can rupture in two ways [1]):

1. Through plastic flow, according to the law of MAXWELL amended by the author, i.e. with perfect plasticity, flowing and contracting like heated glass while the difference between two principal stresses remains equal to the yield stress.

2. With a sudden rupture, according to the law of MARIOTTE–PONCELET, when the greatest strain (extension) reaches the breaking strain. What happens when the load on our structural part is gradually increased depends on which of the criteria is first fulfilled.

At the free surface of a mild steel beam one of the principal stresses, called ϱ_3, is zero.

When ϱ_2, another principal stress, is also zero, the material flows when the only principal stress ϱ is equal to the yield stress. If another principal stress exists, the greatest strain is even less and the danger of sudden rupture is still smaller. Hence we come to the peculiar conclusion, that a break-down in plastic material, for instance in mild steel, will never start at the surface. But a brittle rupture, a sudden laceration, may occur in the interior of such material at a spot where three-dimensional tension prevails.

Imagine a hot-rolled steel beam rapidly cooled (fig. 1). It may be that

Fig. 1.

the thin rib first solidifies and then the tips of the flanges. These parts cool down further while cores at *a* of the section are still red-hot. When

[1]) See chapters 20 and 21 of Traité de Plasticité pour l'Ingénieur; Liège 1944; or Plasticiteitsleer, Luik en Deventer 1945.

these cores also become solid they fit in under a neglegible tension. Let us suppose that the cores are only 100° C hotter than the material further off which is already blue-warm. But then the temperature equalizes. The core tends to shrink and with the modulus of elasticity $E = 2.000.000$ kg/cm². Poissons modulus $m = \frac{10}{3}$ and the coefficient of expansion $\alpha = 0{,}00001$, the tension in all directions would be 5000 kg/cm². This is not so harmful as it looks because the ideal tension accordingly to MARIOTTE–PONCELET would only be $\left(1 - \frac{2}{m}\right) 5000 = 2000$ kg/cm². But this high tension in all directions makes the mild steel beam liable to brittle fracture. By superposing a load on the structure there is a chance that the breaking strain may be reached before plastic flow sets in, and in fact this often happens. The author has seen sixteen ruptures when the flanges of four beams were cut at both ends by the oxy-acetylene torch, as represented in figure 2. The rents in the rib starting from the danger spot were due to the expansion of the flanges when they were heated.

Fig. 2.

But why does a brittle rupture often spread in a plastic material? The question is so important in engineering, that it must be amply discussed.

Brittle fracture in mild steel indicates tension-stress in every direction. We therefore have to infer the occurrence of three-dimensional stress near the ends of cracks. In order to prove the existence of such stresses we start from the stress distribution according to the laws of elastic deformation and tensions near the bottom of incisions. Since Inglis solved this problem by means of elliptic co-ordinates we may refer to literature [2]).

Let figure 3 represent a stretched bar of great length in the direction of the X-axis. The grooves at both sides are hyperbolic, according to the formula

$$\frac{y^2}{a^2} - \frac{z^2}{b^2} = 1.$$

If we consider a section of this bar of length t and suppose the pull to be P, then the average tension in the weakened section is $p = \frac{P}{2at}$.

[2]) H. NEUBER, Kerbspannungslehre, Berlin 1937, p. 33: Die beiderseitige Aussenkerbe.

The radius of curvature in the bottom of the groove is $\varrho = \frac{b^2}{a}$.

Transformed from elliptic co-ordinates to orthogonal co-ordinates the principal tensions in the danger section are:

$$\sigma_z = \frac{(a^2 + 2a\varrho - y^2)(a^2 + a\varrho)}{(a^2 + a\varrho - y^2)^{\frac{3}{2}} \left\{(a+\varrho) \operatorname{arc\,tg} \sqrt{\frac{a}{\varrho}} + \sqrt{a\varrho}\right\}} p$$

$$\sigma_y = \frac{(a^2 - y^2)(a^2 + a\varrho)}{(a^2 + a\varrho - y^2)^{\frac{3}{2}} \left\{(a+\varrho) \operatorname{arc\,tg} \sqrt{\frac{a}{\varrho}} + \sqrt{a\varrho}\right\}} p$$

$$\sigma_x = 0{,}6 \frac{a^2 + a\varrho}{(a^2 + a\varrho - y^2)^{\frac{3}{2}} \left\{(a+\varrho) \operatorname{arc\,tg} \sqrt{\frac{a}{\varrho}} + \sqrt{a\varrho}\right\}} p.$$

This last tension σ_x in the direction of the X-axis is derived from σ_z and σ_y assuming that the width w and the breadth t of the bar and the

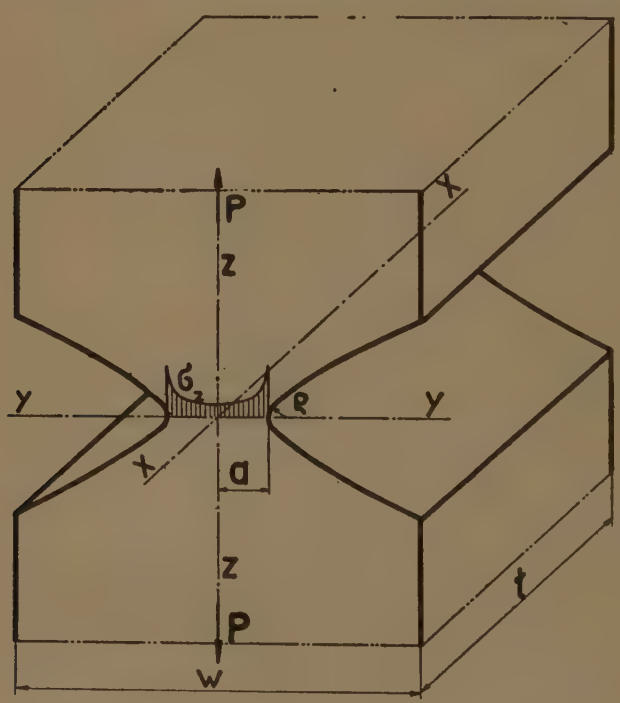

Fig. 3.

incision are so great that a contraction in the length w of the endangered section may be neglected. Then $\sigma_x = \frac{\sigma_z + \sigma_y}{m}$. For steel $m = \frac{10}{3}$.

In figures 4, 5 and 6 the distribution of the three principal stresses is shown for half the section of the incised bar having hyperbolic grooves

with curvatures at the vertex respectively

$$\varrho = \frac{a}{4}, \quad \varrho = \frac{a}{16} \quad \text{and} \quad \varrho = \frac{a}{100}.$$

We see from these figures that three-dimensional tensile-stress occurs in grooved bars below the bottom of the notch and that the tensions tend to infinity when ϱ decreases. This was to be proved.

Now we return to the theory of plasticity.

At the bottom of the groove $\sigma_y = 0$ and, as in plastic flow two tensions become equal, $\sigma_x = 0$.

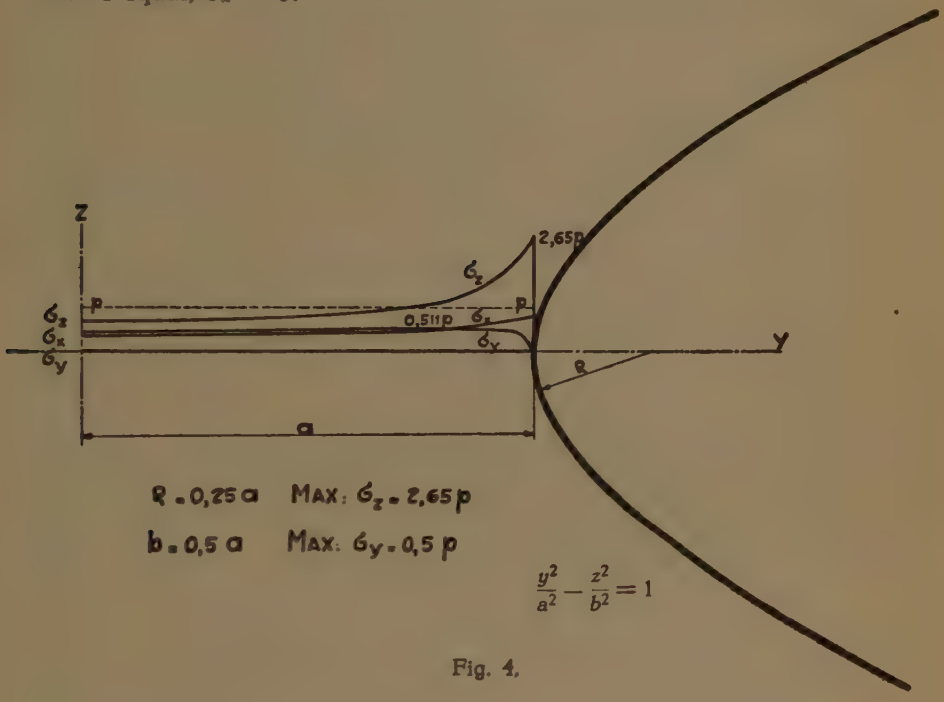

Fig. 4.

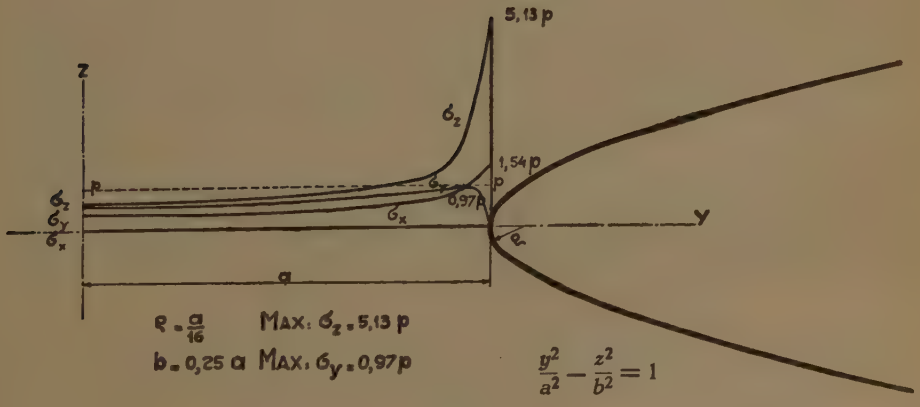

Fig. 5.

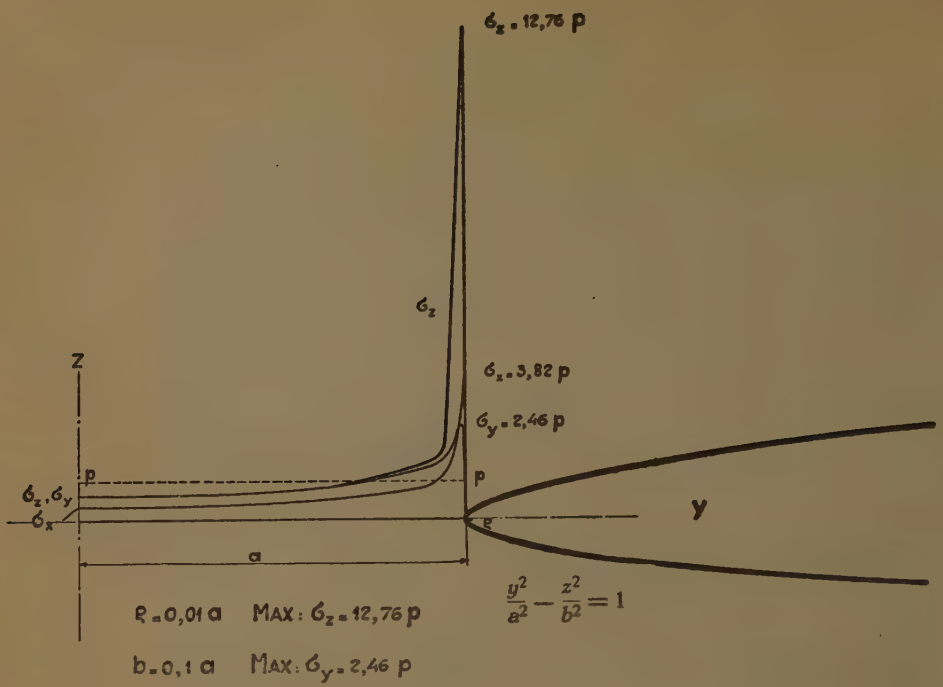

Fig. 6.

But all three tensions rapidly increase when we enter the material. For cylindrical grooves (figure 7) of curvature r_0 this increase of tensions takes place according to the law $\sigma_t = \sigma_0 \left(1 + ln \frac{r}{r_0}\right)$,

$$\sigma_r = \sigma_0 \, ln \frac{r}{r_0} \quad \text{and} \quad \sigma_x = \sigma_0 \, ln \frac{r}{r_0}.$$

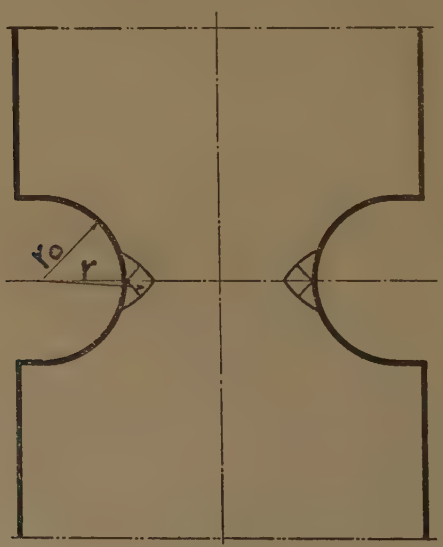

Fig. 7.

Fig. 8.

Although the bridge resisted to heavy traffic and was built of excellent mild steel the rupture occurred without any deformation. The texture of the fracture shows a brittle appearance.

σ_t is double the elastic limit for $r = 2{,}832\ r_0$. This is very near the bottom of the groove when r_0 is small.

The differences in tension are $\sigma_t - \sigma_z = \sigma_0$, and $\sigma_t - \sigma_x = \sigma_0$ where σ_0 represents the elastic limit or yield stress.

The steel flows at the bottom of a rounded notch, but at some distance, the three-dimensional tensions become so large, that the material no longer flows but breaks in a brittle manner. For sharp rents this rupture occurs very near the end of the rent. Often the rupture is explosionlike and spreads far into the material.

Figure 8 is a photograph of a rent in a blown-up bridge girder.

Mathematics. — *On sets of integers.* (First communication.) By J. G. VAN DER CORPUT.

(Communicated at the meeting of February 22, 1947.)

§ 1. *The $\alpha + \beta$-hypothesis.*

Consider two systems A and B, both consisting of integers $\geqq 0$, for instance

$$A \ldots 0, 1, 2, 6, 7, 12, 19.$$
$$B \ldots 0, 1, 2, 6, 7, 8, 12, 20.$$

The "sumset" $A + B$ is the system formed by the integers $a + b$, where a runs through the elements of A and b through the elements of B. In our example $A + B$ consists of the integers

$$0, 1, 2, 3, 4, 6, 7, 8, 9, 10, 12, 13, 14, 15, 18, 19, 20, 21, 22, 24, 25, 26, 27, 31, 32, 39.$$

In this section I denote by $A(m)$ the number of positive integers $\leqq m$ occurring in A; thus zero is never counted. Similarly we define $B(m)$ and $(A + B)(m)$. In our example we have

$$A(12) = 5; \quad B(12) = 6; \quad (A + B)(12) = 10;$$
$$A(13) = 5; \quad B(13) = 6; \quad (A + B)(13) = 11;$$
$$A(18) = 5; \quad B(18) = 6; \quad (A + B)(18) = 14.$$

Herefrom we learn that $(A + B)(m)$ may be less than, equal to or greater than $A(m) + B(m)$.

The mathematicians E. LANDAU [1], I. SCHUR [2] and A. KHINTCHINE [3] have stated the following remarkable $\alpha + \beta$-hypothesis:

If both systems A and B contain the number zero and if there exists a positive integer g, such that

$$A(m) \geqq \alpha m \text{ and } B(m) \geqq \beta m \qquad (m = 1, \ldots, g),$$

where α and β are numbers independant of m, satisfying the inequality $\alpha + \beta \leqq 1$, then

$$(A + B)(m) \geqq (\alpha + \beta) m \qquad (m = 1, \ldots, g).$$

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[1]) Die Goldbachsche Vermutung und der Schnirelmannsche Satz. Göttinger Nachrichten, Math. Phys. Klasse (1930), 255—276.

[2]) Ueber den Begriff der Dichte in der additiven Zahlentheorie. Sitzungsberichte der preuss. Akad. der Wiss., Math. Phys. Klasse (1936), 269—297.

[3]) Zur additiven Zahlentheorie. Matematiceski Sbornik, **39**, 27—34 (1932).

In our example

$$A(m) \geqq \tfrac{4}{11} m \text{ and } B(m) \geqq \tfrac{2}{5} m \qquad (m = 1, \ldots, 12),$$

so that the $\alpha + \beta$-hypothesis gives

$$(A + B)(m) \geqq (\tfrac{4}{11} + \tfrac{2}{5}) m \qquad (m = 1, \ldots, 12).$$

This inequality is rather strong, for instance

$$(A + B)(12) = 10 \text{ and } (\tfrac{4}{11} + \tfrac{2}{5}) 12 = 9, 1 \ldots$$

Several great mathematicians have in vain devoted their efforts to this problem, till finally in 1942 H. B. MANN [4]) found a very ingenious proof of this hypothesis. He even proved more:

Theorem 1. *If both systems A and B contain the number zero, and if there exists a positive integer g, such that*

$$A(m) + B(m) \geqq \gamma m \qquad (m = 1, \ldots, g), \quad . \quad . \quad . \quad . \quad (1)$$

where $\gamma \leqq 1$, then

$$(A + B)(m) \geqq \gamma m \qquad (m = 1, \ldots, g).$$

In our example we have

$$A(m) + B(m) \geqq \tfrac{4}{5} m \qquad (m = 1, \ldots, 12),$$

hence

$$(A + B)(m) \geqq \tfrac{4}{5} m \qquad (m = 1, \ldots, 12),$$

particularly

$$(A + B)(12) = 9, 6;$$

this inequality is stronger than the result furnished by the $\alpha + \beta$-hypothesis.

In the special case $\gamma = 1$ the theorem of MANN gives the result, that $A + B$ contains all positive integers $\leqq g$. To show that this result is obvious, I suppose that there exists a positive integer $m \leqq g$, which does not occur in $A + B$. The set B contains $B(m-1)$ positive integers $\leqq m-1$ and each of these integers is different from the $A(m-1)$ positive integers $\leqq m-1$ of the form $m-a$, where a is an element $\leqq m-1$ of A, consequently

$$B(m-1) + A(m-1) \leqq m-1;$$

since m is not an element of $A + B$, it occurs neither in A nor in B, hence

$$A(m) + B(m) = A(m-1) + B(m-1) \leqq m-1, \quad . \quad . \quad . \quad (2)$$

so that (1) is not true with $\gamma = 1$.

The proof given by MANN of his theorem is rather difficult and even

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[4]) A proof of the fundamental theorem on the density of sums of sets of positive integers. Annals of Math. **43**, 523—529 (1942).

the simplified demonstrations, the first of which is due to E. ARTIN and P. SCHERK [5]), whereas we owe the second to F. J. DYSON [6]), are not easy to follow. Nevertheless the principle, which forms the base of the argument, is very simple. I will illustrate this by our example.

I denote on two vertical lines the numbers $\leqq 12$, occurring respectively in A and B. First I cancel in B the numbers not occurring in A and I add these numbers to A. The only integer, which enters here into consideration,

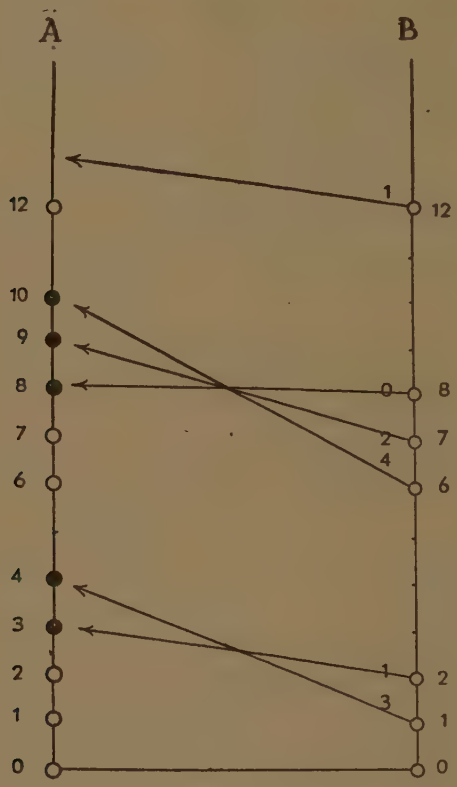

Fig. 1

is the number 8. In this manner A is transformed into the set A^*, which contains all elements $\leqq 12$ of A and moreover the number 8, whereas B becomes the system B^*, which contains all elements $\leqq 12$ of B, the integer 8 excepted. I say that the couple A, B is transformed by an elementary transformation into the couple A^*, B^*. In figure 1 this elementary transformation is denoted by an arrow, which connects the integers 8, situated on the vertical lines. I have provided this arrow with the number 0, because the integer cancelled in B is replaced by an equal integer, which

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[5]) On the sum of two sets of integers, Annals of Math. **44**, 138—142 (1943).

[6]) A theorem on the densities of sets of integers. Journal of the London Math. Society **20**, 8—15 (1945).

we have added to A. It is clear that the sum $A(m) + B(m)$ does not change by this transformation, i.e.

$$A^*(m) + B^*(m) = A(m) + B(m).$$

It is easy to see, that each element $a^* + b^* \leqq 12$ of $A^* + B^*$ is also an element of $A + B$, hence

$$(A + B)(m) \geqq (A^* + B^*)(m) \qquad (m = 1, \ldots, 12).$$

In fact, if $a^* \neq 8$, then a^* and b^* are elements respectively of A and B and if $a^* = 8$, then $a^* + b^* = a + 8$, where 8 is an element of B and $a = b^*$ is an element of A (otherwise b^* would have been cancelled in B), so that $a^* + b^*$ occurs in $A + B$.

Now, having transferred to A all elements $\leqq 12$ of B, which do not occur in A, I cancel in B^* each element $b \leqq 12$ such that $b + 1$ is either > 12 or an integer $\leqq 12$ not occurring in A^*. I add these numbers $b + 1$ to A^*, as far as they are $\leqq 12$. In the whole argument I restrict myself to integers $\leqq 12$, so that the behaviour of the numbers > 12 does not interest me. The only numbers, which are cancelled in B^*, are 2 and 12 and A^* gains the integer 3. By this transformation the couple A^*, B^* is transformed into another couple A^{**}, B^{**}. This transformation consists of two elementary transformations, the first of which cancels in B the element 12, the second of which cancels 2 in B and adds 3 to A. In figure 1 these two elementary transformations are denoted by two arrows, each provided with the number 1. Each element $a^{**} + b^{**} \leqq 12$ of $A^{**} + B^{**}$ is an element of $A^* + B^*$. In fact, if $a^{**} \neq 3$, then a^{**} and b^{**} are elements, respectively of A^* and B^*, and if $a^{**} = 3$, then $a^{**} + b^{**} = a^* + b^*$, where $b^* = 2$ is an element of B^* and $a^* = b^{**} + 1$ is an element of A^* (otherwise b^{**} would have been cancelled in B^*). Consequently $a^{**} + b^{**}$ occurs in $A^* + B^*$. In this manner we obtain

$$(A^* + B^*)(m) \geqq (A^{**} + B^{**})(m) \qquad (m = 1, \ldots, 12).$$

It is not true that $A^*(m) + B^*(m)$ is invariant with regard to this transformation. On the contrary, by this transformation $A^*(m) + B^*(m)$ is diminished by 1 for $m = 2$ and also for $m = 12$. Happily we may make use of the following remarkable phenomenon, which I will prove presently:

The inequalities

$$A^*(m) + B^*(m) \geqq \gamma m \qquad (m = 1, \ldots, 12),$$

where $\gamma \leqq 1$, imply the stronger inequalities

$$A^{**}(m) + B^{**}(m) \geqq \gamma m \qquad (m = 1, \ldots, 12).$$

Further we cancel in B^{**} each element $b \leqq 12$, such that $b + 2$ is either > 12 or an integer $\leqq 12$ not occurring in A^{**}. I add these numbers $b + 2$ to A^{**} as far as they are $\leqq 12$. This means here, that we cancel 7 in B^{**}

and add 9 to A^{**}. Continuing in this manner [7]) we find successively the couples A^{***}, B^{***} and A^{****}, B^{****}, which satisfy for $m = 1, \ldots, 12$ the inequalities

$$(A + B)(m) \geqq (A^* + B^*)(m) \geqq \ldots \geqq (A^{****} + B^{****})(m)$$

and

$$(A^{****} + B^{****})(m) \geqq \gamma\, m.$$

Here the process comes to an end, since all positive elements $\leqq 12$ of B are cancelled. But now we are ready. In fact, for $m = 1, \ldots, 12$ we find $B^{****}(m) = 0$, hence

$$(A + B)(m) \geqq (A^{****} + B^{****})(m) = A^{****}(m) + B^{****}(m) \geqq \gamma\, m.$$

In the general case the proof is analogous. Without loss of generality I may assume that the systems A and B are formed by integers $\leqq g$, since the numbers $> g$ do not interest us.

Be e the smallest element of A, to which corresponds a positive element b of B, such that $e + b$ does not belong to A. Such an element e of A exists, if B contains at least one positive integer b; in fact the greatest element a of A has the property that $a + b$ does not belong to A.

After having fixed e, I cancel in B a positive element b, such that $e + b$ does not belong to A. I add to A the integer $e + b$, if it is $\leqq g$. If A and B are transformed in this manner into A' and B', I say that the couple A, B is transformed by an elementary transformation into A', B'. Thus B' is the set of the elements $\neq b$ of B; if $e + b > g$, then $A' = A$, and if $e + b \leqq g$, then A' is the system formed by $e + b$ and the elements of A. The principal idea of our argument may be found in the following lemma.

Lemma 1. *If A contains zero and if*

$$A(m) + B(m) \geqq \gamma\, m \qquad (m = 1, \ldots, g), \quad . \quad . \quad . \quad . \quad (3)$$

then each elementary transformation transforms A, B into a couple A', B' with the property

$$A'(m) + B'(m) \geqq \gamma\, m \qquad (m = 1, \ldots, g) \quad . \quad . \quad . \quad . \quad (4)$$

Let us suppose that this assertion is not true. Be k the smallest positive integer $\leqq g$, for which the lemma is not valid, more precisely: the positive integer $k \leqq g$ possesses the following properties:

1. It is possible to find a system A containing zero and a system B satisfying the inequalities

$$A(m) + B(m) \geqq \gamma\, m \qquad (m = 1, \ldots, k), \quad . \quad . \quad . \quad . \quad (5)$$

---

[7]) In our example zero is an element of A, moreover 1 an element of A^*, etc. As it will appear, that is the reason why we cancelled in B the elements b such that $b \dotplus 0$ does not occur in A, further in B^* the elements $b + 1$ such that $b + 1$ does not occur in A^*, etc. If for instance A^{**} does not contain the integer 2, we cancel in B^{**} the elements b such that $b \dotplus e$ does not occur A^{**}, where e is the smallest element of A^{**} to which corresponds at least one element b of B^{**} with this property.

such that a convenient elementary transformation transforms A, B into a couple A', B', for which at least one of the inequalities

$$A'(m) + B'(m) \geqq \gamma\, m \qquad (m = 1, \ldots, k) \quad . \quad . \quad . \quad . \quad (6)$$

is not true.

2. Be l an arbitrary positive integer $< k$. If C contains zero and if

$$C(m) + D(m) \geqq \gamma\, m \qquad (m = 1, \ldots, l),$$

then each elementary transformation transforms C, D into a couple C', D' such that

$$C'(m) + D'(m) \geqq \gamma\, m \qquad (m = 1, \ldots, l). \quad . \quad . \quad . \quad . \quad (7)$$

The special case $l = k - 1$, $C = A$ and $D = B$ gives that the inequalities (6) are true for $m = 1, \ldots, k-1$. Consequently (6) is not valid for $m = k$. Hence from (5) we obtain

$$A'(k) + B'(k) < \gamma\, k \leqq A(k) + B(k) \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The case $b > k$ is excluded. In fact, in that case we would not have cancelled in B any element $\leqq k$, nor added to A any element $\leqq k$, so that

$$A'(k) + B'(k) = A(k) + B(k). \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

contrary to (8). Similarly the case $e + b \leqq k$ is excluded. In fact, in that case we would have cancelled in B one and only one integer $\leqq k$ (namely b), and added to A one and only one integer $\leqq k$ (namely $e + b$), so that we would find (9) again.

Consequently $b < k \leqq e + b$, hence $e > 0$. I have indicated this result in figure 2.

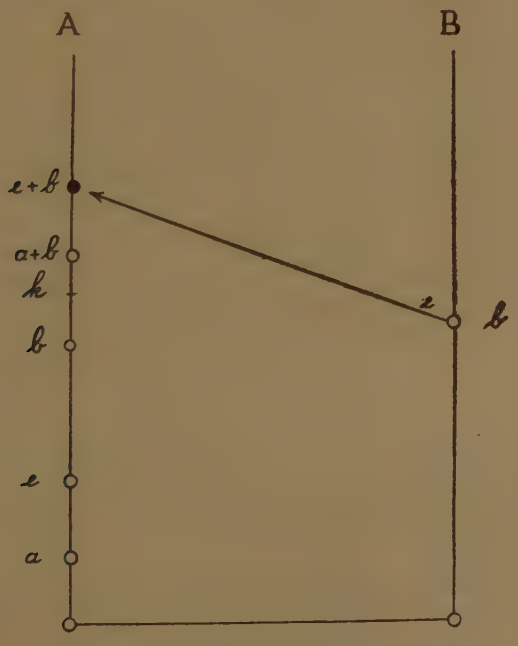

Fig. 2

In this argument e is the smallest element of A, such that $e + b$ does not belong to A. Consequently each element $a < e$ of A (hence also each element $a \leqq k - b$ of A) has the property that $a + b$ belongs to A; in particular we find that b belongs to A. This system A contains $A(k-b)$ positive elements $a \leqq k - b$ and each of them gives an element $a + b$ of A, which is $\geqq b + 1$ and $\leqq k$, hence $A(k) - A(b) \geqq A(k-b)$. Since A contains also the element b, we obtain $A(b) = 1 + A(b-1)$, hence

$$A(k) - A(b-1) \geqq 1 + A(k-b).$$

The inequalities (6) are true for $m = 1, \ldots, k-1$, hence

$$A'(b-1) + B'(b-1) \geqq \gamma(b-1), \quad . \quad . \quad . \quad . \quad . \quad (10)$$

since this inequality is also valid for $b = 1$. The proof is established, if we show

$$A(k-b) \geqq \gamma(k-b) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

In fact, then we find, in virtue of $A'(m) = A(m)$ $(m \leqq k)$,

$$A'(k) + B'(k) \geqq A(k) + B'(b-1) \geqq \gamma(b-1) + 1 + \gamma(k-b) \geqq \gamma k, \quad . \quad (12)$$

contrary to (8).

How to prove (11)? We know

$$A(m) + B(m) \geqq \gamma m \qquad (m = 1, \ldots, k-b),$$

Happily these apparently much too weak inequalities suffice. They show namely that there exists a subset D of B with the properties

$$A(m) + D(m) \geqq \gamma m \quad . \quad (m = 1, \ldots, k-b); \quad . \quad . \quad . \quad (13)$$

for instance $D = B$ has the required properties. I chose the subset D of B with these properties, such that the number of elements of D is as small as possible. We are ready if we have shown that D does not contain any positive element; in fact, then $D(k-b) = 0$ and (11) follows from (13).

Let us suppose for a moment, that D contains at least one positive element. Be e' the smallest element of A to which corresponds a positive element d of D such that $e' + d$ does not belong to A. Since d belongs to D, therefore also to B, we have $e' \geqq e$. Thus

$$e' + d > e' \geqq e \geqq k - b.$$

The couple A, D is transformed by an elementary transformation (with e' instead of e) into a couple C', D'. This transformation cancels in D the integer d and does not add to A an integer $\leqq k - b$, hence

$$C'(m) = A(m) \qquad (m = 1, \ldots, k-b).$$

From (13) we see that we may apply (7) with $C = A$ and $l = k - b$, thus

$$A(m) + D'(m) \geqq \gamma m \qquad (m = 1, \ldots, k-b). \quad . \quad . \quad . \quad (14)$$

These inequalities are impossible, since D' is a subset of B and contains less elements than D. This proves the lemma.

Above we considered a transformation in which exactly one element of B was cancelled and at most one element was added to A. Such a transformation has the disadvantage that it may increase $(A + B)(m)$. For instance, if both A and B consist of the numbers 0, 1 and 3, then the elementary transformation, which cancels the element 1 of B and adds the element 2 to A, transformes the couple A, B into the couple A', B', where A' consists of the elements 0, 1, 2, 3 and B' of the elements 0 and 3; $A + B$ consists of the numbers 0, 1, 2, 3, 4, 6, but $A' + B'$ of the numbers 0, 1, 2, 3, 4, 5 and 6, hence $(A + B)(5) < (A' + B')(5)$. This is an inequality in the wrong direction; hence it is not sufficient to consider only elementary transformations. But now we combine all elementary transformations in which the same number e occurs, just as in our example we combined the two elementary transformations with $e = 2$. In the general case our argument is as follows:

Be e again the smallest element of A to which corresponds at least one element b of B, such that $e + b$ does not belong to A. In B we cancel all elements b with the property that $e + b$ does not belong to A, and to A we add these numbers $e + b$ as far as they are $\leqq g$. In this manner the sets A and B are transformed into the systems A^* and B^*. I say that the couple A, B is transformed into the couple A^*, B^* by an e-transformation. I prove that the couple A, B may be transformed into the couple A^*, B^* by a finite number of elementary transformations. This is evident if in B only one element is cancelled, for then one elementary transformation is sufficient. Let us now assume that at least two elements b and b' are cancelled in B. If b is cancelled in B and if $e + b$, as far as it is $\leqq g$, is added to A, the couple A, B is transformed into the couple A', B' by an elementary transformation. Here e is an element of A (hence of A'), to which an element b_1 of B corresponds such that $e + b_1$ does not belong to A; in fact $b_1 = b'$ satisfies. I assert that e is the smallest element of A' with this property. For, if A' contained an element $e' < e$ with this property, then e' would not be the number added to A (because the number possibly added to A is $\geqq e$). In that case e' would be an element $< e$ of A, to which an element b_1 of B (since b_1 belongs to B') corresponds, such that $e' + b_1$ does not belong to A. This contradicts the condition, that e is the smallest number with this property. Consequently: If in B' the element b' is cancelled and if the number $e + b'$, as far as it is $\leqq g$, is added to A', then A', B' is transformed by an elementary transformation into a couple A'', B''. If in B only the elements b and b' are cancelled, we are ready. If not, we proceed in the same manner applying elementary transformations untill all elements in question are cancelled.

Since an e-transformation is composed of a finite number of elementary transformations, lemma 1 gives immediately:

Lemma 2. *If the set A, containing 0, satisfies*

$$A(m) + B(m) \geqq \gamma m \qquad (m \geqq 1, \ldots, g),$$

where $\gamma \leqq 1$ and if the couple A, B is transformed by the corresponding e-transformation into the couple A^*, B^* then

$$A^*(m) + B^*(m) \geqq \gamma m \qquad (m = 1, \ldots, g).$$

Now the final step.

Lemma 3. *If the couple A, B is transformed into the couple A^*, B^* by the corresponding e-transformation, then $A + B$ contains each element $u \leqq g$ of $A^* + B^*$.*

In fact, that number u may be written as $a^* + b^*$, where a^* and b^* belong respectively to A^* and B^*. If a^* belongs to A, then $a^* + b^*$ is an element of $A + B$, since b^* belongs to B^*, consequently to B. Hence only the case has to be considered in which a^* is one of the elements added to A; hence a^* may be written in the form $e + b$, where b is a positive element of B. Then $a = e + b^*$ belongs to A. For, if a would not belong to A, then $b^* \neq 0$, since e is an element of A and the positive number b^* would have been cancelled in B by the e-transformation, therefore not belong to B^*. Herefrom it follows that a is an element of A, so that $a^* + b^* = a + b$ belongs to $A + B$.

The proof of the theorem of MANN may now be given in a few lines:

I may assume that B contains at least one positive element, since otherwise the assertion is evident in virtue of

$$(A + B)(m) = A(m) = A(m) + B(m) \quad . \quad . \quad . \quad . \quad (15)$$

Further I may assume the theorem proved, if B is replaced by a system containing less elements than B. By an e-transformation we may transform the couple A, B into a couple A^*, B^*, which satisfies by lemma 2 the inequalities

$$A^*(m) + B^*(m) \geqq \gamma m \qquad (m = 1, \ldots, g).$$

Since B^* contains less elements than B, we have by induction

$$(A^* + B^*)(m) \geqq \gamma m \qquad (m = 1, \ldots, g),$$

wherefrom the assumption follows by lemma 3.

The condition that B contains the number zero is only used in the proof of (15).

Above the function γm occurred, which may be replaced by an arbitrary real function $\varphi(m)$ which satisfies some general conditions. Since (10) must be valid for $b = 1$, we must suppose $\varphi(0) \leqq 0$, and for (12) I suppose

$$\varphi(u + v + 1) \leqq \varphi(u) + \varphi(v) + 1, \quad . \quad . \quad . \quad . \quad . \quad (16)$$

for all integers u and v both $\geqq 0$ with $u + v + 1 \leqq g$. If γm is replaced

by a function $\varphi(m)$, which satisfies these two conditions, the above argument remains valid. This gives the general theorem:

Theorem 2. *Let be $\varphi(0) \leqq 0$ and let the inequality (16) be valid for all integers $u \geqq 0$ and $v \geqq 0$ with $u + v + 1 \leqq g$. If the sets A and B both contain zero and if*

$$A(m) + B(m) \geqq \varphi(m) \qquad (m = 1, \ldots, g),$$

then also

$$(A + B)(m) \geqq \varphi(m) \qquad (m = 1, \ldots, g).$$

For instance we may put $\varphi(m) = \gamma m + \gamma - 1$, where $\gamma \leqq 1$. This gives:

Theorem 3. *If the sets A and B both contain zero and if*

$$A(m) + B(m) \geqq \gamma(m + 1) - 1 \qquad (m = 1, \ldots, g),$$

where $\gamma \leqq 1$, then

$$(A + B)(m) \geqq \gamma(m + 1) - 1 \qquad (m = 1, \ldots, g).$$

In our above example

$$A(m) + B(m) \geqq \tfrac{5}{6}(m + 1) - 1 \qquad (m = 1, \ldots, 12),$$

hence

$$(A + B)(12) \geqq \tfrac{5}{6} \cdot 13 - 1 = 9\tfrac{5}{6};$$

this inequality is still sharper than MANN's.

The preceding theorem also gives useful results, if the number zero belongs only to one or to neither of the sets A and B. Let us assume f.i. that A contains the number 0 and B the number 1 (it is unimportant whether 1 is contained in A or 0 in B). If b runs through the system B, then $b - 1$ runs through a system, which I denote by $\breve{B}$. If s runs through the natural numbers $\leqq g$, which belong to $A + B$, then $s - 1$ runs through the number 0 and the elements $\leqq g - 1$ of $A + \breve{B}$, hence

$$(A + B)(m) = 1 + (A + \breve{B})(m-1) \qquad (m \geqq 2).$$

To apply our last theorem I suppose

$$A(m) + \breve{B}(m) \geqq \gamma(m + 1) - 1 \qquad (m = 1, \ldots, g-1),$$

which is identical with

$$A(m) + B(m + 1) \geqq \gamma(m + 1) \qquad (m = 1, \ldots, g-1).$$

In this manner we obtain:

Theorem 4. *If A contains the number 0 and B the number 1 and if*

$$A(m-1) + B(m) \geqq \gamma m \qquad (m = 2, \ldots, g),$$

where $\gamma \leqq 1$, then

$$(A + B)(m) \geqq \gamma m \qquad (m = 1, \ldots, g).$$

This inequality holds also for $m = 1$, because then we get 1 at the left hand side and γ at the right hand side.

Aerodynamics. — *Aerodynamical problems connected with the motion of a cloud of gas emitted by Nova Persei.* I. By J. M. BURGERS. (Mededeling no. 49 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft.)

(Communicated at the meeting of February 22, 1947.)

1. *Introduction.* — The present paper is a continuation of a preliminary investigation concerning cases of motion presented by interstellar gas clouds, published last year [1]). Its aim is to consider one particular example, referring to the motion of a presumably very thin cloud or shell of gas emitted by Nova Persei and retarded by the resistance of interstellar material encountered in its course; it will be attempted to work out in some detail various features which had been indicated in outline only in the former publication. It is found that the resistance experienced in retarded motion cannot be calculated immediately from the instantaneous value of the velocity, as it is influenced by expansion phenomena; this has the consequence that the law of retardation now obtained is different from that provisionally assumed in §§ **6** and **7** of the paper mentioned. Attention further must be given to the transmission of heat by conduction from the compressed gas to the advancing cloud; and it has been attempted to work out the energy balance for the cloud with the aid of a somewhat improved form of the method of calculation, introduced *l.c.* in § **6**.

Like the former paper the present one owes its conception and its astrophysical data to Professor dr. J. H. OORT at Leiden. Together we have considered various aspects of the motion, and OORT's constant advice was a valuable guide in framing the problem in such a way that it became tractable. It will be seen that several links are still missing; this is due in part to difficulties of the mathematical problems involved, in part to lack of sufficient data, in particular referring to physical properties of the interstellar material. It is hoped that the investigation of cases of motion, such as the one considered here, although the picture developed is an extreme simplification of what perhaps may happen in reality, nevertheless can be of help in elucidating the phenomena.

Astronomical data. — Several years after the appearance and consequent decay of Nova Persei 1901 a gas cloud has been observed, which moves away from the star with gradually decreasing velocity. In 1934 the velocity was estimated at 1200 km/sec $= 1{,}2 \cdot 10^8$ cm/sec; the retardation amounted to 0,11 cm/sec^2. It is supposed that the cloud has the form of a

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[1]) See J. M. BURGERS, Some problems of the motion of interstellar gas clouds. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **49**, 589 (1946).

very thin shell or sheet, and that the retardation is due to resistance experienced from interstellar material upon which this sheet has struck. The cross dimensions of the sheet are of the order $2 \cdot 10^{17}$ cm^2; its area has been estimated at $F = 4{,}8 \cdot 10^{34}$ cm^2, whereas its thickness D probably is of the order $10^{13} - 10^{14}$ cm. As the interstellar gas upon which the cloud has struck is compressed, the temperature of this gas will be raised considerably (formulae referring to motion with constant velocity have been given in the preceding paper). Nevertheless in consequence of its very small density this interstellar gas does not produce observable radiation. It can be expected, however, that some of the heat generated is transmitted to the cloud, which has a density about a thousand times higher; the radiation emitted by this cloud is observed and the spectrum has been measured. It is estimated that the total amount of radiation is 10^{34} erg/sec, while the temperature of the cloud is estimated at 50.000°. Starting from these data OORT has deduced the following estimates concerning thickness and density of the cloud: $D = 7{,}5 \cdot 10^{13}$ cm; ϱ aver. $= 1{,}1 \cdot 10^4$ hydrogen atoms per cm^3 $= 1{,}83 \cdot 10^{-20}$ gr/cm^3; and density of the original interstellar gas (before the compression) $\varrho_0 = 7{,}3$ hydrogen atoms per cm^3 $= 12{,}1 \cdot 10^{-24}$ gr/cm^3. We shall come back to these quantities afterwards.

In framing the problem in such a way that it becomes amenable to mathematical treatment, far going simplifications have been introduced. The motion is considered as taking place in one dimension only; the cloud is assumed to be a plane sheet, perpendicular to the direction of the motion. It is supposed that the interstellar gas originally was at rest and presented a plane front parallel to that of the cloud, upon which the cloud at a certain instant suddenly impinged with its full velocity. In view of the high forward velocity of the cloud the expansion velocity of the gas at a temperature of 10.000° ($3{,}5 \cdot 10^6$ cm/sec) can be neglected. Hence this part of the problem can be compared with the following mechanical case: a cylindrical tube, extending indefinitely into one direction and filled with a gas of uniform density and temperature, is closed near the origin by a movable piston; originally the whole system is at rest; at the instant $t = 0$ a very high velocity is suddenly given to the piston, after which it is left to itself. The sudden motion of the piston produces a compression wave in the gas, which wave moves outward; the high pressure generated causes a retardation of the motion of the piston (it is assumed that on the other side of the piston there is a vacuum, so that the piston does not experience any force at its back). In consequence of the decrease of the velocity of the piston expansion occurs in the gas adjoining the piston. The expansion waves overtake the compression wave and interfere with it; it is asked to describe the ensuing state of motion of the gas. In view of its general interest this problem is treated at some length in sections 2—5 of the paper. It may be objected that the assumption of a discrete front of the interstellar material, which is considered originally to represent a homogeneous

mass, will be far from the truth; however, the assumption of a gradual increase of density (in such a way that the advancing cloud at first should not suffer an appreciable retardation, and then would gradually be obstructed by denser material), or other suppositions of similar nature, bring a twofold difficulty: the exact distribution of material is unknown, and the mathematical treatment becomes far more complicated. Another point is that the cloud will suffer a deformation during its motion; hence its reaction to the resistance offered by the interstellar gas may differ from the reaction of a solid body like a piston. It will appear, however, in a later part of the paper, that the difference is of small importance.

The problem in this form should be fully determined by the following data: original density ϱ_0 of the interstellar gas; original temperature T_0 of this gas; velocity V_0 of the cloud at the moment it strikes upon the gas; mass $\bar{\varrho}$ of the cloud per cm² of its frontal area. From astrophysical considerations T_0 is assumed to be 10.000°; as to the other data, which play a basic part in the treatment of the problem, their values have not been given explicitly beforehand and should be deduced afterwards from the observational data.

The gas will be treated as an ideal gas, to which the equations of aerodynamics can be applied; it is supposed to consist of monatomic hydrogen, having the ratio of the specific heats $k = 5/3$. At the very high temperatures generated by the compression the gas will be gradually ionised; however, according to calculations by OORT, the time elapsed until now is far from sufficient to reach equipartition of energy between protons and electrons, so that for general thermodynamical properties we will neglect the electrons. The change of state of every element of volume is assumed to take place without loss of heat through radiation or conduction; the latter point, however, is one of the serious difficulties of the problem.

2. *Equation of motion for the gas in front of the advancing cloud.* — We use the LAGRANGIAN form of the equations and describe the motion of a layer of the gas by:

$$x = \varphi(s; t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where s is a parameter distinguishing the various layers and x is the path travelled by any layer. We take $s = 0$ for the layer immediately adjoining the advancing cloud; for all other layers $s > 0$. The way in which the values of s are assigned to the consecutive layers is arbitrary; a particular way leading to a simplification of the equations will be indicated below. The velocity of a layer is given by:

$$u = dx/dt = \partial\varphi/\partial t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

As the expansion of a layer is determined by the behaviour of $\partial\varphi/\partial s$ as a function of the time for constant s, we have for the density:

$$\varrho = \theta_1(s) / (\partial\varphi/\partial s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and for the pressure (the expansion of every layer being assumed to follow POISSON's law):

$$p = \theta_2(s) / (\partial\varphi/\partial s)^{5/3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The functions $\theta_1(s)$ and $\theta_2(s)$ occurring in these expressions are independent of the time; they will be given below. With the aid of these formulae the equation of motion:

$$\varrho \,.\, (du/dt) = -\,\partial p/\partial x$$

can be brought into the form:

$$\frac{\partial^2 \varphi}{\partial t^2} = -\,\frac{1}{\theta_1(s)}\,\frac{\partial}{\partial s}\left[\frac{\theta_2(s)}{(\partial\varphi/\partial s)^{5/3}}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The motion of the front of the advancing cloud (compare fig. 1, which may help to visualize the relations) is determined by:

$$x_{\text{front}} = \varphi(0;t) \quad ; \quad V = (dx/dt)_{\text{front}} = (\partial\varphi/\partial t)_{s=0}.$$

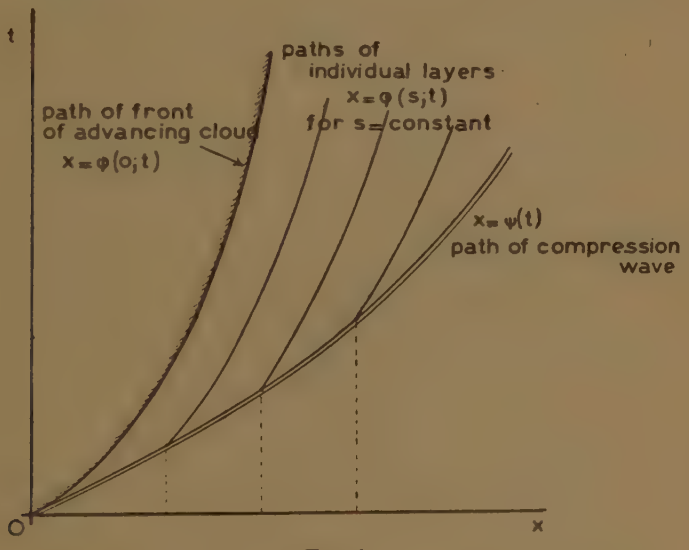

Fig. 1.

As the cloud for the present is treated as a solid body, having a mass $\bar{\varrho}$ per unit area of frontal surface, its equation of motion is given by:

$$\left(\frac{\partial^2 \varphi}{\partial t^2}\right)_{s=0} = -\,\frac{p_{(s=0)}}{\bar{\varrho}} = -\,\frac{1}{\bar{\varrho}}\,\frac{\theta_2(0)}{(\partial\varphi/\partial s)^{5/3}_{s=0}} \quad . \quad . \quad . \quad . \quad (6)$$

A second boundary condition refers to the compression wave, which is the shock wave at the head of the region influenced by the advancing cloud. The path of this compression wave will be represented by $x = \psi(t)$; the function ψ provisionally is unknown. Consider the point of intersection X, T, of a particular path $\varphi(s;t)$ with the path of this compression wave; then:

$$\psi(T) = \varphi(s;T) \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

When ψ should be known, this equation would determine T in function of s. Now from the formulae valid for shock waves [2]) it follows that in the present case of a gas with $k = 5/3$ moving with a velocity far surpassing the velocity of sound (as calculated for the original state of the gas), the following relations hold:

(*a*) the velocity of propagation of the shock wave $(d\psi/dT)$ is equal to 4/3 times the velocity $(\partial\varphi/\partial t)_{t=T}$ of the gas immediately at the back of the shock wave;

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[2]) See: J. M. BURGERS, Over de eendimensionale voortplanting van drukstoringen in een ideaal gas, Versl. Nederl. Akad. v. Wetensch., Amsterdam 52, 476 (1943). The velocity ξ of a shock wave propagating itself through a gas at rest, is given by the formula:

$$\xi = \beta u + \sqrt{c_0^2 + \beta^2 u^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(I)}$$

where c_0 is the velocity of sound in the gas at rest; $\beta = (k+1)/4 = 2/3$ in the present case, and u is the velocity of the gas at the back of the shock wave. The density of the gas at the back is given by:

$$\varrho_1 = \varrho_0 \, \xi/(\xi - u) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II)}$$

the pressure by:

$$p_1 = p_0 + \varrho_0 \, \xi \, u \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III)}$$

and the velocity of sound in the compressed gas by:

$$c_1^2 = c_0^2 + [(k-1)/8\,\beta^2]\{k\,\xi^2 - (k-1)\,c_0^2 - c_0^4/\xi^2\} \quad . \quad . \quad . \quad \text{(IV)}$$

The increase of temperature follows from the ratio c_1^2/c_0^2.

When the velocity u considerably exceeds c_0 these formulae reduce to (with $k = 5/3$):

$$\xi = \tfrac{4}{3} u \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(Ia)}$$

$$\varrho_1 = 4\,\varrho_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IIa)}$$

$$p_1 = \tfrac{4}{3}\,\varrho_0 \, u^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IIIa)}$$

$$c_1^2 = \tfrac{5}{9}\,u^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(VIa)}$$

from which:

$$T_1 = u^2/3\,R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(Va)}$$

The same results can be deduced by means of simple kinetic considerations. An observer moving with the compressed gas at the back of the shock wave, sees the uncompressed gas rushing towards himself with the velocity u. As the collision of the uncompressed gas with the compressed gas brings the former to rest for the observer, the translational motion is wholly transformed into heat motion; hence the mean square velocity of the heat motion after the shock must be equal to u^2, from which it follows that the temperature of the (monatomic) gas will be equal to $u^2/3R$, R being the gas constant. When the density of the compressed gas is denoted by ϱ_1, its pressure will be equal to $p_1 = \varrho_1 \, u^2/3$. This pressure also can be calculated from the loss of momentum, which is given by the product $\varrho_0 \, \xi \, u$, where $\varrho_0 \, \xi$ is the mass taken in per unit area of the wave front in unit time, and u is the change of velocity. As the equation of continuity gives: $\varrho_0 \, \xi = \varrho_1 \, (\xi - u)$, we find $\xi = 4\,u/3$.

For an observer stationary with respect to the uncompressed gas the material through which the shock wave has passed has obtained a kinetic energy of translation $u^2/2$ per unit of mass and an energy of the heat motion of the same amount. Hence the total increase of energy per unit of area of the advancing wave front in unit time is equal to $\varrho_0 \, \xi \, u^2$. This is equal to the work performed by the pressure acting at the back of the compressed gas, which moves with the velocity u.

(*b*) the density of the compressed gas immediately at the back of the shock wave has the value: $\varrho_1 = 4\,\varrho_0$;

(*c*) the pressure immediately at the back of the shock wave is equal to:

$$p_1 = \tfrac{4}{3}\,\varrho_0\,(\partial\varphi/\partial t)^2_{t=T}.$$

The first relation gives:

$$d\psi/dT = \tfrac{4}{3}\,(\partial\varphi/\partial t)_{t=T}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Taken in conjunction with (7) this equation will make it possible to find the function ψ and the relation between s and T when the function φ should be known. From (*b*) and (*c*) we can deduce the values of $\theta_1(s)$, $\theta_2(s)$:

$$\theta_1(s) = 4\,\varrho_0 \left(\frac{\partial\varphi}{\partial s}\right)_{t=T};\ \theta_2(s) = \tfrac{4}{3}\,\varrho_0 \left(\frac{\partial\varphi}{\partial t}\right)^2_{t=T} \left(\frac{\partial\varphi}{\partial s}\right)^{5/3}_{t=T} . \quad . \quad (9)$$

It is useful now to define the parameter s in such a way that along the path of the compression wave we shall have:

$$s = \tfrac{1}{4}\,\psi(T) = \tfrac{1}{4}\,\varphi(s;\,T) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

As the initial slope of the path of a layer of gas (*i.e.* its initial velocity, immediately after the shock wave has moved over it) is given by:

$$(dx/dt)_{t=T} = (\partial\varphi/\partial t)_{t=T} = \tfrac{3}{4}\,d\psi/dT,$$

equation (7) leads to:

$$(\partial\varphi/\partial s)_{t=T} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This makes it possible to simplify formulae (9). In particular: $\theta_1(s) = 4\,\varrho_0 = constant$, while in consequence of (4) the pressure in an arbitrary point of a path s is given by:

$$p = \tfrac{4}{3}\,\varrho_0 \left(\frac{\partial\varphi}{\partial t}\right)^2_{t=T} \left(\frac{\partial\varphi}{\partial s}\right)^{-5/3}.$$

As a result the equation of motion (5) assumes the form:

$$\frac{\partial^2\varphi}{\partial t^2} = -\tfrac{1}{3}\,\frac{\partial}{\partial s}\left[\frac{(\partial\varphi/\partial t)^2_{t=T}}{(\partial\varphi/\partial s)^{5/3}}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (5a)$$

while at the same time the equation of motion (6) for the front of the advancing cloud (or of the piston in the mechanical analogue) is reduced to:

$$\left(\frac{\partial^2\varphi}{\partial t^2}\right)_{s=0} = -\tfrac{4}{3}\,\frac{\varrho_0}{\varrho}\left[\frac{(\partial\varphi/\partial t)^2_{t=T}}{(\partial\varphi/\partial s)^{5/3}}\right]_{s=0} . \quad . \quad . \quad . \quad . \quad (6a)$$

Comparison of the two formulae shows that the boundary condition at $s = 0$ can be written:

$$\frac{\partial}{\partial s}\,ln\left[\frac{(\partial\varphi/\partial t)^2_{t=T}}{(\partial\varphi/\partial s)^{5/3}}\right]_{s=0} = \frac{4\,\varrho_0}{\varrho} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

3. *Transformation of the equation of motion. Initial solution.* — It is convenient to take a new dependent variable: $\Phi = \partial\varphi/\partial s$, and to introduce a diagram having s and t as coordinates (fig. 2). The boundary $s = 0$

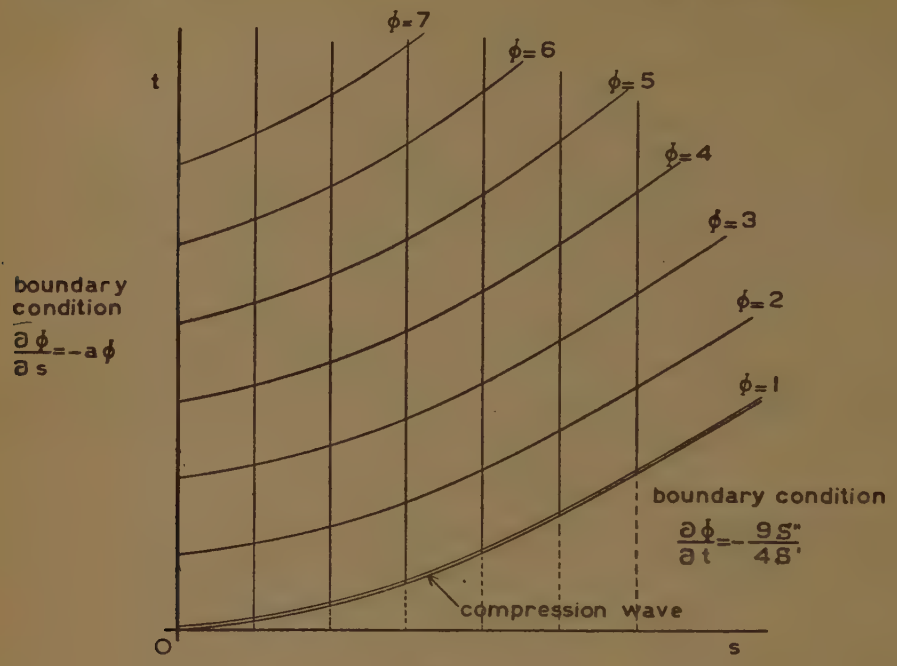

Fig. 2.

then becomes a straight line, while the path of the compression front (shock wave) will be represented by a curve $s = S(t)$. From (8) and (10) it can be deduced that along this curve:

$$S' = dS/dt = \tfrac{1}{3}\,(\partial\varphi/\partial t)_{t=\tau}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

When the differentiation with respect to s occurring in (12) is worked out, there appears a term:

$$\frac{d}{ds}\,ln\,(\partial\varphi/\partial t)^2_{t=\tau} = \frac{d}{ds}\,ln\,(S')^2 = \frac{2\,S''}{S'^2}$$

(primes indicating differentiations with respect to t). Hence the boundary condition (12) can be written:

$$(2\,S''/S'^2)_{s=0} - \tfrac{5}{3}\,(\partial\,ln\,\Phi/\partial s)_{s=0} = 4\,\varrho_0/\bar{\varrho},$$

from which:

$$\partial\Phi/\partial s = -\,a\,\Phi \quad (\text{for } s=0). \quad . \quad . \quad . \quad . \quad . \quad . \quad (14a)$$

where:

$$a = \tfrac{3}{5}\,[4\,\varrho_0/\bar{\varrho} - (2\,S''/S'^2)_{s=0}] \quad . \quad . \quad . \quad . \quad . \quad . \quad (14b)$$

so that a is a constant independent of the time.

Differentiating equation (5a) with respect to s and making use of (13)

we obtain the following equation for Φ, which can be taken as the central equation of the problem:

$$\frac{\partial^2 \Phi}{\partial t^2} = -3 \frac{\partial^2}{\partial s^2} [S'^2 \Phi^{-5/3}] \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

In this equation S', which originally had been defined as a function of t, must be recalculated as a function of s, making use of the formula $s = S(t)$ valid along the path of the shock wave. The boundary conditions for equation (15) are first, condition (14a) referring to $s = 0$. As to the condition at the path of the shock wave, it follows from (11) that this path now is determined by:

$$\Phi = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

When a solution of (15) should have been found, this path can be drawn in the diagram; in consequence of (10) the value of the function $\varphi(s; t)$ can then be obtained from:

$$\varphi(s; t) = 4S - \int_s^S ds\, \Phi(s, t) \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where the integration with respect to s must be performed along a line of constant t, S indicating the value of s at the point of intersection of this line with the curve defined by (16). It must be observed that in consequence of the dependence of S upon t the function $\varphi(s; t)$ defined by (17) will not automatically satisfy the original differential equation (5a). By inserting (17) into this equation and performing the necessary differentiations, making use of (16), it is found that the following condition must be satisfied along the line defined by (16):

$$(\partial \Phi / \partial t)_{s=S} = -\tfrac{9}{4} S''/S' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

This equation, taken together with (16), constitutes the second boundary condition for Φ [3]).

That these conditions are sufficient to make the problem fully determinate, can be seen by calculating a series development of the function Φ in the form:

$$\Phi = 1 + (a_1 t + a_2 s) + (b_1 t^2 + b_2 t s + b_3 s^2) + \quad . \quad . \quad . \quad (19)$$

---

[3]) It is possible to transform equation (15) into the following equation in which t is taken as the dependent variable, considered as a function of s and Φ (indices denoting derivatives):

$$t_{\Phi\Phi}(t_s^2 - \Phi^{8/3}/5\,\theta) - 2\, t_{\Phi s}\, t_\Phi\, t_s + t_{ss}\, t_\Phi^2 = \\ = -\tfrac{8}{5}\, t_\Phi\, t_s^2/\Phi - 2(\theta'/\theta)\, t_\Phi^2\, t_s - \tfrac{3}{5}(\theta''/\theta)\, \Phi\, t_\Phi^3,$$

where θ has been written for S'^2, which is a function of s. The boundary conditions for t then refer to fixed lines in a Φ, s-diagram; they have the form:

$$\text{along } \Phi = 1 : t_\Phi = 4\, t_s^2/9\, t_{ss};$$

$$\text{along } s = 0 : t_s = a\, \Phi\, t_\Phi.$$

valid for small values of s and t (the point $s = 0$, $t = 0$ is taken at the intersection of the path of the advancing front $s = 0$ and the path of the shock wave; *i.e.* at the point from which the whole process originates). We can immediately apply eq. (14*a*); next from (16) we calculate the function $S(t)$ in the form:

$$s = S(t) = a\,t + \beta\,t^2 + \gamma\,t^3 + \ldots\ldots \quad (20)$$

which series also can be inverted in order to determine t in function of S. From (20) the quantities S' and S'' can be obtained; from (19) we calculate the value of $\partial\Phi/\partial t$ along the line (20), expressing it likewise as a function of t; then the second boundary condition (18) can be applied. Finally the differential equation (15) is worked out. It is found that all coefficients appearing in the series can be expressed as functions of the constant a introduced in (14*a*) — (14*b*) and of one arbitrary parameter, for which we take the velocity V_0 of the front of the advancing cloud (or, in the mechanical analogue, of the piston) at the instant $t = 0$. Putting $a\,V_0 t/3 = \tau$, $a\,s = \sigma$, the first terms of the series are:

$$\left.\begin{aligned}
\Phi &= 1 + \tau - \sigma - \tfrac{7}{90}\tau^2 - \tau\sigma + \tfrac{77}{90}\sigma^2 + 0{,}0368\,\tau^3 + \\
&\quad + 0{,}0778\,\tau^2\sigma + 0{,}8517\,\tau\sigma^2 - 0{,}6898\,\sigma^3 - \ldots \\
a\,S &= \tau - \tfrac{2}{9}\tau^2 + \tfrac{16}{135}\tau^3 - 0{,}0818\,\tau^4 + \ldots \\
a\,\varphi &= 3\tau + \sigma - \tfrac{7}{6}\tau^2 + \tau\sigma - \tfrac{1}{2}\sigma^2 + \tfrac{35}{54}\tau^3 - \tfrac{7}{90}\tau^2\sigma - \tfrac{1}{2}\tau\sigma^2 + \\
&\quad + \tfrac{77}{270}\sigma^3 - 0{,}4573\,\tau^4 + 0{,}0368\,\tau^3\sigma + 0{,}0389\,\tau^2\sigma^2 + \\
&\quad + 0{,}2839\,\tau\sigma^3 - 0{,}1724\,\sigma^4 + \ldots
\end{aligned}\right\} \quad (21)$$

An investigation concerning the domain of convergence of these series has not been made; as a division had to be carried out in (15) with the series for Φ it will be necessary to restrict the application to values of τ and σ for which $\Phi < 2$.

The value of the parameter a can be obtained from (14*b*) by inserting the expression for $(2\,S''/S'^2)_{s=0}$; the result is:

$$a = 36\,\varrho_0/7\,\bar{\varrho} \quad \ldots\ldots\ldots \quad (22)$$

The path of the front of the advancing cloud is given by the expression for φ when σ is taken zero. Differentiating with respect to t we obtain the expression for the velocity:

$$V = V_0\,(1 - \tfrac{7}{9}\tau + \tfrac{35}{54}\tau^2 - 0{,}610\,\tau^3 + \ldots) \quad \ldots \quad (23)$$

from which the retardation is found to be:

$$dV/dt = -\tfrac{7}{27}\,a\,V_0^2\,(1 - \tfrac{5}{9}\tau + 2{,}35\,\tau^2 - \ldots) \quad \ldots \quad (23a)$$

It will be seen that when the values of V_0, ϱ_0 and $\bar{\varrho}$ should be known, the velocity V can be calculated for values of t within the domain of convergence of the series. On the other hand the observed values of the velocity and the retardation at a single epoch are not sufficient to enable

us to calculate the three unknowns V_0, a and the epoch t, unless some further assumption is introduced. Such an assumption may refer to the original value of the velocity V_0. When we assume that this value was 1,5 times that of V in 1934, so that $V_0 = 1{,}8 \cdot 10^8$ cm/sec, we find as rough approximations [with a provisional estimate for the remainder of the series (23a)]:

$$\tau \cong 0{,}55; \quad a V_0^2 \cong 0{,}85,$$

from which:

$$a \cong 2{,}6.10^{-17}; \quad \bar{\varrho}/\varrho_0 \cong 2.10^{17}; \quad t \cong 3{,}5.10^8 \text{ sec } (= 11 \text{ years}).$$

The series for Φ (with $\sigma = 0$) then gives: $\Phi = 1{,}53$. As $\varrho = 4\,\varrho_0/\Phi$; $p = 12\,\varrho_0 S'^2/\Phi^{5/3}$, we find:

$$RT = 3\,S'^2/\Phi^{2/3},$$

which at the front of the cloud ($\sigma = 0$) reduces to $RT = \frac{1}{3} V_0^2/\Phi^{2/3}$. This gives: $RT \cong 0{,}81 \cdot 10^{16}$, from which T comes out roughly as 100.000.000°.

(To be continued.)

Zoology. — *The Rhizocephalan parasites of the crab Actaea hirsutissima (Rüpp.).* By H. BOSCHMA.

(Communicated at the meeting of January 25, 1947.)

The present paper contains notes on four different species of Rhizocephala which may occur as parasites of the crab *Actaea hirsutissima* Rüpp. One of these is described here as a new species, *Sacculina ignorata;* previously specimens of this parasite had been wrongly identified. It proved to be distinct as it shows retinacula of a kind not yet known in parasites with more or less corresponding characters as far as concerns other parts of the body. Another species, *Loxothylacus variabilis,* until now was not recorded as a parasite of *Actaea hirsutissima.*

The four parasites dealt with here may be characterized in the following manner:

Sacculina brevispina V. K. and B., 1925 (diagnosis copied from BOSCHMA, 1937): Male genital organs in the posterior part of the body, outside the visceral mass, in their dorsal part largely united, forming a common wide sac. Testes abruptly passing into the vasa deferentia, which, especially in their ventral part, are rather narrow. Colleteric glands with comparatively few canals, arranged in a single row parallel to the surface of the visceral mass. External cuticle of the mantle rather sparsely covered with small blunt spines of approximately 15 μ length, which may possess a few minute hairs. Internal cuticle of the mantle with rows of retinacula, each consisting of a basal part and a variable number (1 to 4) of spindles. The latter are of variable size; they may reach a length of 20 μ.

Type on *Actaea hirsutissima* (Rüpp.); type-locality Sanguisiapo, Sulu Archipelago. The species is not known to occur on other hosts.

Sacculina microthrix Boschma, 1931*c* (diagnosis copied from BOSCHMA, 1937): Male genital organs in the visceral mass, completely separated. Testes more or less globular, rather abruptly passing into the comparatively wide vasa deferentia. Colleteric glands with a moderate number of canals. External cuticle of the mantle densely covered with small thin hairs which have a length of 3 to 8 μ. The structure of the chitin of these hairs is not different from that of the main layers. Retinacula unknown.

Type on *Actaea hirsutissima* (Rüpp.); type-locality Banda Neira. The species is not known to occur on other hosts.

Loxothylacus variabilis Boschma, 1940: Male genital organs of equal size or left small and right large. Curvature of male organs distinct, narrow or wide, or male organs slightly bent, or male organs practically straight. Colleteric glands with a moderate to fairly large number of branched canals. External cuticle densely covered with comparatively small hairs (minimum and maximum measurements 6 and 52 μ). Between these hairs

there are larger spines in far smaller numbers (minimum and maximum measurements 30 and 186 μ). The excrescences have undivided tips or are irregularly divided into smaller branches. Retinacula with 1 to 5 spindles which may show barbs and vary in length from 9 to 13 μ.

Type on *Chlorodiella nigra* (Forsk.); type-locality near Koepang, Timor. The species moreover is known as a parasite of *Actaea rüppellii* (Krauss) and of another, unidentified Xanthid crab. To these hosts now *Actaea hirsutissima* (Rüpp.) must be added.

Sacculina ignorata nov. spec.: Male genital organs in the posterior part of the body, outside the visceral mass, completely separated. Testes more or less globular, abruptly passing into the wide vasa deferentia. Colleteric glands with a small number of canals (less than 10 canals in the most strongly branched region of these glands). External cuticle of the mantle with groups of hyaline spines, consisting of a kind of chitin which is different from that of the main layers of the cuticle. As a rule the spines of each group remain isolated, though they may be united on a common basal part. The length of the spines is 30 to 80 μ. Retinacula with single spines of a length of 10 to 15 μ, occurring in groups or rows on the surface of the internal cuticle.

Type on *Actaea hirsutissima* (Rüpp.); type-locality Ghardaqa, Red Sea (from edge of shore reef by the Laboratory, collected by Mahmûd Eff. Ramadan). The species is not known to occur on other hosts.

Some more details of the specimens examined follow here.

A well preserved specimen, from which a series of longitudinal sections has been made (no. 943 A), is selected as the type of the species. Its measurements are: greater diameter 9 mm, antero-posterior diameter 6½ mm, thickness 4 mm.

The male genital organs have the same shape and structure as those of numerous other parasites which possess excrescences of the external cuticle consisting of groups of hyaline spines. The male organs are found in the posterior part of the body, outside the visceral mass (fig. 1*a*). The vasa deferentia are rather wide, their lumen is divided into a number of cavities on account of a great number of ridges on the inner wall (fig. 1*b*). The testes are united to the vasa deferentia by a narrow canal with a chitinous wall. In fig. 1*c* this narrow canal is seen in the anterior wall of the right testis, the testis itself is surrounded by a thin muscular layer. The connection of the left vas deferens with its testis is represented in fig. 1*d*, here the narrow canal is found between the lumina of the vas deferens and of the testis. In the latter section a more dorsal part of the right testis is shown. As usual in the parasites of this group one of the testes is found at a slight distance behind the other.

The colleteric glands (fig. 2) are comparatively small. They contain few canals only (less than 10 in the most strongly branched region of the glands). The canals are neatly arranged in a single row along the surface of the visceral mass.

The external cuticle of the mantle is covered with excrescences in the shape of spines which consist of a hyaline, hard kind of chitin, distinctly

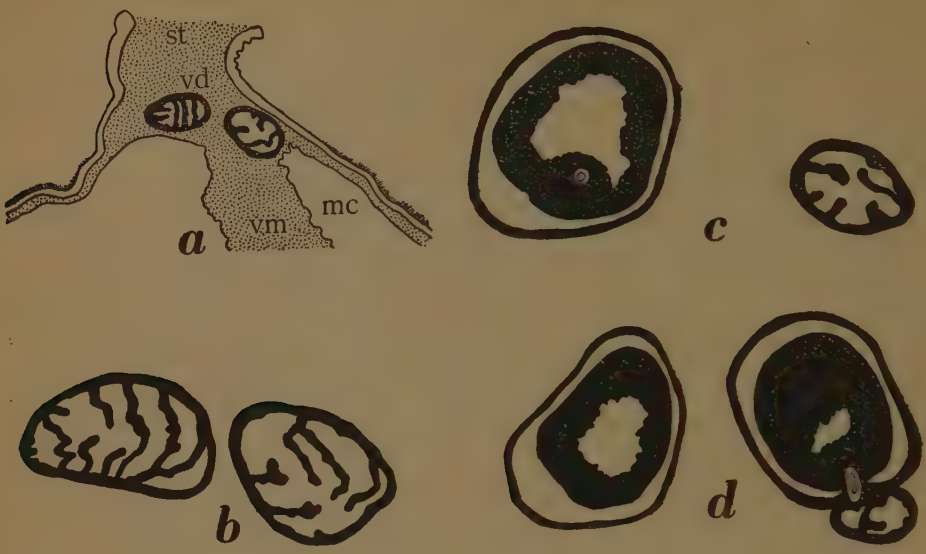

Fig. 1. *Sacculina ignorata*. *a*, longitudinal section through the stalk; *b*—*d*, parts of longitudinal sections showing the transverse section of the male genital organs in different planes; *b*, through the vasa deferentia, each following section from a more dorsal region. *a*, × 18, *b*—*d*, × 45.

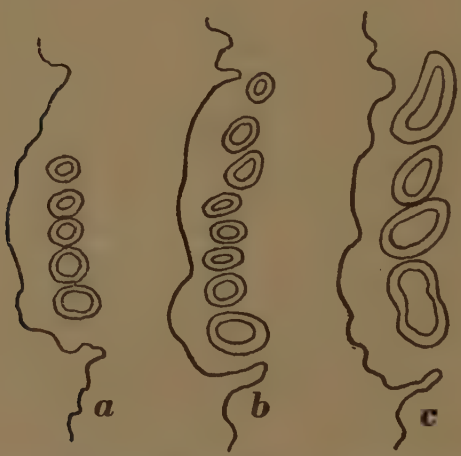

Fig. 2. *Sacculina ignorata*, longitudinal sections of one of the colleteric glands; *a*, of the peripheral region, each following section from a more central part. × 72.

different from that of the main layers of the cuticle. The spines are more or less arranged in groups of a few spines each (fig. 3). Generally the individual spines remain completely isolated, rather exceptionally two or three spines may be united in their basal part. In some regions of the

cuticle all the spines remain isolated, in other parts compounds of two or three spines are rather common. The length of the excrescences varies, in

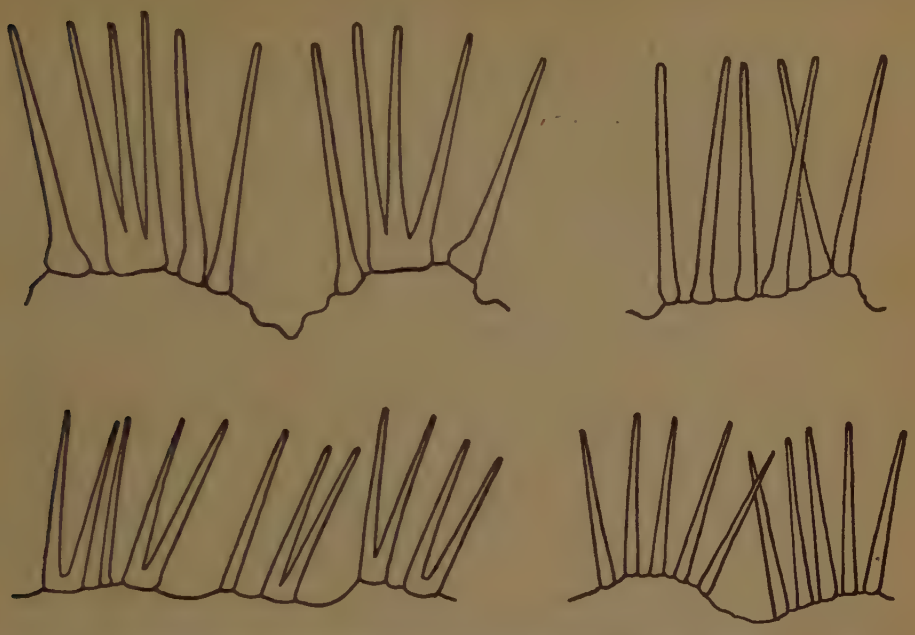

Fig. 3. *Sacculina ignorata*, excrescences on the surface of various parts of the external cuticle. × 530.

some regions of the mantle the spines are 30 to 50 μ, in other regions 55 to 65 μ.

On the internal cuticle of the mantle there occur retinacula which are arranged in groups or rows on the surface (fig. 4). Each retinaculum

Fig. 4. *Sacculina ignorata*, retinacula as they are arranged on the internal cuticle. × 530.

consists of a single spindle which has a length of 10 to 15 μ. The spindles are not barbed.

In previous papers remarks have been made on specimens of parasites of *Actaea hirsutissima* which undoubtedly belong to *Sacculina ignorata*. These specimens were collected in the Red Sea at Jiddah in 1881, their state of preservation was not sufficient for a distinct study of the internal organs. The excrescences of these specimens were described by VAN KAMPEN and BOSCHMA (1925), here already attention was drawn to the fact that the spines for the greater part are not united into compounds with a common basal part. In another paper (BOSCHMA, 1928, fig. 2*d*) an excrescence is represented consisting of three united spines. Later (BOSCHMA, 1931b, fig. 11*d*) a section of the posterior part of the body was represented, showing one of the testes. In the same paper (l.c., fig. 13) drawings of groups of excrescences as they occur on the external cuticle of the mantle are figured. The spines of these two specimens may attain a length of 80 μ, so that they may become slightly larger than those of the type specimen.

Various species of crabs may be found to be infested by parasites which completely correspond with *Sacculina ignorata* as far as concerns the shape of the male genital organs and that of the colleteric glands whilst moreover they show excrescences of the external cuticle of similar structure. Now in *S. ignorata* as a rule the excrescences consist of isolated or very little united spines whilst as a rule in the other forms the spines are united into compounds. But in this respect *S. ignorata* is but gradually distinct from the other forms. The retinacula of *S. ignorata,* however, form a character of a high order and on account of these excrescences the species is easily to be distinguished from the other forms with similar male and female organs. In these latter forms a thorough search has been made for retinacula on the surface of the internal cuticle; as they never were found it seems a safe conclusion to state that these forms are devoid of retinacula.

Sacculina phacelothrix Boschma, 1931a, closely corresponds with *S. ignorata* in the shape of the male genital organs, of the colleteric glands, and of the excrescences of the external cuticle, which consist of isolated hyaline spines arranged in small groups. The two species show, however, striking differences in the shape of their retinacula, which in *S. ignorata* have a single spindle each, whilst in *S. phacelothrix* each retinaculum has 3 to 5 spindles. These differences are so pronounced that they give proof of the specific distinction of the two forms.

The characters of *Sacculina microthrix* (cf. BOSCHMA, 1931c) seem to be of distinct specific value. As, however, the specimens on which the description is based still are immature it is not quite certain that the definite specific characters, especially those of the excrescences of the cuticle, as yet are known. The male organs of *S. microthrix* closely correspond with those of *S. ignorata*, the colleteric glands have a far larger number of canals (cf. BOSCHMA, 1937).

The other two parasites of *Actaea hirsutissima* are strongly different from *S. ignorata*. In *S. brevispina* the male genital organs are largely united and the excrescences of the cuticle are different, this holds for the spines

as well as for the retinacula. In *S. brevispina* the spines of the external cuticle are far more sparsely distributed than those of *S. ignorata*. In both species the retinacula are arranged in rows, but whilst in *S. ignorata* all retinacula consist of a single spindle, in *S. brevispina* two to four spindles may be united on a common basal part. *Loxothylacus variabilis* differs from *Sacculina ignorata* besides by its generic characters by the peculiar shape of its excrescences.

It is interesting that in the specimen of *L. variabilis* on *Actaea hirsutissima* two layers of external cuticle are present (fig. 5). Both layers have a quite similar structure, the outer layer is thinner than the inner, and the smaller excrescences are more densely arranged on the inner layer than on

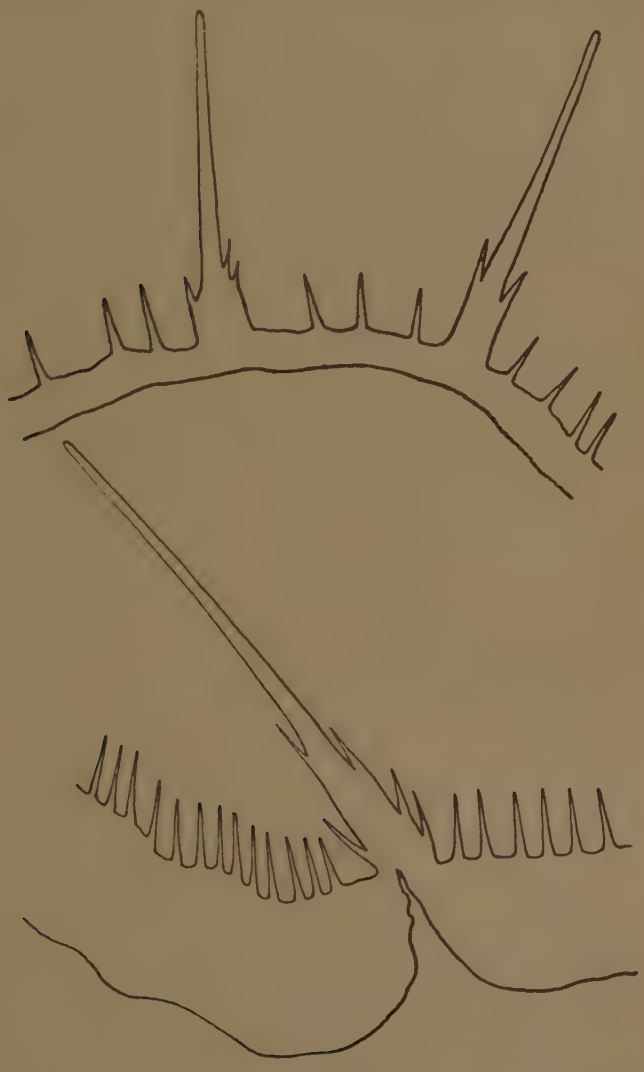

Fig. 5. *Loxothylacus variabilis*, specimen on *Actaea hirsutissima*, section of the two layers of external cuticle. × 530.

the outer. As far as concerns the male genital organs this specimen shows a strong resemblance to the type (BOSCHMA, 1940), as these organs are not strongly unequal in size and possess a distinct curvature.

The pronounced differences in the characters of the four species of Sacculinidae known to occur as parasites of *Actaea hirsutissima* show that these differences really are due to specific qualities of the parasites themselves. The influence of the host is not evident in these characters.

In contradistinction to the three other species which till now are known as parasites of *Actaea hirsutissima* (Rüpp.) only, *Loxothylacus variabilis* is known to infest four different species of crabs. As, however, all these crabs belong to the family Xanthidae it is not astonishing that they may have the same species of parasite.

LITERATURE.

BOSCHMA, H., 1928. The Rhizocephala of the Leiden Museum. Zool. Meded., vol. 11.

————, 1931 a. Rhizocephala. Papers from Dr. Th. Mortensen's Pacific Expedition 1914—16, LX. Vidensk. Medd. Dansk naturh. Foren., vol. 89.

————, 1931 b. Die Rhizocephalen der Siboga-Expedition. Supplement. Siboga-Exp., monogr. 31 bis.

————, 1931 c. Rhizocéphales. Résult. scient. Voy. Ind. Orient. Néerl. de LL.AA.RR. le Prince et la Princesse Léopold de Belgique. Mém. Mus. Roy. Hist. Nat. Belg., hors série, vol. 3.

————, 1937. The Species of the Genus Sacculina (Crustacea Rhizocephala). Zool. Meded., vol. 19.

————, 1940. Biological Results of the Snellius Expedition, VIII. Some Rhizocephala of the Genus Loxothylacus. Temminckia, vol. 5.

KAMPEN, P. N. VAN and H. BOSCHMA, 1925. Die Rhizocephalen der Siboga-Expedition. Siboga-Exp., monogr. 31 bis.

Mathematics. — *Inverse homomorphisms of the homotopy sequence.* By SZE—TSEN HU. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of February 22, 1947.)

1. Introduction.

In 1935 W. HUREWICZ [4] imparted new vitality to the homotopy theory, founded by L. E. J. BROUWER, by introducing the homotopy groups of an arcwise connected space. Later this notion was relativized and the relative homotopy groups modulo an arcwise connected subset were independently defined by HUREWICZ—STEENROD [5], B. ECKMANN [2], and J. H. C. WHITEHEAD [8] during their investigations on fibre spaces. Let Y be an arcwise connected space and Y_0 be an arcwise connected subset of Y. Choose a point $y_0 \in Y_0$ as the base point and denote by $\pi^n(Y)$, $\pi^n(Y_0)$, $\pi^n(Y, Y_0)$ the homotopy groups of Y, Y_0, Y modulo Y_0 respectively. According to J. H. C. WHITEHEAD [8, Theorem 13] the fundamental relation concerning these groups is a chain of natural homomorphisms, now called *the homotopy sequence* of the pair (Y, Y_0):

$$\to \pi^n(Y_0) \xrightarrow{j} \pi^n(Y) \xrightarrow{k} \pi^n(Y, Y_0) \xrightarrow{\partial} \pi^{n-1}(Y_0) \xrightarrow{j} \dots \xrightarrow{j} \pi^1(Y),$$

which is *exact* in the sense that the kernel of each homomorphism is exactly the image of the preceding.

Let E^n denote the n-element in the HILBERT space R^ω with coordinates system $x = (x_1, x_2, \dots)$ defined by

$$E^n: \; x_1^2 + x_2^2 + \dots + x_n^2 \leqslant 1, \quad x_1 = 0 \quad (i > n).$$

Let $S^{n-1} = \dot{E}^n$ denote the boundary sphere of E^n. Let $\xi_0 = (1, 0, \dots)$ and define a mapping $h: R^\omega \to R^\omega$ by taking $h(x) = x' \in R^\omega$ for each $x \in R^\omega$, where $x' = (x_1', x_2', \dots)$ is given by

$$x_i' = 2|x|^2 - 1, \quad (|x|^2 = x_1^2 + x_2^2 + \dots),$$

$$x_{i+1}' = \begin{cases} 2\sqrt{1 - |x|^2} \cdot x_i & (|x|^2 \leqslant 1, \quad i = 1, 2, \dots) \\ 2\sqrt{|x|^2 - 1} \cdot x_i & (|x|^2 \geqslant 1, \quad i = 1, 2, \dots). \end{cases}$$

Then $|x'|^2 = 1$ if $|x|^2 \leqslant 1$ and $|x'|^2 > 1$ if $|x|^2 > 1$. For each $n \geqslant 1$ h maps S^{n-1} on the point ξ_0 and the interior of E^n topologically onto $S^n - \xi_0$. Further the inverse mapping h^{-1} is defined over $S^n - \xi_0$ for each $n \geqslant 1$ and maps $S^n - \xi_0$ topologically onto the interior of E^n.

Let $\alpha \in \pi^n(Y_0)$ be represented by a mapping $f: S^n \to Y_0$ with $f(\xi_0) = y_0$ then $j(\alpha) \in \pi^n(Y)$ is represented also by f. Let $\alpha \in \pi^n(Y)$ be represented by a mapping $f: S^n \to Y$ with $f(\xi_0) = y_0$, then $k(\alpha) \in \pi^n(Y, Y_0)$ is represented by the mapping $fh: E^n \to Y$. Let $\alpha \in \pi^n(Y, Y_0)$ be represented by a mapping $f: E^n \to Y$ with $f(\dot{E}^n) \subset Y_0$ and $f(\xi_0) = y_0$, then $\partial(\alpha) \in \pi^{n-1}(Y_0)$ is represented by the partial mapping $f|\dot{E}^n$.

In a number of special cases, e.g. (1) when Y_0 is contractible to a point in Y, (2) when Y_0 is a retract of Y, and (3) when Y is deformable into Y_0, there are some homomorphisms inverse (in a certain sense) to those of the homotopy sequence. The aim of the present note is to carry out this idea. Our main results are the statements (2.4), (3.4), (3.5), (4.3), (4.4).

A subset M of a space X is said to have *the homotopy extension property* in X relative to Y, if any partial homotopy $f_t: M \to Y$ $(0 \leqq t \leqq 1)$ of an arbitrary mapping $f_0: X \to Y$ has an extension $f_t^*: X \to Y$ $(0 \leqq t \leqq 1)$ such that $f_0^* = f_0$. In particular, we have two important special cases. (i) If Y is an absolute neighbourhood retract [7, p. 58], then every closed set M of a normal space X has the homotopy extension property in X relative to Y [6, p. 86, BORSUK's theorem]. (ii) If X, M are polyhedra, then M has the homotopy extension property in X relative to any topological space Y [1, p. 501].

2. Y_0 contractible in Y.

Y_0 is said to be *contractible in* Y if there exists a homotopy $\theta_t: Y_0 \to Y$ $(0 \leqq t \leqq 1)$ such that θ_0 is the identity and $\theta_1(Y_0)$ is a single point.

In the topological product $X \times I$ of a space X and the closed interval $I = <0, 1>$ of real numbers, let us identify each point $x \in X$ with the point $(x, 0) \in X \times I$, and all the point $(x, 1)$ of $X \times 1$ to a single point v. The resulting space is called *the cone* $c(X)$ *over* X, and v, *its vertex*.

(2.1) *If Y_0 is contractible in Y and $y_0 \in Y_0$ an arbitrary point, then there exists a homotopy $\delta_t: Y_0 \to Y$ $(0 \leqq t \leqq 1)$ such that δ_0 is the identity, $\delta_1(Y_0) = y_0$, and $\delta_t(y_0) = y_0$ for each $0 \leqq t \leqq 1$, provided that the set $Y_0 \cup c(y_0)$ has the homotopy extension property in $c(Y_0)$ relative to Y.*

Proof. Let $\psi: Y_0 \times I \to c(Y_0)$ denote the mapping of identification, then ψ maps $Y_0 \times 1$ on the vertex v and elsewhere topologically. Define a mapping $f: c(Y_0) \to Y$ by taking $f\psi(y, t) = \theta_t(y)$ for each $y \in Y_0$ and $t \in I$. Define a homotopy $f_\lambda: Y_0 \cup c(y_0) \to Y$ $(0 \leqq \lambda \leqq 1)$ by taking

$$f_\lambda = f \text{ on } Y_0,$$

$$f_\lambda \psi(y_0, t) = f\psi(y_0, \lambda t), \quad 0 \leqq t \leqq 1.$$

Since $f_1 = f \mid Y_0 \cup c(y_0)$, it follows from the homotopy extension property that f_λ has an extension $f_\lambda^*: c(Y_0) \to Y$ $(0 \leqq \lambda \leqq 1)$. In particular, $f_0^* = f$ on Y_0 and $f_0^*(c(y_0)) = y_0$. Define a homotopy $\delta_t: Y_0 \to Y$ $(0 \leqq t \leqq 1)$ by taking

$$\delta_t(y) = f_0^* \psi(y, t), \quad (y \in Y_0, \quad 0 \leqq t \leqq 1)$$

Then δ_t is the homotopy required. Q. E. D.

Now assume Y_0 to be contractible in Y and let $\theta_t\colon Y_0 \to Y$ be a given contraction, i.e. a homotopy such that θ_0 is the identity and $\theta_1(Y_0)$ is a point. For each $n>1$, θ_t determines a homomorphism $\bar{\partial}\colon \pi^{n-1}(Y_0) \to \pi^n(Y, Y_0)$ described as follows. Let $\alpha \in \pi^{n-1}(Y_0)$ be represented by a mapping $f\colon \dot{E}^n \to Y_0$ with $f(\xi_0)=y_0$. Using polar coordinates (ϱ, ξ) in E^n where $\xi \in \dot{E}^n$ and $0 \leqq \varrho \leqq 1$, we define a mapping $\Phi\colon E^n \to Y$ by taking

$$\Phi(\varrho, \xi) = \theta_{1-\varrho} f(\xi) \quad (\xi \in \dot{E}^n, \quad 0 \leqq \varrho \leqq 1).$$

Then $\Phi = f$ on $\dot{E}^n$. Φ represents an element $\beta \in \pi^n(Y, Y_0)$ which is clearly independent of the choice of the representative f for α. It can be verified that the transformation $\alpha \to \beta$ is a homomorphism $\bar{\partial}$. Since $\Phi \mid \dot{E}^n = f$, we have $\partial\bar{\partial}(\alpha) = \alpha$ for each $\alpha \in \pi^{n-1}(Y_0)$ and each $n>1$. Hence $\bar{\partial}$ is a *right inverse* of the boundary homomorphism ∂. It follows that ∂ is a *homomorphism onto* and $\bar{\partial}$ is an *isomorphism into*. Let $\hat{\pi}^n(Y, Y_0)$ denote the kernel of ∂ and $\bar{\pi}^n(Y, Y_0)$ the image of $\bar{\partial}$. We have proved the following statement:

(2.2) *If Y_0 is contractible in Y, then there exists a subgroup $\bar{\pi}^n(Y, Y_0)$ of $\pi^n(Y, Y_0)$ which depends on the contraction θ_t and is mapped isomorphically onto $\pi^{n-1}(Y_0)$ by the boundary homomorphism ∂.*

Next, let $\beta \in \pi^n(Y, Y_0)$ be represented by a mapping $\Phi\colon E^n \to Y$ with $\Phi(\dot{E}^n) \subset Y_0$ and $\Phi(\xi_0) = y_0$. Using polar coordinates (ϱ, ξ) in E^n, we define a mapping $\psi\colon E^n \to Y$ by taking

$$\psi(\varrho, \xi) = \begin{cases} \Phi(3\varrho, \xi) & (\xi \in \dot{E}^n, \quad 0 \leqq \varrho \leqq 1/3) \\ \theta_{3\varrho-1}\Phi(1, \xi) & (\xi \in \dot{E}^n, \quad 1/3 \leqq \varrho \leqq 2/3) \\ \theta_{3-3\varrho}(y_0) & (\xi \in \dot{E}^n, \quad 2/3 \leqq \varrho \leqq 1\ ). \end{cases} \quad \ldots \quad \text{(i)}$$

Then $\psi(\dot{E}^n) = y_0$. Define a mapping $g\colon S^n \to Y$ by taking

$$g(\xi) = \begin{cases} \psi h^{-1}(\xi) & (\xi \in S^n - \xi_0) \\ y_0 & (\xi = \xi_0\ ). \end{cases} \quad \ldots\ldots \quad \text{(ii)}$$

g represents an element $\nu \in \pi^n(Y)$ which is clearly independent of the choice of the representative Φ for β. It can be seen that the transformation $\beta \to \nu$ is a homomorphism $\bar{k}\colon \pi^n(Y, Y_0) \to \pi^n(Y)$.

Now let $\nu \in \pi^n(Y)$ be represented by a mapping $f\colon S^n \to Y$ with $f(\xi_0) = y_0$. Then $k(\nu) \in \pi^n(Y, Y_0)$ is represented by $\Phi = fh$. Since $fh(\dot{E}^n) = y_0$, the mapping $\psi\colon E_n \to Y$ in (i) becomes

$$\psi(\varrho, \xi) = \begin{cases} \Phi(3\varrho, \xi) & (\xi \in \dot{E}^n, \quad 0 \leqq \varrho \leqq 1/3) \\ \theta_{3\varrho-1}(y_0) & (\xi \in \dot{E}^n, \quad 1/3 \leqq \varrho \leqq 2/3) \\ \theta_{3-3\varrho}(y_0) & (\xi \in \dot{E}^n, \quad 2/3 \leqq \varrho \leqq 1\ ). \end{cases}$$

Define a homotopy $\psi_t : E^n \to Y$ $(0 \leqq t \leqq 1)$ by taking

$$\psi_t(\varrho, \xi) = \begin{cases} \Phi\left(\dfrac{3\varrho}{1+2t}, \xi\right) & \left(0 \leqq \varrho \leqq \dfrac{1+2t}{3}\right) \\ \theta_{3\varrho-2t-1}(y_0) & \left(\dfrac{1+2t}{3} \leqq \varrho \leqq \dfrac{2+t}{3}\right) \\ \theta_{3-3\varrho}(y_0) & \left(\dfrac{2+t}{3} \leqq \varrho \leqq 1\right) \end{cases}$$

Then $\psi_0 = \psi$, $\psi_1 = \Phi$ and $\psi_t(\dot{E}^n) = y_0$ for each $0 \leqq t \leqq 1$. Hence $f \simeq g$ rel. ξ_0 and $\bar{k}k(\nu) = \nu$ for $\nu \in \pi^n(Y)$ and each $n > 1$. Thus $\bar{k}$ is a *left inverse* of the injection homomorphism k. It follows that k is an *isomorphism into* and $\bar{k}$ is a *homomorphism onto*.

(2.3) *$\bar{k}$ maps $\hat{\pi}^n(Y, Y_0)$ isomorphically onto $\pi^n(Y)$, and the kernel of $\bar{k}$ is $\bar{\pi}^n(Y, Y_0)$.*

Proof. The first statement follows from the fact that $\bar{k}k$ is the identity. It remains to study the kernel of $\bar{k}$.

Let $\alpha \in \pi^{n-1}(Y_0)$ be represented by a mapping $f : \dot{E}^n \to Y_0$, then $\bar{\partial}(\alpha) \in \bar{\pi}^n(Y, Y_0)$ is represented by the mapping $\Phi : E^n \to Y$ given by $\Phi(\varrho, \xi) = \theta_{1-\varrho} f(\xi)$ for each $\xi \in \dot{E}^n$ and $0 \leqq \varrho \leqq 1$. Then the mapping $\psi : E^n \to Y$ in (i) which represents $\bar{k}\bar{\partial}(\alpha)$ becomes

$$\psi(\varrho, \xi) = \begin{cases} \theta_{1-3\varrho} f(\xi) & (\xi \in \dot{E}^n, \quad 0 \leqq \varrho \leqq 1/3) \\ \theta_{3\varrho-1} f(\xi) & (\xi \in \dot{E}^n, \quad 1/3 \leqq \varrho \leqq 2/3) \\ \theta_{3-3\varrho}(y_0) & (\xi \in \dot{E}^n, \quad 2/3 \leqq \varrho \leqq 1\ ). \end{cases}$$

It becomes clear that ψ is nullhomotopic relative to $\dot{E}^n$ and the mapping g defined in (ii) is nullhomotopic relative to ξ_0. Hence $\bar{\pi}^n(Y, Y_0)$ is contained in the kernel of $\bar{k}$.

Conversely, suppose $\beta \in \pi^n(Y, Y_0)$ be an element of the kernel of $\bar{k}$ represented by $\Phi : E^n \to Y$. Then the mapping ψ given by (i) is nullhomotopic relative to $\dot{E}^n$, i.e. there exists a homotopy $\psi_t : E^n \to Y$ $(0 \leqq t \leqq 1)$ such that $\psi_0 = \psi$, $\psi_1(E^n) = y_0$, and $\psi_t(\dot{E}^n) = y_0$ for each $0 \leqq t \leqq 1$. By the aid of ψ_t one can construct a homotopy $\Phi_t : E^n \to Y$ $(0 \leqq t \leqq 1)$ such that

$$\Phi_0 = \Phi, \quad \Phi_1(\varrho, \xi) = \theta_{1-\varrho}\Phi(1, \xi), \quad \Phi_t(1, \xi) = \Phi(1, \xi)$$

for each $\xi \in \dot{E}^n$ and $0 \leqq t, \varrho \leqq 1$. Hence it follows that $\beta = \bar{\partial}\partial(\beta)$ which belongs to $\bar{\pi}^n(Y, Y_0)$. Q. E. D.

The following main theorem of this paragraph is a generalization of a theorem due to HUREWICZ—STEENROD [5, Theorem 3]. The original proof given by them is also available for the general case.

(2.4) *If Y_0 is contractible in Y, then $\pi^n(Y, Y_0)$ is the direct product of $\hat{\pi}^n(Y, Y_0)$ and $\bar{\pi}^n(Y, Y_0)$; hence*

$$\pi^n(Y, Y_0) \approx \pi^n(Y) \times \pi^{n-1}(Y_0).$$

Proof. Let $\alpha \in \pi^n(Y, Y_0)$. Let $\nu = \bar{\partial}\partial(\alpha) \in \bar{\pi}^n(Y, Y_0)$ and $\beta = \alpha\nu^{-1}$. Then it is clear that $\beta \in \hat{\pi}^n(Y, Y_0)$. Hence α can be expressed in the form $\alpha = \beta\nu$. Suppose $\alpha = \beta^*\nu^*$ where $\beta^* \in \hat{\pi}^n(Y, Y_0)$, $\nu^* \in \bar{\pi}^n(Y, Y_0)$; then $\partial(\nu) = \partial(\alpha) = \partial(\nu^*)$. Since ∂ maps $\bar{\pi}^n(Y, Y_0)$ isomorphically, we have $\nu^* = \nu$ and $\beta^* = \beta$. The expression $\alpha = \beta\nu$ is unique.

Since $\bar{\pi}^n(Y, Y_0)$ is the kernel of $\bar{k}$, it is a normal subgroup of $\pi^n(Y, Y_0)$. Hence $\beta\nu\beta^{-1} = \nu^* \in \bar{\pi}^n(Y, Y_0)$ and $\partial(\nu^*) = \partial(\nu)$. Then it follows that $\nu^* = \nu$ and $\beta\nu = \nu\beta$. This completes the proof. The last argument is necessary only for $\pi^2(Y, Y_0)$ which is in general non-commutative.

Q. E. D.

3. Y_0 a retract of Y.

We recall that a *retraction* of Y onto Y_0 is a mapping $\theta: Y \to Y_0$ such that $\theta(y) = y$ for each $y \in Y_0$. Y_0 is said to be a *retract* of Y if there exists a retraction of Y onto Y_0. Throughout the present paragraph, we assume Y_0 to be a retract of Y with a given retraction $\theta: Y \to Y_0$.

Let $\pi: S^n \to E^n$ denote the mapping defined by $\pi(x) = x' \in E^n$ for each $x \in S^n$, where $x'_i = x_i$ $(i = 1, 2, \ldots, n)$ and $x'_i = 0$ $(i > n)$. Let D_1^n, D_2^n denote the subsets of S^n defined by $x_{n+1} \geqq 0$, $x_{n+1} \leqq 0$ respectively. Then π maps D_i^n $(i = 1, 2)$ topologically onto E^n and is the identity on S^{n-1}.

Let $\alpha \in \pi^n(Y, Y_0)$ be represented by a mapping $f: E^n \to Y$ with $f(\dot{E}^n) \subset Y_0$ and $f(\xi_0) = y_0$. Define a mapping $\Phi: S^n \to Y$ by taking

$$\Phi(x) = \begin{cases} \theta f \pi(x) & (x \in D_1^n) \\ f\pi(x) & (x \in D_2^n). \end{cases}$$

Since $\Phi(\xi_0) = y_0$, Φ represents an element $\beta \in \pi^n(Y)$ which is clearly independent of the choice of the representative f for α. It can be verified that the transformation $\alpha \to \beta$ is a homomorphism which will be denoted by $\bar{k}: \pi^n(Y, Y_0) \to \pi^n(Y)$.

(3.1) *$\bar{k}$ is a right inverse of k, i.e. $k\bar{k}(\alpha) = \alpha$ for each $\alpha \in \pi^n(Y, Y_0)$.*

Proof. Let E_1^n, E_2^n be the subsets of E^n defined by $x_n \geqq 0$, $x_n \leqq 0$ respectively. The element $k\bar{k}(\alpha) \in \pi^n(Y, Y_0)$ is represented by the mapping $g: E^n \to Y$ given by

$$g(x) = \begin{cases} \theta f \pi h(x) & (x \in E_1^n) \\ f\pi h(x) & (x \in E_2^n). \end{cases}$$

Define a homotopy $\delta_t: E^n \to E^n$ $(0 \leqq t \leqq 1)$ by taking $\delta_t(x) = x$ if $x \in E_2^n$ and $\delta_t(x) = (x_1, x_2, \ldots, x_{n-1}, tx_n, 0, \ldots)$ if $x = (x_1, x_2, \ldots, x_n, 0, \ldots) \in E_1^n$.

Define a homotopy $g_t\colon E^n \to Y$ $(0 \leqslant t \leqslant 1)$ by taking $g_t = g\,\delta_t$; then $g_1 = g$, $g_t(\dot{E}^n) \subset Y_0$ and $g_t(\xi_0) = y_0$ for each $0 \leqslant t \leqslant 1$. Hence $g_0 = f\pi h\,\delta_0$ represents the element $k\bar{k}(\alpha)$.

Consider the mapping $\psi = \pi h\,\delta_0\colon E^n \to E^n$. Denote by D_1^{n-1}, D_2^{n-1} the subsets of S^{n-1} defined by $x_n \geqslant 0$, $x_n \leqslant 0$. Then $\psi(D_2^{n-1}) = \xi_0$ and ψ maps $S^{n-1} - D_2^{n-1}$ topologically onto $S^{n-1} - \xi_0$ with degree $+1$. Hence there exists a partial homotopy $\psi_t\colon S^{n-1} \to S^{n-1}$ $(0 \leqslant t \leqslant 1)$ such that $\psi_0 = \psi \mid S^{n-1}$, ψ_1 is the identity, and $\psi_t(\xi_0) = \xi_0$ for each $0 \leqslant t \leqslant 1$. Since $\psi_0 = \psi$ is defined over E^n with values in E^n, it follows from the homotopy extension property of polyhedra that ψ_t has an extension $\psi_t^*\colon E^n \to E^n$ $(0 \leqslant t \leqslant 1)$ with $\psi_0^* = \psi$.

Define a homotopy $\eta_t\colon E^n \to E^n$ $(0 \leqslant t \leqslant 1)$ by taking $\eta_t(x)$ to be the point which divides the line-segment joining $\psi_1^*(x)$ to x in the ratio $t\colon(1-t)$. Then we have $\eta_0 = \psi_1^*$, η_1 is the identity, and $\eta_t(x) = x$ for each $x \in \dot{E}^n$ and $0 \leqslant t \leqslant 1$.

Define a homotopy $f_t\colon E^n \to Y$ $(0 \leqslant t \leqslant 1)$ by taking

$$f_t(x) = \begin{cases} f\psi_{2t}(x) & (0 \leqslant t \leqslant 1/2,\quad x \in E^n) \\ f\eta_{2t-1}(x) & (1/2 \leqslant t \leqslant 1,\quad x \in E^n). \end{cases}$$

Then $f_0 = g_0$, $f_1 = f$, $f_t(\dot{E}^n) \subset Y_0$ and $f_t(\xi_0) = y_0$ for each $(0 \leqslant t \leqslant 1$. Hence $k\bar{k}(\alpha) = \alpha$. Q. E. D.

Let $\bar{\pi}^n(Y)$ denote the image of $\bar{k}$, then we deduce from (3.1) the following statement.

(3.2) *If Y_0 is a retract of Y, then for each $n \geqslant 2$ there exists a subgroup $\bar{\pi}^n(Y)$ of $\pi^n(Y)$ which depends on the retraction θ and is mapped isomorphically onto $\pi^n(Y, Y_0)$ by the injection homomorphism k.*

The retraction θ induces a homomorphism $\bar{j}\colon \pi^n(Y) \to \pi^n(Y_0)$. If $\beta \in \pi^n(Y)$ is represented by a mapping $f\colon S^n \to Y$ with $f(\xi_0) = y_0$, then $\bar{j}(\beta)$ is represented by θf. It is trivial that $\bar{j}$ is a *left inverse* of j, i.e. $\bar{j}j(\nu) = \nu$ for each $\nu \in \pi^n(Y_0)$. Let $\hat{\pi}^n(Y)$ denote the image of j.

(3.3) *$\bar{j}$ maps $\hat{\pi}^n(Y)$ isomorphically onto $\pi^n(Y_0)$, and the kernel of $\bar{j}$ is $\bar{\pi}^n(Y)$.*

Proof. The first statement follows from the fact that $\bar{j}j$ is the identity. It remains to study the kernel of $\bar{j}$.

Let $\alpha \in \pi^n(Y, Y_0)$ be represented by a mapping $f\colon E^n \to Y$ with $f(\dot{E}^n) \subset Y_0$ and $f(\xi_0) = y_0$, then $\bar{j}\,\bar{k}(\alpha)$ is represented by $\psi = \theta f\pi\colon S^n \to Y_0$. The mapping $\pi\colon S^n \to E^n$ has evidently an extension $\pi^*\colon E^{n+1} \to E^n$. Hence ψ has an extension $\psi^* = \theta f\pi^*$ of E^{n+1} into Y_0, and $\bar{j}\,\bar{k}(\alpha)$ is the neutral element of $\pi^n(Y_0)$. Thus we have proved that $\bar{\pi}^n(Y)$ is contained in the kernel of $\bar{j}$.

Conversely, let $\beta \in \pi^n(Y)$ be represented by a mapping $\phi: S^n \to Y$ with $\phi(\xi_0) = y_0$. We may assume that $\phi(D_1^n) = y_0$. Suppose β is an element of the kernel of $\bar{j}$, then $\theta\phi$ is a representative of the neutral element of $\pi^n(Y_0)$ and hence of $\pi^n(Y)$. Let $\tau: S^n \to S^n$ be the mapping which maps each point $\xi \in S^n$ on its mirror image $\tau(\xi)$ in the hyperplane $x_{n+1} = 0$. Define a mapping $\psi: S^n \to Y$ by taking

$$\psi(\xi) = \begin{cases} \theta\phi\tau(\xi) & (\xi \in D_1^n) \\ \phi(\xi) & (\xi \in D_2^n). \end{cases}$$

Then ψ also represents β. Since π maps D_2^n topologically onto E^n, the inverse π^{-1} of the partial mapping $\pi \mid D_2^n$ is defined. Define a mapping $f: E^n \to Y$ by taking $f = \phi\pi^{-1}$. Then we have

$$\psi(\xi) = \begin{cases} \theta f\pi(\xi) & (\xi \in D_1^n) \\ f\pi(\xi) & (\xi \in D_2^n). \end{cases}$$

f represents an element $\alpha \in \pi^n(Y, Y_0)$ and $\beta = \bar{k}(\alpha)$. Hence it follows that $\beta \in \bar{\pi}^n(Y)$.

The following main theorem of this paragraph can be proved by the argument used in (2.4).

(3.4) *If Y_0 is a retract of Y, then for each $n \geqq 2$, $\pi^n(Y)$ is the direct product of $\hat{\pi}^n(Y)$ and $\bar{\pi}^n(Y)$; hence*

$$\pi^n(Y) \approx \pi^n(Y_0) \times \pi^n(Y, Y_0).$$

It remains to consider the fundamental group $\pi^1(Y)$. Y is said to be *1-simple relative to Y_0* if $\hat{\pi}^1(Y)$ is a normal subgroup of $\pi^1(Y)$. Let $\bar{\pi}^1(Y)$ denote the kernel of $\bar{j}$. Then the following statement can be easily proved.

(3.5) *If Y_0 is a retract of Y and Y is 1-simple relative to Y_0, then $\pi^1(Y)$ is the direct product of $\hat{\pi}^1(Y)$ and $\bar{\pi}^1(Y)$, where $\hat{\pi}^1(Y) \approx \pi^1(Y_0)$.*

4. Y deformable into Y_0.

Y is said to be *deformable into* Y_0 if there exists a homotopy $\theta_t: Y \to Y$ $(0 \leqq t \leqq 1)$ such that θ_0 is the identity and $\theta_1(Y) \subset Y_0$. Throughout the present paragraph, we assume Y to be deformable into Y_0 with a given homotopy θ_t. Further, we may suppose $\theta_1(y_0) = y_0$ provided that we assume y_0 to have the homotopy extension property in Y relative to Y_0.

Let $\sigma: I \to Y$ denote the (closed) continuous path defined by $\sigma(t) = \theta_t(y_0)$ for each $t \in I = <0, 1>$. According to S. EILENBERG [3], σ induces an automorphism (stilled denoted by σ) of the group $\pi^n(Y)$ described as follows. Let $\alpha \in \pi^n(Y)$ be represented by $f: S^n \to Y$, then $\sigma(\alpha) \in \pi^n(Y)$ is the element represented by a mapping $g: S^n \to Y$ such that there

exists a homotopy h_t: $S^n \to Y$ $(0 \leqq t \leqq 1)$ with $h_0 = f$, $h_1 = g$, and $h_t(\xi_0) = \sigma(t)$ for each $0 \leqq t \leqq 1$.

Now let $\alpha \in \pi^n(Y)$ be an arbitrary element, and f: $S^n \to Y$ with $f(\xi_0) = y_0$ be a representative of α. Define a homotopy h_t: $S^n \to Y$ $(0 \leqq t \leqq 1)$ by taking $h_t = \theta_t f$ for each $0 \leqq t \leqq 1$. Since $h_1(S^n) \subset Y_0$ and $h_1(\xi_0) = y_0$, h_1 represents an element $\beta \in \pi^n(Y_0)$. It is obvious that the transformation $\alpha \to \beta$ is the homomorphism induced by the mapping θ_1: $Y \to Y_0$. Let us denote this homomorphism by $\bar{j}$: $\pi^n(Y) \to \pi^n(Y_0)$. Since $h_t(\xi_0) = \theta_t(y_0) = \sigma(t)$, we have

(4. 1) $j\bar{j}(\alpha) = \sigma(\alpha)$ *for each* $\alpha \in \pi^n(Y)$.

Since σ is an automorphism, it follows that $\bar{j}$ is an *isomorphism into* and j is a *homomorphism onto*. Let $\bar{\pi}^n(Y_0)$ denote the image of $\bar{j}$, then we have

(4. 2) j *maps* $\bar{\pi}^n(Y_0)$ *isomorphically onto* $\pi^n(Y)$.

Since j is a homomorphism onto, it follows from the exactness of the homotopy sequence that the boundary homomorphism ∂ is an *isomorphism into*. Since $\pi^n(Y_0)$ is commutative when $n \geqq 2$ or when $n = 1$ and Y_0 is 1-simple in the sense of S. EILENBERG [3]. We have the following statements.

(4. 3) *If* Y *is deformable into* Y_0, *then for each* $n \geqq 2$ $\pi^n(Y_0)$ *is the direct product of* $\hat{\pi}^n(Y_0)$ *and* $\bar{\pi}^n(Y_0)$; *hence*

$$\pi^n(Y_0) \approx \pi^{n+1}(Y, Y_0) \times \pi^n(Y).$$

(4. 4) *If* Y *is deformable into* Y_0 *and* Y_0 *is 1-simple then* $\pi^1(Y_0)$ *is the direct product of* $\hat{\pi}^1(Y_0)$ *and* $\bar{\pi}^1(Y_0)$; *hence*

$$\pi^1(Y_0) \approx \pi^2(Y, Y_0) \times \pi^1(Y).$$

5. Combinations.

The following statements, which are combinations of the above cases two by two, are immediate consequences of the exactness of the homotopy sequence.

(5. 1) *If* Y_0 *is contractible in itself, then* $\pi^n(Y_0) = 0$ *for each* $n \geqq 1$ *and* $\pi^n(Y) \approx \pi^n(Y, Y_0)$ *for each* $n \geqq 2$.

(5. 2) *If* Y *is contractible, then* $\pi^n(Y) = 0$ *for each* $n \geqq 1$ *and* $\pi^n(Y, Y_0) \approx \pi^{n-1}(Y_0)$ *for each* $n \geqq 2$.

(5. 3) *If* Y_0 *is a deformation retract of* Y, *then* $\pi^n(Y, Y_0) = 0$ *for each* $n \geqq 2$ *and* $\pi^n(Y) \approx \pi^n(Y_0)$ *for each* $n \geqq 1$.

BIBLIOGRAPHY.

1. ALEXANDROFF–HOPF, Topologie I.
2. ECKMANN, B., Zur Homotopietheorie gefaster Räume, Comm. Math. Helv., **14**, 141—192 (1941).

3. EILENBERG, S., On the relation between the fundamental group of a space and higher homotopy groups, Fund. Math., **32**, 167—175 (1939).
4. HUREWICZ, W., Beiträge zur Topologie der Deformationen, I—IV, Proc. Kon. Akad. v. Wetensch., Amsterdam, **38**, 112—119, 521—528 (1935), **39**, 117—125, 215—223 (1936).
5. HUREWICZ–STEENROD, Homotopy relations in fibre spaces, Proc. Nat. Acad. Sci. U.S.A., **27**, 60—64 (1941).
6. HUREWICZ–WALLMAN, Dimension theory (Princeton 1941).
7. LEFSCHETZ, S., Topics in topology (Princeton 1942).
8. WHITEHEAD, J. H. C., On the groups $\pi_r(V_{n,m})$ and sphere-bundles, Proc. Lond. Math. Soc., (2) **48**, 243—291 (1944).

Mathematics. — *Sur les systèmes d'équations aux dérivées partielles, qui, comme les systèmes normaux, comportent autant d'équations que de fonctions inconnues* (troisième communication). By A. FINZI. (Communicated by Prof. L. E. J. BROUWER.)

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§ VI.

Systèmes auxquels on ne peut pas appliquer les considérations du § II ou du § V.

On a vu dans le paragraphe précédent que s'il est possible de déduire des équations (1) une système normal d'ordre $h^{(l)}$, les (1) admettent une solution et une seule quand on donne des valeurs initiales des φ et de leurs $h^{(l)}-1$ premières dérivées satisfaisant à $n(h^{(l)}-h)+l$ conditions. Quand $l > hn$, le nombre des conditions deviendrait plus grand que celui des fonctions à donner.

Mais nous montrerons que si les hn premiers systèmes, qui se déduisent du système donné par la méthode des §§ II et V, ne sont pas normaux, il existe entre les F, les Φ et leurs dérivées une identité de la forme

$$\Psi(F_1, F_2, \ldots, F_n, \ldots, \Phi, \ldots, x) = 0. \qquad (12)$$

Si une Φ_μ ($\mu = 1, 2, \ldots, nh+1$) est identiquement nulle, la proposition est démontrée, parce que Φ_μ elle-même constitue la relation cherchée. Dans le cas contraire nous savons que des (1) on peut déduire les $nh+1$ combinaisons $\Phi_1, \Phi_2, \ldots, \Phi_{nh+1}$ d'ordre $h^{(1)}, h^{(2)}, \ldots, h^{(hn+1)}$.

Φ_1 est, comme on l'a vu, une combinaison linéaire entre les dérivées des F d'ordre $h^{(1)}+1$ en φ, Φ_2 est une combinaison entre les dérivées de Φ_1 et des F d'ordre $h^{(2)}+1$ en φ; enfin Φ_{nh+1} est une combinaison entre les dérivées de Φ_{nh} et des F d'ordre $h^{(nh+1)}+1$ en φ.

Choisissons maintenant un nombre

$$\bar{h} \geqq h^{(nh+1)}$$

et considérons le nombre des fonctions indépendantes des F et des leurs dérivées, qui ne contiennent pas de dérivées des inconnues φ d'ordre supérieur a $\bar{h}$.

Parmi ces fonctions nous pouvons prendre en premier lieu les F elles-mêmes et leurs dérivées jusqu'à l'ordre $\bar{h}-h$.

Prenons ensuite les dérivées d'ordre $\bar{h}-h^{(1)}$ de Φ_1; celles d'ordre inférieur ne sont en effet que des combinaisons entre les dérivées des F d'ordre, par rapport aux inconnues φ, inférieur où égal à $\bar{h}$ et ne peuvent donc pas constituer de nouvelles expressions indépendantes.

Puis prenons les dérivées d'ordre $\bar{h}-h^{(2)}$ de Φ_2, puisque celles d'ordre inférieur sont des combinaisons des dérivées de Φ_1 et des F d'ordre, par rapport à φ, ne dépassant pas $\bar{h}$.

Procédons d'une manière analogue pour les relations successives et ainsi, finalement, prenons les dérivées de Φ_{nh+1} d'ordre $\overline{h} - h^{(nh+1)}$.

Nous voulons montrer que lorsqu'on fait augmenter $\overline{h}$, le nombre des expressions considérées arrive à dépasser le nombre total des φ et de leurs dérivées dont elles dépendent. Ces arguments sont en nombre égal à celui des dérivées d'ordre non supérieur à $\overline{h}$ d'une fonction de $m + 1$ variables, augmenté de 1 et multiplié par n. Les F et leurs dérivées jusqu'à l'ordre $\overline{h} - h$ sont, d'autre part, en nombre égal à celui des dérivées d'ordre non supérieur à $\overline{h} - h$ d'une fonction de $m + 1$ variables, augmenté aussi de 1 et multiplié par n.

La différence de ces deux nombres est égale au nombre total des dérivées d'ordre $\overline{h} - h + 1$, $\overline{h} - h + 2$, ..., $\overline{h}$ d'une fonction de $m + 1$ variables, multiplié par n: il est certainement plus petit que celui des dérivées d'ordre $\overline{h}$ de la fonction, multiplié par hn, c'est à dire $hn \dfrac{(m+\overline{h})!}{m!\,\overline{h}!}$.

Il y a aussi, parmi les expressions à considérer, et comme on l'a vu, les dérivées d'ordre $\overline{h} - h^{(1)}$ de Φ_1, celles d'ordre $\overline{h} - h^{(2)}$ de Φ_2 et, enfin, celles d'ordre $\overline{h} - h^{(nh+1)}$ de Φ_{nh+1}: leur nombre total sera certainement supérieur au nombre des dérivées d'ordre $\overline{h} - h^{(nh+1)}$ d'une fonction de $m + 1$ variables, multiplié par $nh + 1$, c.-à-d. $(nh + 1) \dfrac{(m+\overline{h}-h^{(nh+1)})!}{m!\,(\overline{h}-h^{(nh+1)})!}$.

Lorsque $\overline{h}$ augmente indéfiniment, le rapport entre les nombres

$$(nh+1)\frac{(m+\overline{h}-h^{(nh+1)})!}{m!\,(\overline{h}-h^{(nh+1)})!} \text{ et } nh\frac{(m+\overline{h})!}{m!\,\overline{h}!} \text{ tend vers } \frac{nh+1}{nh} > 1.$$

Le nombre des expressions considérées devient donc supérieur à celui des arguments dont elles dépendent: il doit donc exister entre les F, les Φ et leurs dérivées une relation identique comme la (12).

§ VII.

Existence des solutions pour les systèmes considérés au § précédent.

Comme nous avons vu au paragraphe précédent, la relation (12) est une identité en φ; elle doit donc être, en particulier, satisfaite si l'on prend pour les φ une solution éventuelle des (1).

Cela équivaut à dire que (12) elle-même est nécessairement vérifiée par des valeurs nulles des F, des Φ et de leurs dérivées: au cas contraire elle constituerait une relation absurde pour le système donné qui ne pourrait pas admettre de solution.

Si nous supposons, comme cela est toujours possible, que la caractéristique de la matrice obtenue en supprimant la première ligne de Ω est égale à

$n-1$, nous pouvons certainement construire une expression différentielle convenable Z, d'ordre h en φ, telle que le système

$$F_1^* = F_1 + Z = 0, \tag{13}$$
$$F_i = 0 \qquad (i = 2, 3, \ldots, n)$$

soit normal.

Puisque, comme nous l'avons vu, la relation (12) doit être vérifiée par les valeurs nulles des F, des Φ et de leurs dérivées, l'équation

$$\Psi(F_1^*, F_2, \ldots, F_n, \ldots, \Phi^*, \ldots, x) = 0 \tag{12^1}$$

(Φ^* étant les combinaisons qu'on obtient des Φ en substituant F_1^* à F_1) sera satisfaite si l'on prend pour les φ une solution du système normal (13).

La relation (12^1) constitue donc une équation différentielle qui doit être satisfaite par Z, définie par la solution du même système (13).

Cette équation doit admettre nécessairement la solution nulle: en effet quand on prend

$$Z = 0, \tag{14}$$

(12^1) se réduit à (12) et par conséquent est certainement satisfaite. Désignons maintenant par h^* l'ordre de la dérivée la plus élevée de F_1 en (12) et supposons que (12^1), considérée comme une équation en Z, soit normale par rapport à z^0.

Il découle alors de ce qui a été dit que si Z est nulle ainsi que ses premières h^*-1 dérivées sur la surface

$$z^0(x^0, x^1, \ldots, x^m) = 0 \tag{6}$$

pour une solution du système normal (13), il devra être partout

$$Z = 0 \tag{14}$$

et la même solution vérifiera aussi le système donné (1).

On voit tout de suite la différence substantielle entre le cas présent et celui étudié aux § II et V.

Nous devons maintenant assigner, sur la surface portante, les φ et leurs premières $h-1$ dérivées assujetties à h^* conditions convenables; mais la solution du système donné n'est pas déterminée d'une manière univoque. On peut, au contraire, imposer à la solution elle-même une autre condition différentielle.

$$Z = 0 \tag{14}$$

dans tout le domaine de définition.

Comme on l'a observé au numéro 5 de la préface, un système d'un tel type peut donc être comparé, de ce point de vue, à un système de $n-1$ équations de n fonctions inconnues.

Mais naturellement si $h^* > hn$ on devra prendre Z de façon que, par un choix convenable des valeurs initiales, il soit possible de satisfaire, avec la solution du système normal, les h^* conditions en question.

§ VIII.

Lemme relatif aux combinaisons Φ.

On a vu au § V que le système (1) admet une solution et une seule quand on se donne, sur $x^0 = a$, les φ et leurs $h^{(l)} - 1$ dérivées assujetties à $(h^{(l)} - h)n + l$ conditions.

L'analogie avec la théorie des systèmes normaux conduit à se demander s'il suffit de se donner les φ et leurs $h - 1$ premières dérivées seulement, assujetties naturellement à l conditions.

Nous verrons au paragraphe suivant que la réponse est affirmative quand la condition suivante est satisfaite:

Les coefficients des dérivées des φ d'ordre h, que nous désignerons maintenant par $E_{(0)}$, ne contiennent pas de dérivées des φ d'ordre supérieur à $h - l + 1$; les coefficients des dérivées des φ d'ordre $h - 1$, que nous désignerons par $E_{(1)}$, ne contiennent pas de dérivées d'ordre supérieur à $h - l + 2$, et en général les coefficients des dérivées d'ordre $h - r$, que nous désignerons par $E_{(r)}$, ne contiennent pas de dérivées d'ordre supérieur à $h - l + r + 1$. Dans cette hypothèse plus générale rentre en particulier le cas d'un système d'équations linéaires et celui de $l = 1$.

Nous devrons faire, dans ce paragraphe, quelques considérations préliminaires.

Etant donnée, en premier lieu, une équation différentielle satisfaisante à la condition énoncée, supposons d'abord, pour plus de simplicité, qu'elle ne comporte que des dérivées ordinaires

$$\Sigma_j E^j_{(0)} \frac{\partial^h \varphi_j}{\partial x^h} + \ldots + \Sigma_j E^j_{(s)} \frac{\partial^{h-s} \varphi_j}{\partial x^{h-s}} + \ldots + \Sigma_j E^j_{(r)} \frac{\partial^{h-r} \varphi_j}{\partial x^{h-r}} + \ldots = 0.$$

Sa dérivée d'ordre t sera donnée par

$$\Sigma_j E^j_{(0)} \frac{\partial^{h+t} \varphi_j}{\partial x^{h+t}} + \ldots + \Sigma_j \left[\binom{t}{s} \frac{\partial^s E^j_{(0)}}{\partial x^s} + \ldots + E^j_{(s)} \right] \frac{\partial^{h+t-s} \varphi_j}{\partial x^{h+t-s}} + \ldots +$$

$$+ \Sigma_j \left[\binom{t}{r} \frac{\partial^r E^j_{(0)}}{\partial x^r} + \ldots + \binom{t}{r-s} \frac{\partial^{r-s} E^j_{(s)}}{\partial x^{r-s}} + \ldots + E^j_{(r)} \right] \frac{\partial^{h+t-r} \varphi_j}{\partial x^{h+t-r}} + \ldots = 0.$$

Nous pouvons observer que les coefficients des dérivées des φ_j d'ordre $h + t - r$ ne peuvent pas contenir de dérivées des $E_{(s)}$ d'ordre supérieur à $r - s$, et que, ensuite les $E_{(r)}$ y entrent seulement en termes finis.

Une conclusion analogue est aussi valable pour une équation aux dérivées partielles.

Nous nous servirons de cette simple observation pour démontrer un lemme relatif à la structure des expressions Φ_μ.

Φ_μ peut être regardée comme une combinaison entre les dérivées des F jusqu'à celles d'ordre $h^{(\mu)} - h + \mu$.

Nous voulons maintenant démontrer que le coefficient d'une dérivée d'ordre $h^{(\mu)} - h + \mu - \nu$ peut contenir les $E_{(\nu)}$ seulement en termes finis et les dérivées des $E_{(\nu-p)}$ seulement jusqu'à l'ordre p.

Le lemme est certainement exact pour Φ_1: en effet Φ_1 est une combinaison entre les dérivées des F d'ordre $h^{(1)} - h + 1$ et les coefficients de celles-ci contiennent les $E_{(0)}$ en termes finis.

Il reste donc à démontrer que si notre lemme est exact pour Φ_μ il l'est aussi pour $\Phi_{\mu+1}$.

Il s'agit évidemment de faire voir que dans $\Phi_{\mu+1}$ les coefficients des dérivées des F d'ordre $h^{(\mu+1)} - h + \mu - \nu + 1$ peuvent contenir les dérivées des $E_{(\nu-p)}$ seulement jusqu'à l'ordre p.

Montrons d'abord que dans Φ_μ le coefficient d'une dérivée d'une des φ, d'ordre $h^{(\mu)}$ $\dfrac{\partial^{h^{(\mu)}} \varphi_j}{\partial x^{0h_0^{(\mu)}} \partial x^{1h_1^{(\mu)}} \partial x^{mh_m^{(\mu)}}}$ ne peut pas contenir de dérivées des $E_{(\mu-p)}$ d'un ordre supérieur à p.

Ce coefficient est donné en effet par la somme de ceux de

$$\frac{\partial^{h^{(\mu)}} \varphi_j}{\partial x^{0h_0^{(\mu)}} \partial x^{1h_1^{(\mu)}} \ldots \partial x^{mh_m^{(\mu)}}}$$

dans les dérivées des F, multipliés par les coefficients de ces dérivées dans Φ_μ.

Les dérivées des F qui entrent en Φ_μ et qui peuvent contenir

$$\frac{\partial^{h^{(\mu)}} \varphi_j}{\partial x^{0h_0^{(\mu)}} \partial x^{1h_1^{(\mu)}} \ldots \partial x^{mh_m^{(\mu)}}}$$

sont évidemment seulement celles d'ordre $h^{(\mu)} - h + \mu$, $h^{(\mu)} - h + \mu - 1$, ..., $h^{(\mu)} - h$, et, d'après la remarque faite au commencement du paragraphe, les coefficients de $\dfrac{\partial^{h^{(\mu)}} \varphi_j}{\partial x^{0h_0^{(\mu)}} \partial x^{1h_1^{(\mu)}} \ldots \partial x^{mh_m^{(\mu)}}}$ dans ces dérivées ne peuvent pas contenir de dérivées des $E_{(\mu-p)}$ d'un ordre supérieur à p, $p-1$, ... respectivement.

D'autre part, ayant supposé notre lemme vrai pour Φ_μ, dans une telle combinaison les coefficients des dérivées des F d'un ordre non inférieur à $h^{(\mu)} - h$ ne peuvent pas contenir de dérivées des $E_{(\mu-p)}$ d'un ordre supérieur à p.

Nous avons donc effectivement démontré que le coefficient de $\dfrac{\partial^{h^{(\mu)}} \varphi_j}{\partial x^{0h_0^{(\mu)}} \partial x^{1h_1^{(\mu)}} \ldots \partial x^{mh_m^{(\mu)}}}$ dans Φ_μ ne peut pas contenir de dérivées des $E_{(\mu-p)}$ d'un ordre supérieur à p.

La dernière ligne de $\Omega^{(\mu)}$ contient donc les dérivées des $E_{(\mu-p)}$ seulement jusqu'à l'ordre p; les autres lignes du déterminant sont proportionelles à celles de Ω et contiennent donc seulement les $E_{(0)}$ en termes finis.

$\Phi_{\mu+1}$ est une combinaison entre les dérivées de F_i $(i = 2, 3, \ldots, n)$ d'ordre $h^{(\mu+1)} - h + 1$ et les dérivées de Φ_μ d'ordre $h^{(\mu+1)} - h^{(\mu)} + 1$; les coefficients des premières sont les coefficients des puissances des p dans le développement du mineur de $\Omega^{(\mu)}$ obtenu en négligeant la i-ième ligne

et la première colonne, alors que les coefficients des secondes sont les coefficients des puissances des p dans le développement du mineur de $\Omega^{(\mu)}$, obtenu en négligeant la dernière ligne et la première colonne.

Les coefficients des premières contiennent donc les dérivées des $E_{(\mu-p)}$ jusqu'à l'ordre p seulement et satisfont par conséquent au lemme, alors que les coefficients des secondes contiennent seulement les $E_{(0)}$ en termes finis. Φ_μ, d'après notre lemme, est une combinaison, dont chaque terme est constitué par une dérivée de F_i $(i = 1, 2, \ldots, n)$, d'ordre $h^{(\mu)} - h + \mu - \tau$, multipliée par un coefficient, qui peut contenir les dérivées des $E_{(\tau-p)}$ jusqu'à l'ordre p seulement.

Si nous dérivons un tel terme $h^{(\mu+1)} - h^{(\mu)} + 1$ fois, nous obtenons un premier terme constitué par une dérivée de F_i d'ordre $h^{(\mu+1)} - h + \mu - \tau + 1$, multipliée par un coefficient, qui peut contenir les dérivées des $E_{(\tau-p)}$ jusqu'à l'ordre p, d'autres termes constitués par une dérivée de F_i d'ordre $h^{(\mu+1)} - h + \mu - \tau$, multipliée par un coefficient, qui peut contenir les dérivées des $E_{(\tau-p)}$ jusqu'à l'ordre $p + 1$, etc.

Evidemment tous ces termes satisfont aussi notre lemme.

Ce lemme est donc vrai pour $\Phi_{\mu+1}$, et, par conséquent, pour toutes les Φ. *Lorsque le système donné satisfait la condition énoncée au commencement du paragraphe, le coefficient dans Φ_μ d'une dérivée de F_i d'ordre $h^{(\mu)} - h + \mu - \nu$ ne pourra pas contenir de dérivées des φ d'un ordre supérieur à $h - l + \nu + 1$.*

§ IX.

Nouvelle forme du théorème d'existence pour une classe de systèmes non normaux.

Revenons maintenant à la question, que nous avons posée au commencement du paragraphe précédent et supposons que le système donné vérifie la condition énoncée.

Nous devons substituer aux anciennes $(h^{(l)} - h)n + l$ conditions, autant d'autres conditions équivalentes.

l de celles-ci ne doivent pas contenir de dérivées par rapport à x^0 d'ordre supérieur à $h - 1$, de façon à constituer autant de conditions pour les nouvelles données initiales: les φ et leurs premières $h - 1$ dérivées. Il devra y avoir ensuite n équations qui permettent de déterminer les $\dfrac{\partial^h \varphi_j}{\partial x^{0^h}}$ en fonction de φ et de leurs premières $h - 1$ dérivées par rapport à x^0, n qui permettent de déterminer les $\dfrac{\partial^{h+1} \varphi_j}{\partial x^{0^{h+1}}}$ en fonction des h premières dérivées, et ainsi de suite jusqu'à l'ordre $h^{(l)} - 1$.

La relation

$$\Gamma = 0 \tag{4}$$

obtenue en éliminant les $\dfrac{\partial^h \varphi_j}{\partial x^{0^h}}$ entre les équations du système donné constitue une première relation pour les nouvelles valeurs initiales.

La condition de rendre nulles les F_i $(i = 2, 3, \ldots, n)$ et leurs premières $h^{(l)} - h - 1$ dérivées apporte ensuite $n - 1$ équations indépendantes pour déterminer les $\dfrac{\partial^h \varphi_j}{\partial x^{0h}}$ et autant, respectivement, pour les

$$\frac{\partial^{h+1} \varphi_j}{\partial x^{0h+1}}, \ldots, \frac{\partial^{h^{(l)}-1} \varphi_j}{\partial x^{0h^{(l)}-1}}.$$

Il suffira donc de trouver $l - 1$ conditions pour les valeurs initiales et une équation en termes finis en $\dfrac{\partial^h \varphi_j}{\partial x^{0h}}$.

De cette dernière nous pourrons en effet obtenir, par dérivations successives par rapport à x^0, une équation pour les $\dfrac{\partial^{h+1} \varphi_j}{\partial x^{0h+1}}$, une pour les $\dfrac{\partial^{h+2} \varphi_j}{\partial x^{0h+2}}$, et ainsi de suite jusqu'à l'ordre $h^{(l)} - 1$.

A partir de Φ_μ $(\mu = 1, 2, \ldots, l)$ nous définissons la nouvelle combinaison Φ_μ^*, dans laquelle les coefficients des dérivées des $\dfrac{\partial^{h^{(\mu)}-h} F_i}{\partial x^{0h^{(\mu)}-h}}$ sont les mêmes que dans Φ_μ, alors que, au contraire, les coefficients des $\dfrac{\partial^{h^{(\mu)}-h} F_i}{\partial x^{0h^{(\mu)}-h}}$ et de toutes les autres dérivées des F_i sont nuls.

Les dérivées des $\dfrac{\partial^{h^{(\mu)}} \varphi_j}{\partial x^{0h^{(\mu)}}}$ peuvent apparaître seulement dans les dérivées des $\dfrac{\partial^{h^{(\mu)}-h} F_i}{\partial x^{0h^{(\mu)}-h}}$; leurs coefficients sont donc les mêmes dans Φ_μ et dans Φ_μ^*. Puis, comme coefficients de dérivées d'un ordre supérieur à $h^{(\mu)}$, ils sont certainement nuls dans Φ^μ, ils doivent donc être nuls dans Φ_μ^*.

Un terme quelconque de Φ_μ^* sera donné par

$$a \frac{\partial^{h^{(\mu)}-h+t}}{\partial x^{0h^{(\mu)}-h+t}} (\ldots). \tag{15}$$

mettant entre parenthèse une des F_i ou une de leurs dérivées par rapport aux variables $x^1, x^2, \ldots, x^m$.

D'après la remarque faite à la fin du paragraphe précedent a ne pourra pas contenir de dérivées d'un ordre supérieur à $h - l + \mu - t + 1$ dans le premier cas, et à $h - l + \mu - t$ dans le second.

Construisons maintenant une troisième combinaison Φ_μ^{**} en substituant à (15) dans le premier cas

$$\begin{aligned} a \frac{\partial^t}{\partial x^{0t}} (\ldots) - \binom{h^{(\mu)}-h}{1} \frac{\partial a}{\partial x^0} \frac{\partial^{t-1}}{\partial x^{0t-1}} (\ldots) + \binom{h^{(\mu)}-h+1}{2} \frac{\partial^2 a}{\partial x^{02}} (\ldots) + \\ + \ldots + (-1)^{t-1} \binom{h^{(\mu)}-h+t-2}{t-1} \frac{\partial^{t-1} a}{\partial x^{0t-1}} \frac{\partial}{\partial x^0} (\ldots), \end{aligned} \tag{16}$$

et dans le second

$$a\frac{\partial^t}{\partial x^{0t}}(\ldots)-\binom{h^{(\mu)}-h}{1}\frac{\partial a}{\partial x^0}\frac{\partial^{t-1}}{\partial x^{0t-1}}(\ldots)+\binom{h^{(\mu)}-h+1}{2}\frac{\partial^2 a}{\partial x^{02}}\frac{\partial^{t-2}}{\partial x^{0t-2}}(\ldots)+$$
$$+\ldots+(-1)^t\binom{h^{(\mu)}-h+t-1}{t}\frac{\partial^t a}{\partial x^{0t}}(\ldots). \tag{17}$$

Φ^{**}_{μ}, ainsi définie, est une combinaison entre les dérivées des F jusqu'à l'ordre μ et peut donc contenir les dérivées des φ jusqu'à l'ordre $h+\mu$.

En dérivant (16) $h^{(\mu)}-h$ fois par rapport à x^0 on obtient

$$a\frac{\partial^{h^{(\mu)}-h+t}}{\partial x^{0h^{(\mu)}-h+t}}(\ldots)+(-1)^{t-1}\binom{h^{(\mu)}-h+t-1}{t}\frac{\partial^t a}{\partial x^{0t}}\frac{\partial^{h^{(\mu)}-h}}{\partial x^{0h^{(\mu)}-h}}(\ldots)+$$
$$+\ldots+(-1)^{t-1}\binom{h^{(\mu)}-h+t-2}{t-1}\frac{\partial^{h^{(\mu)}-h+t-1}a}{\partial x^{0h^{(\mu)}-h+t-1}}\frac{\partial}{\partial x^0}(\ldots). \tag{16'}$$

Puisque, en vertu de l'identité

$$\binom{s}{r}-\binom{s}{r-1}\binom{s}{1}+\binom{s}{r-2}\binom{s+1}{2}-\binom{s}{r-3}\binom{s+2}{3}+$$
$$+\ldots+(-1)^r\binom{s}{0}\binom{s+r-1}{r}=0$$

tous les coefficients de dérivées de $(\ldots)$ d'ordre $h^{(\mu)}-h+t-1$, $h^{(\mu)}-h+t-2\ldots h^{(\mu)}-h+1$ sont nuls.

De la même façon, en dérivant (17), on obtient

$$a\frac{\partial^{h^{(\mu)}-h+t}}{\partial x^{0h^{(\mu)}-h+t}}(\ldots)+(-1)^t\binom{h^{(\mu)}-h+t}{t+1}\frac{\partial^{t-1}a}{\partial x^{0t+1}}\frac{\partial^{h^{(\mu)}-h-1}}{\partial x^{0h^{(\mu)}-h-1}}(\ldots)+$$
$$+\ldots+(-1)^t\binom{h^{(\mu)}-h+t-1}{t}\frac{\partial^{h^{(\mu)}-h+t}a}{\partial x_0{}^{h^{(\mu)}-h+t}}(\ldots). \tag{17'}$$

Les expressions (16') et (17') diffèrent de (15) par des termes, qui peuvent contenir les $\dfrac{\partial^{h^{(\mu)}}\varphi_j}{\partial x^{0h^{(\mu)}}}$ (si $\mu=l$), mais pas leurs dérivées.

Il en résulte que dans l'équation, qu'on obtient en dérivant $\Phi^{**}_{\mu}=0$ $h^{(\mu)}-h$ fois par rapport à x^0, les coefficients des dérivées des $\dfrac{\partial^{h^{(\mu)}}\varphi_j}{\partial x^{0h^{(\mu)}}}$ sont les mêmes que dans $\Phi^{*}_{\mu}=0$ et sont par conséquent tous nuls.

Donc, dans $\Phi^{**}_{\mu}=0$ les coefficients des dérivées des $\dfrac{\partial^h\varphi_j}{\partial x^{0h}}$ doivent être nuls, de sorte que cette équation constitue, par rapport aux mêmes $\dfrac{\partial^h\varphi_j}{\partial x^{0h}}$, une relation en termes finis.

Nous montrerons maintenant qu'il est possible d'éliminer les $\frac{\partial^h \varphi_j}{\partial x^{0^h}}$ entre $F_2, F_3, \ldots, F_n$ et chacune des combinaisons $\Phi_1^{**}, \Phi_2^{**}, \ldots, \Phi_{l-1}^{**}$, de manière qu'on arrive à obtenir, comme on le voulait, $l-1$ conditions pour les valeurs initiales.

Il s'agit évidemment de faire voir que le déterminant formé par les coefficients des $\frac{\partial^h \varphi_j}{\partial x^{0^h}}$ en $F_2, F_3, \ldots, F_n, \Phi_\mu^*$ $(\mu = 1, 2, \ldots, l-1)$ est nul.

Pour cela nous notons que, étant donnée la non-normalité du système.

$$\begin{aligned} &\frac{\partial^{h^{(\mu)}-h} F_i}{\partial x^{0^{h^{(\mu)}-h}}} = 0 \qquad (i = 2, 3, \ldots, n), \\ &\Phi_\mu = 0, \end{aligned} \tag{1^\mu}$$

le déterminant des coefficients des $\frac{\partial^{h^{(\mu)}} \varphi_j}{\partial x^{h^{(\mu)}}}$ est nul.

D'autre part d'après (16′) et (17′) et d'après ce qui a été dit, on voit facilement que quand $\mu < l$, comme c'est le cas actuel, les termes pour lesquels Φ_μ diffère de $\frac{\partial^{h^{(\mu)}-h} \Phi_\mu^{**}}{\partial x^{0^{h^{(\mu)}-h}}}$ et qui contiennent les $\frac{\partial^{h^{(\mu)}} \varphi_j}{\partial x^{0^{h^{(\mu)}}}}$, sont tous du type

$$C \frac{\partial^{h^{(\mu)}-h} F_i}{\partial x^{0^{h^{(\mu)}-h}}}$$

où C désigne un coefficient, qui ne contient pas les $\frac{\partial^{h^{(\mu)}} \varphi_j}{\partial x^{0^{h^{(\mu)}}}}$.

Le déterminant formé par les coefficients des

$$\frac{\partial^{h^{(\mu)}} \varphi_j}{\partial x^{0^{h^{(\mu)}}}} \text{ dans } \frac{\partial^{h^{(\mu)}-h} F_2}{\partial x^{0^{h^{(\mu)}-h}}}, \frac{\partial^{h^{(\mu)}-h} F_3}{\partial x^{0^{h^{(\mu)}-h}}}, \ldots, \frac{\partial^{h^{(\mu)}-h} F_n}{\partial x^{0^{h^{(\mu)}-h}}}, \frac{\partial^{h^{(\mu)}} \Phi_\mu^{**}}{\partial x^{0^{h^{(\mu)}-h}}}$$

doit donc être nul.

De cela, comme les coefficients des $\frac{\partial^{h^{(\mu)}} \varphi_j}{\partial x^{0^{h^{(\mu)}}}}$ dans $\frac{\partial^{h^{(\mu)}-h} \Phi_\mu^{**}}{\partial x^{0^{h^{(\mu)}-h}}}$ sont respectivement égaux à ceux des $\frac{\partial^h \varphi_j}{\partial x^{0^h}}$ dans Φ_μ^{**}, il en résulte que le déterminant des coefficients des $\frac{\partial^h \varphi_j}{\partial x^{0^h}}$ dans $F_2, F_3, \ldots, F_n, \Phi_\mu^{**}$ est aussi nul, ce qui est d'accord avec nos intentions.

$\Phi_l^{**} = 0$ constitue ensuite la condition à associer aux équations

$$F_i = 0 \qquad (i = 2, 3, \ldots, n)$$

pour déterminer les $\frac{\partial^h \varphi_j}{\partial x^{0^h}}$ en fonction des données initiales.

En dérivant successivement par rapport à x^0, on tire de $\Phi_l^{**} = 0$ comme on a dit au début du paragraphe, les autres conditions nécessaires pour déterminer les dérivées des φ d'ordre $h+1$, $h+2$, ..., $h^{(l)}-1$.

Nous sommes arrivés ainsi à construire les nouvelles $(h^{(l)}-h)n+l$ conditions pour les φ et leurs dérivées et il est facile de se rendre compte de ce qu'elles sont absolument équivalentes aux anciennes.

On doit observer que le résultat obtenu peut être établi, avec des considérations semblables, également sous des conditions différentes de celle énoncée au commencement du paragraphe précédent.

Cela est vrai, par exemple, et je me contenterai ici de l'affirmer, sous la condition suivante, qui, en quelque sorte, a un caractère moins restrictif que celle dont il a été question jusqu'ici:

Les coefficients des dérivées des φ, d'ordre, par rapport à x^0, égal à h, ne contiennent pas de dérivées d'ordre, par rapport à x^0, supérieur à $h-l$; les coefficients des dérivées, d'ordre, par rapport à x^0, égal à $h-1$, ne contiennent pas de dérivées d'ordre, par rapport à x^0, supérieur à $h-l+1$; et en général les coefficients des dérivées d'ordre, par rapport à x^0, $h-r$, ne contiennent pas de dérivées d'ordre, toujours par rapport à x^0, supérieur à $h-l+r$.

Mathematics. — *On the theory of simultaneous linear and quadratic representation.* Part IV. By F. VAN DER BLIJ. (Communicated by Prof. J. G. VAN DER CORPUT.)

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§ 17. *The systems in five and seven variables.*

Now we consider the system

$$\begin{aligned} x_1^2+x_2^2+\ldots+x_r^2&=n,\\ x_1+x_2+\ldots+x_r&=m. \end{aligned} \qquad r=5,7. \quad . \quad . \quad . \quad . \quad (17.1)$$

The system with $r=5$ and $r=7$ can be treated together, so we consider the system with $r=5$ once more. We will use the formulae deduced from the general theory of SIEGEL [14]. Let $A(\mathbf{Q},\Delta)$ denote the number of solutions of the equation

$$\mathbf{y}'\mathbf{Q}\mathbf{y}=r(y_1^2+y_2^2+\ldots y_{r-1}^2)-(y_1+y_2+\ldots y_{r-1})^2=rn-m^2=\Delta. \quad (17.2)$$

If $m\equiv 0(r)$ the number of solutions of the system (17.1) equals $A(\mathbf{Q},\Delta)$. If $m\not\equiv 0(r)$ it equals $\frac{1}{2}A(\mathbf{Q},\Delta)$.

From the theory of SIEGEL we deduce the number $A_0(\mathbf{Q},\Delta)$ which is the average value of the numbers $A(\mathbf{Q}_i,\Delta)$ if $\mathbf{Q}_i$ runs through a complete system of representatives of all classes of the genius of $\mathbf{Q}$. It can be proved by analytic means under certain conditions that this number (the singular series) gives us an asymptotic value of $A(\mathbf{Q},\Delta)$, too.

We may write

$$A_0(\mathbf{Q},\Delta)=\frac{\pi^{\frac{r-1}{2}}\Delta^{\frac{r-3}{2}}}{\Gamma\left(\frac{r-1}{2}\right)|\mathbf{Q}|^{\frac{1}{2}}}\prod_p \alpha_p(\mathbf{Q},\Delta). \quad . \quad . \quad . \quad (17.3)$$

Where, if $p\nmid 2r$ and $p^l \parallel \Delta$

$$\alpha_p(\mathbf{Q},\Delta)=\left(1-\varepsilon p^{-\frac{r-1}{2}}\right)\left\{1+\varepsilon p^{-\frac{r-3}{2}}+\left(\varepsilon p^{-\frac{r-3}{2}}\right)^2+\ldots+\left(\varepsilon p^{-\frac{r-3}{2}}\right)^l\right\}, \quad (17.4)$$

with

$$\varepsilon=\left(\frac{(-1)^{\frac{r-1}{2}}r}{p}\right).$$

Now we calculate $\alpha_2(\mathbf{Q},\Delta)$ and $\alpha_r(\mathbf{Q},\Delta)$. The number $\alpha_p(\mathbf{Q},\Delta)$ was defined by

$$\alpha_p(\mathbf{Q},\Delta)=p^{-\lambda(r-2)}A_{p^\lambda}(\mathbf{Q},\Delta);$$

where $A_q(\mathbf{Q},\Delta)$ denote the number of solutions of the congruence

$$\mathbf{y}'\mathbf{Q}\mathbf{y}\equiv\Delta \pmod q;$$

and λ is an integer satisfying $\lambda>2\mu$ where $p^\mu \parallel 2|\mathbf{Q}|$.

First we calculate $\alpha_r(\mathbf{Q}, \Delta)$. $\alpha_q(\mathbf{Q}, \Delta)$ being a class invariant we replace the quadratic form $\mathbf{y'Qy}$ by $-y_1^2 - r y_2^2 + r(y_3^2 + \ldots + y_{r-1}^2)$.

These two forms are r-adic equivalent since:

i. The matrix $\mathbf{Q}$ has the r-adic rank 1.

ii. The determinant of $\mathbf{Q}$ equals r^{r-2}.

iii. The form $-\mathbf{y'Qy}$ represents only those numbers which are quadratic residues modulo r. Now we calculate

$$\alpha_r(\mathbf{Q}, \Delta) = r^{-\lambda(r-2)} A_{r^\lambda}(\mathbf{Q}, \Delta).$$

$$= r^{-\lambda(r-1)} \sum_{l \bmod r^\lambda} \sum_{y \bmod r^\lambda} e\left(\frac{l}{r^\lambda}[-y_1^2 + r(-y_2^2 + y_3^2 + \ldots) - \Delta]\right).$$

We replace the sum over l modulo r^λ by a sum over k $(0 \leq k \leq \lambda)$ and a sum over h running through a reduced system of residues modulo r^k $(l = h r^{\lambda-k})$. Thus we find

$$\alpha_r(\mathbf{Q}, \Delta) = 1 + \sum_{k=1}^{\lambda} r^{-k(r-1)} \sum'_{h \bmod r^k} e\left(-\frac{h\Delta}{r^k}\right) \sum_{y \bmod r^k} e\left(\frac{h}{r^k}[-y_1^2 + r(-y_2^2 + y_3^2 + \ldots)]\right).$$

$$= 1 + \sum_{k=1}^{\lambda} r^{-k(r-1)} \sum'_{h \bmod r^k} e\left(-\frac{h\Delta}{r^k}\right)\left(\frac{-h}{r^k}\right) r^{\frac{k}{2}} I_r^{k^2} r^{(k+1)\frac{r-2}{2}} \times$$

$$\times \left(\frac{-1}{r}\right)^{k-1} \left(\frac{h}{r}\right)^{(r-2)(k-1)} I_r^{(r-2)(k-1)^2}.$$

$$= 1 + \sum_{k=1}^{\lambda} r^{-\frac{1}{2}(k-1)(r-1)-\frac{1}{2}} \left(\frac{-1}{r}\right) I_r^{(r-2)(k-1)^2+k^2} \sum'_{h \bmod r^k} \left(\frac{h}{r}\right) e\left(-\frac{h\Delta}{r^k}\right).$$

Now let $r^\beta \parallel \Delta$ and Δ being even we introduce $\Delta = 2 r^\beta N_0$. Thus

$$\alpha_r(\mathbf{Q}, \Delta) = 1 + \left(\frac{2}{r}\right)\left(\frac{N_0}{r}\right) r^{-\beta\frac{r-3}{2}} I_r^{(r-1)\beta^2+2\beta+2}.$$

And if $r = 5$ or 7 we may write this

$$\alpha_r(\mathbf{Q}, \Delta) = 1 - \left(\frac{N_0}{r}\right) r^{-\beta\frac{r-3}{2}} \quad \ldots\ldots \quad (17.5)$$

After this we consider $\alpha_2(\mathbf{Q}, \Delta)$.

We replace $\mathbf{Q}$ by a 2-adic equivalent matrix

$$\begin{bmatrix} \mathbf{G}_1 & \mathbf{N} & & \\ \mathbf{N} & \mathbf{G}_2 & & \\ & & \ddots & \\ & & & \mathbf{G}_{\frac{r-1}{2}} \end{bmatrix},$$

where

$$\mathbf{G}_i = \begin{pmatrix} 2A_i & B_i \\ B_i & 2C_i \end{pmatrix}, \quad A_i \equiv B_i \equiv 1 \pmod 2; \quad 4A_i C_i - B_i^2 = D_i;$$

$$D_1 D_2 \ldots \equiv r^{r-2} \equiv r \pmod 8.$$

(vid. MINKOWSKI [10] pag. 18—25.)

$$a_2(\mathbf{Q},\Delta)=1+\sum_{k=1}^{\lambda}2^{-(r-1)k}\sum_{h \bmod 2^k}{}' e\left(-\frac{h\Delta}{2^k}\right)\prod_{i=1}^{\frac{r-1}{2}}\sum_{x_i \bmod 2^k} e\left(\frac{2h}{2^k}[A_i x_i^2+B_i x_i x_{i,1}+C_i x_{i,1}^2]\right)$$

$$=2+\sum_{k=2}^{\lambda}2^{-\frac{r-1}{2}(k-1)}\sum_{h \bmod 2^k}{}' \left(\frac{2}{r}\right)^{k+1} e\left(-\frac{h\Delta}{2^k}\right).$$

After some elementary calculations we obtain, $(2^{\alpha+1} \parallel \Delta)$,

$$a_2(\mathbf{Q},\Delta)=2\left(1-\left(\frac{2}{r}\right)2^{-\frac{r-1}{2}}\right)\sum_{d \mid 2^\alpha}\left(\frac{d}{r}\right)d^{-\frac{r-3}{2}}. \quad . \quad . \quad (17.6)$$

Introducing the results of (17.5) and (17.6) in (17.3) we get

$$A_0(\mathbf{Q},\Delta)=\frac{\pi^{\frac{r-1}{2}}\,2^{\frac{r-1}{2}}\,N^{\frac{r-3}{2}}\sum_{d \mid N}\left(\frac{d}{r}\right)d^{-\frac{r-3}{2}}}{\Gamma\left(\frac{r-1}{2}\right)r^{\frac{r-2}{2}}\,L\left(\frac{r-1}{2},\chi_r\right)}\left\{1-\left(\frac{N_0}{r}\right)r^{-\frac{1}{2}\beta(r-3)}\right\};$$

where $\Delta=2N$, $r^\beta \parallel N$; $N=r^\beta N_0$ and

$$L\left(\frac{r-1}{2},\chi_r\right)=\sum_{q=1}^{\infty}\left(\frac{q}{r}\right)q^{-\frac{r-1}{2}}.$$

The values of the sums L can be found using the "Hilfssatzen" 9 and 10 of KLOOSTERMAN [7] pag. 331—332.

Theorem 7. *The number of solutions of the system* (17.1) *equals if* $r=5$ *or* $r=7$

$$\tfrac{1}{2}A(\mathbf{Q},\Delta)=\frac{rN^{\frac{1}{2}(r-3)}}{8^{\frac{1}{2}(r-5)}}\sum_{d \mid N}\left(\frac{d}{r}\right)d^{-\frac{r-3}{2}} \quad \text{if} \quad N\not\equiv 0 \pmod r.$$

$$A(\mathbf{Q},\Delta)=\frac{rN^{\frac{1}{2}(r-3)}}{8^{\frac{1}{2}(r-5)}}\left\{1-\left(\frac{N_0}{r}\right)r^{-\frac{1}{2}\beta(r-3)}\right\}\sum_{d \mid N}\left(\frac{d}{r}\right)d^{-\frac{r-3}{2}} \quad \text{if} \quad N\equiv 0\,(r)$$

and

$$N=r^\beta N_0,\ (N_0,\ r)=1.$$

In order to prove this theorem we must prove the relation $A_0(\mathbf{Q},\Delta)=A(\mathbf{Q},\Delta)$. This will be done in § 19, where we calculate the measure of the genius of the quadratic form $\mathbf{y}'\mathbf{Q}\mathbf{y}$. From the results of this calculation we will deduce that the genius of $\mathbf{Q}$ contains one class of matrices only. Then it can be deduced from the definition of $A_0(\mathbf{Q},\Delta)$ that $A_0(\mathbf{Q},\Delta)=A(\mathbf{Q},\Delta)$.

§ 18. *The system in six variables.*

Now we consider the system (17.1) with $r=6$. We replace it by

$$\left.\begin{aligned}\mathbf{y}'\,\mathbf{Q}\,\mathbf{y}=6(y_1^2+\ldots y_5^2)-(y_1+\ldots+y_5)^2=6n-m^2=\Delta,\\ y_1+y_2+\ldots+y_5\equiv m \pmod 6.\end{aligned}\right\} \quad (18.1)$$

Once more we make use of the theory of SIEGEL,

$$A_0(\mathbf{Q}, \Delta) = \frac{\pi^2}{27} \Delta^{\frac{1}{2}} \prod_p \alpha_p(\mathbf{Q}, \Delta) \ldots \ldots \quad (18.2)$$

Now let

$$\Delta = 2^\alpha 3^\beta \Delta_3 \Delta_4 ; \quad 2^\alpha \| \Delta, \ 3^\beta \| \Delta ;$$
$$\Delta_3 = p_1^{2\alpha_{p_1}-1} \ldots \qquad \alpha_{p_i} > 0.$$
$$\Delta_4 = q_1^{2\beta_{q_1}} \ldots \qquad \beta_{q_i} > 0.$$
$$\Delta_4 D = \Delta.$$

The letters A_p, B_q were introduced in § 9 (Part II), here we must take $r = 6$. We get:

$$\left.\begin{aligned} &\text{If } p \nmid 6\Delta : \alpha_p(\mathbf{Q}, \Delta) = (1-p^{-4})\left(1-\left(\frac{D}{p}\right)p^{-2}\right)^{-1}. \\ &\text{If } p \mid \Delta_3 : \alpha_p(\mathbf{Q}, \Delta) = (1-p^{-4})\, A_p. \\ &\text{If } q \mid \Delta_4 : \alpha_q(\mathbf{Q}, \Delta) = \left\{1+\left(\frac{D}{q}\right)q^{-2}\right\} B_q. \end{aligned}\right\} \quad (18.3)$$

Now we must calculate $\alpha_3(\mathbf{Q}, \Delta)$. After replacing $\mathbf{y'Qy}$ by

$$-y_1^2 - 3y_2^2 + 3(y_3^2 + y_4^2 + y_5^2)$$

it may be seen readily that we obtain after some calculations, as it was done in the preceding paragraph, the value of $\alpha_3(\mathbf{Q}, \Delta)$. It becomes

If $3^\beta \| \Delta$ and $\Delta = 3^\beta \Delta_0$ we find

$$\left.\begin{aligned} &\text{if } \beta \text{ is odd} \quad \alpha_3(\mathbf{Q}, \Delta) = \tfrac{10}{13}(1+4.3^{-\frac{1}{2}\beta-\frac{1}{2}}). \\ &\text{if } \beta \text{ is even } \alpha_3(\mathbf{Q}, \Delta) = \begin{cases} \tfrac{10}{13}(1-3^{-\frac{1}{2}\beta})\ldots & \text{if } \Delta_0 \equiv 1 \pmod 3. \\ \tfrac{2}{13}(5+8.3^{-\frac{1}{2}\beta})\ldots & \text{if } \Delta_0 \equiv 2 \pmod 3. \end{cases} \\ &\text{if } \beta = 0 \qquad \alpha_3(\mathbf{Q}, \Delta) = 2. \end{aligned}\right\} \quad (18.4)$$

In order to calculate $\alpha_2(\mathbf{Q}, \Delta)$ we replace $\mathbf{y'Qy}$ by

$$6(2y_1^2 + 2y_1y_2 + 2y_2^2) + 6(2y_3^2 + 10y_3y_4 + 20y_4^2) + 5y_5^2.$$

After some elementary reductions we get, $\Delta = 2^\alpha \Delta_1$,

$$\left.\begin{aligned} &\alpha_2(\mathbf{Q}, \Delta) = \tfrac{15}{7}(1-2^{-\frac{1}{2}\alpha+\frac{1}{2}}) \quad \text{if } \alpha \equiv 1 \pmod 2, \ \alpha \geqslant 3. \\ &\alpha_2(\mathbf{Q}, \Delta) = \tfrac{15}{7}(1-2^{-\frac{1}{2}\alpha}) \quad \text{if } \alpha \equiv 0 \pmod 2, \ \Delta_1 \equiv 3 \pmod 4. \\ &\alpha_2(\mathbf{Q}, \Delta) = \tfrac{5}{14}(6+2^{-\frac{1}{2}\alpha}) \quad \text{if } \alpha \equiv 0 \pmod 2, \ \Delta_1 \equiv 1 \pmod 8. \\ &\alpha_2(\mathbf{Q}, \Delta) = \tfrac{3}{14}(10-3.2^{-\frac{1}{2}\alpha}) \quad \text{if } \alpha \equiv 0 \pmod 2, \ \Delta_1 \equiv 5 \pmod 8. \end{aligned}\right\} \quad (18.5)$$

We introduce these results in (18.2) and we are led to

Theorem 8. *The number of solutions of the system* (17.1) *with* $r = 6$ *equals*

$$\varepsilon_3 A(\mathbf{Q}, \Delta) = \frac{18}{5\pi}\, a_2(\mathbf{Q}, \Delta)\, a_3^*(\mathbf{Q}, \Delta)\, \Delta^{\frac{3}{2}} \prod_{p \mid \Delta_3} A_p \prod_{q \mid \Delta_4} B_q \sum_{\substack{k=1 \\ (k, 6D)=1}}^{\infty} \left(\frac{D}{k}\right) k^{-2}.$$

$a_3^*(\mathbf{Q}, \Delta) = a_3(\mathbf{Q}, \Delta)$ (*vid.* 18.4) *if* $\Delta \equiv 0 \pmod 3$.

$a_3^*(\mathbf{Q}, \Delta) = 1$ *if* $\Delta \not\equiv 0 \pmod 3$.

$a_2(\mathbf{Q}, \Delta)$ (*vid.* 18.5).

In order to complete the proof we shall prove in the next paragraph $A_0(\mathbf{Q}, \Delta) = A(\mathbf{Q}, \Delta)$ where $\mathbf{Q}$ was defined in (18.1).

§ 19. *The measure of some genera of quadratic forms.*

In order to prove that there is only one class in the genius of the matrix $\mathbf{Q}$ defined by (17.2) or by (18.1), we calculate the measure of the genera of these matrices. This number equals in the three considered cases the inverse of the number of automorphs of the matrix $\mathbf{Q}$, that is to say the inverse of the number of matrices $\mathbf{V}$ with determinant unity, satisfying $\mathbf{V}'\mathbf{Q}\mathbf{V} = \mathbf{Q}$.

Then it is proved that the genius of $\mathbf{Q}$ contains one class only. From the definition of $A_0(\mathbf{Q}, \Delta)$ it may then be seen readily that $A_0(\mathbf{Q}, \Delta) = A(\mathbf{Q}, \Delta)$.

The measure of the genius can be calculated using

$$M(\mathbf{Q}) = \frac{2\,\Gamma(\frac{1}{2})\,\Gamma\left(\frac{2}{2}\right) \ldots \Gamma\left(\frac{r-1}{2}\right) |\mathbf{Q}|^{\frac{r}{2}}}{\pi^{\frac{1}{2}r(r-1)} \prod_{p} a_p(\mathbf{Q}, \mathbf{Q})} \quad . \quad . \quad . \quad . \quad (19.1)$$

(Vid. formula 72 pag. 568 SIEGEL [14]). Further we find by SIEGEL the value of those $a_p(\mathbf{Q}, \mathbf{Q})$ for which $(p, 2\,|\,\mathbf{Q}\,|) = 1$.

We now first calculate $a_p(\mathbf{Q}, \mathbf{Q})$ where p denotes the odd prime divising $|\,\mathbf{Q}\,|$. In the field of the p-adic numbers we may replace $\mathbf{y}'\mathbf{Q}\mathbf{y}$ by $-y_1^2 + p(-y_2^2 + y_3^2 + \ldots) = \mathbf{y}'\mathbf{R}\mathbf{y}$ since these forms are equivalent ones. We use the definition

$$a_p(\mathbf{R}, \mathbf{R}) = \tfrac{1}{2}\, q^{-\frac{(r-1)(r-2)}{2}} A_q(\mathbf{R}, \mathbf{R}); \quad q = p^\lambda, \lambda > 2r;$$

where $A_q(\mathbf{R}, \mathbf{R})$ denotes the number of solutions $\mathbf{V}$ modulo q of the congruence $\mathbf{V}'\mathbf{R}\mathbf{V} \equiv \mathbf{R} \pmod q$. We use the formula

$$A_{p^\lambda}(\mathbf{R}, \mathbf{R}) = A_{p^\lambda}(\mathbf{R}, -1) \cdot A_{p^{\lambda-1}}(\mathbf{R}_1, \mathbf{R}_1) \cdot p^{(r-2)\lambda}$$

where $A_{p\lambda}(\mathbf{R}, -1)$ denotes the number of solutions of $\mathbf{z}'\mathbf{R}\mathbf{z} = -1$ and

$$R = \begin{bmatrix} -1 & \mathbf{n} \\ \mathbf{n} & p\mathbf{R}_1 \end{bmatrix}.$$

This formula is a special case of a more general one. The proof of it is simple. Let first $\mathbf{c}$ be a solution of the congruence $\mathbf{z}'\mathbf{R}\mathbf{z} \equiv -1 \pmod{p^\lambda}$. Now we determine a matrix $\mathbf{U}_0$ in such a way that the matrix $(\mathbf{c}\mathbf{U}_0)$ has

determinant unity. If we replace $\mathbf{U}_0$ by $\mathbf{U} = \mathbf{cf}' + \mathbf{U}_0\mathbf{H}$, with arbitrary integral $\mathbf{f}$ we have $|(\mathbf{cU})| = 1$ and it is possible to chose $\mathbf{f}$ in such a way that $\mathbf{c}'\mathbf{RU} \equiv \mathbf{n} \pmod{p^\lambda}$ since

$$\mathbf{c}'\,\mathbf{R}\,\mathbf{U} = \mathbf{c}'\,\mathbf{R}\,(\mathbf{c}\,\mathbf{f}' + \mathbf{U}_0\,\mathbf{H}) =$$
$$= \mathbf{c}'\,\mathbf{R}\,\mathbf{c}\cdot\mathbf{f}' + \mathbf{c}'\,\mathbf{R}\,\mathbf{U}_0\,\mathbf{H} \equiv -\mathbf{f}' + \mathbf{c}'\,\mathbf{R}\,\mathbf{U}_0\,\mathbf{H} \pmod{p^\lambda}.$$

Those solutions $\mathbf{V}_0$ of $\mathbf{V}'\mathbf{RV} \equiv \mathbf{R} \pmod{p^\lambda}$, the first column of which is $\equiv \mathbf{c} \pmod{p^\lambda}$ can be written as $\mathbf{V}_0 = (\mathbf{cU}) \begin{pmatrix} 1 & \mathbf{n}' \\ \mathbf{n} & \mathbf{W} \end{pmatrix}$.

Thus to every $\mathbf{V}_0$ with first column congruent $\mathbf{c}$, there exists a matrix $\mathbf{W}$ satisfying

$$\mathbf{W}'\,\mathbf{R}_2\,\mathbf{W} \equiv \mathbf{R}_2 \pmod{p^\lambda} \quad , \quad \mathbf{R}_2 = \mathbf{U}'\,\mathbf{R}\,\mathbf{U}$$

since

$$\mathbf{V}_0'\,\mathbf{R}\,\mathbf{V}_0 = \begin{pmatrix} 1 & \mathbf{n}' \\ \mathbf{n} & \mathbf{W}' \end{pmatrix} \begin{pmatrix} -1 & \mathbf{n}' \\ \mathbf{n} & \mathbf{U}'\mathbf{R}\mathbf{U} \end{pmatrix} \begin{pmatrix} 1 & \mathbf{n}' \\ \mathbf{n} & \mathbf{W} \end{pmatrix} =$$
$$= \begin{pmatrix} -1 & \mathbf{n}' \\ \mathbf{n} & \mathbf{W}'\mathbf{R}_2\mathbf{W} \end{pmatrix} \equiv \begin{pmatrix} -1 & \mathbf{n}' \\ \mathbf{n} & p\mathbf{R}_1 \end{pmatrix} \pmod{p^\lambda}.$$

The p-adic rank of $\mathbf{R}_2$ is zero, thus $\mathbf{R}_2 = p\mathbf{R}_3$ and $\mathbf{R}_3$ is equivalent with $\mathbf{R}_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. (Vid. SIEGEL [14] Hilfssatz 12).

Now we have

$$\alpha_p(\mathbf{Q},\mathbf{Q}) = \alpha_p(\mathbf{R},\mathbf{R}) = \tfrac{1}{2}p^{-\lambda\frac{(r-1)(r-2)}{2}} A_{p^\lambda}(\mathbf{R},\mathbf{R}).$$
$$= \tfrac{1}{2}p^{-\lambda\frac{(r-1)(r-2)}{2}} A_{p^\lambda}(\mathbf{R},-1)\cdot A_{p^{\lambda-1}}(\mathbf{R}_1,\mathbf{R}_1)\cdot p^{(r-2)^2}.$$
$$= \tfrac{1}{2}p^{-\lambda\frac{(r-1)(r-2)}{2}} A_{p^\lambda}(\mathbf{R},-1)\,2\cdot p^{(\lambda-1)\frac{(r-2)(r-3)}{2}} \alpha_p(\mathbf{R}_1,\mathbf{R}_1)\cdot p^{(r-2)^2}.$$
$$= 2\,p^{\frac{(r-1)(r-2)}{2}} \alpha_p(\mathbf{R}_1,\mathbf{R}_1).$$

If we introduce the values of $\alpha_p(\mathbf{R}_1\,\mathbf{R}_1)$, we obtain

$$\left.\begin{aligned} &\text{If } r=5 \;:\; \alpha_p(\mathbf{Q},\mathbf{Q}) = 2\cdot 5^6\cdot(1-5^{-2}). \\ &\text{If } r=6 \;:\; \alpha_p(\mathbf{Q},\mathbf{Q}) = 2\cdot 3^{10}\cdot(1+3^{-2})(1-3^{-2}). \\ &\text{If } r=7 \;:\; \alpha_p(\mathbf{Q},\mathbf{Q}) = 2\cdot 7^{15}\cdot(1-7^{-2})(1-7^{-4}). \end{aligned}\right\} \quad . \quad . \quad (19.2)$$

The calculation of $\alpha_2(\mathbf{Q},\mathbf{Q})$ is somewhat complicated. We refer to a general formula of MINKOWSKI [11] pag. 181—182. Using this formula we are able to calculate $\alpha_2(\mathbf{Q},\mathbf{Q})$ and we find

$$\left.\begin{aligned} &\text{if } r=5 \;:\; \alpha_2(\mathbf{Q},\mathbf{Q}) = 15. \\ &\text{if } r=6 \;:\; \alpha_2(\mathbf{Q},\mathbf{Q}) = 45\cdot 2^8. \\ &\text{if } r=7 \;:\; \alpha_2(\mathbf{Q},\mathbf{Q}) = \frac{3^2\cdot 5\cdot 7}{2^3}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (19.3)$$

After this we determine the number of integral rational automorphs of $\mathbf{Q}$. We use the *lemme*:

The number of automorphs of the quadratic form $r \sum_{i=1}^{r-1} x_i^2 - \left(\sum_{i=1}^{r-1} x_i \right)^2$ equals the number of those of the couple of forms $\sum_{i=1}^{r} x_i^2$, $\sum_{i=1}^{r} x_i$.

First we determine to every automorph of the couple an automorph of the quadratic form.

If

$$\mathbf{W}'(r\,\mathbf{E}^{(r)} - \mathbf{e}\,\mathbf{e}')\,\mathbf{W} = \begin{pmatrix} r\,\mathbf{E}^{(r-1)} - \mathbf{e}\,\mathbf{e}' & \mathbf{n} \\ \mathbf{n}' & 0 \end{pmatrix},$$

and

$$\begin{aligned} \mathbf{U}'\,\mathbf{E}^{(r)}\,\mathbf{U} &= \mathbf{E}^{(r)}, \\ \mathbf{U}'\,\mathbf{e}^{(r)} &= \mathbf{e}^{(r)}, \end{aligned}$$

we write

$$\mathbf{W}^{-1}\,\mathbf{U}\,\mathbf{W} = \begin{bmatrix} \mathbf{V} & \mathbf{a} \\ \mathbf{b}' & \gamma \end{bmatrix}$$

thus the matrix $\mathbf{V}$ is an automorph of the quadratic form

$$\mathbf{V}'(r\,\mathbf{E}^{(r-1)} - \mathbf{e}^{(r-1)}\,\mathbf{e}^{(r-1)'})\,\mathbf{V} = r\,\mathbf{E}^{(r-1)} - \mathbf{e}\,\mathbf{e}' \quad . \quad . \quad . \quad (19.4)$$

Now we try to determine an automorph of the couple if a matrix $\mathbf{V}$, satisfying (19. 4) is given. We introduce arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$ and element γ, and we write

$$\begin{bmatrix} \mathbf{V} & \mathbf{a} \\ \mathbf{b}' & \gamma \end{bmatrix} = \mathbf{W}^{-1}\,\mathbf{U}\,\mathbf{W}.$$

Now we chose the vectors $\mathbf{a}$ and $\mathbf{b}$ and the number γ in such a way that the matrix $\mathbf{U}$ satisfies the equations

$$\begin{aligned} \mathbf{U}'(r\,\mathbf{E}^{(r)} - \mathbf{e}\,\mathbf{e}')\,\mathbf{U} &= r\,\mathbf{E}^{(r)} - \mathbf{e}\,\mathbf{e}', \\ \mathbf{U}'\,\mathbf{e} &= \mathbf{e}. \end{aligned}$$

Then we have constructed an automorph of the couple.

The elements of $\mathbf{a}$, $\mathbf{b}$ and γ must satisfy the three equations

$$\begin{aligned} \mathbf{a}'(r\,\mathbf{E}^{(r-1)} - \mathbf{e}\,\mathbf{e}') &= \mathbf{n}', \\ \mathbf{V}'\,\mathbf{e}^{(r-1)} + r\,\mathbf{b} &= \mathbf{e}^{(r-1)}, \\ \mathbf{a}'\,\mathbf{e}^{(r-1)} + \gamma\,r &= r. \end{aligned}$$

It is quite easy to prove that these equations have only one solution for $\mathbf{a}$, $\mathbf{b}$ and γ. Thus the lemme is proved.

It may be seen readily that the couple

$$\left.\begin{aligned} x_1^2 + x_2^2 + \ldots + x_r^2 \\ x_1 + x_2 + \ldots + x_r \end{aligned}\right\}$$

has just $r!$ trivial automorphs and no other ones.

Thus the number of the integral rational automorphs of the quadratic form

$$\mathbf{x}'\mathbf{Q}\mathbf{x} = r\sum_{i=1}^{r-1} x_i^2 - \left(\sum_{i=1}^{r-1} x_i\right)^2$$

equals $r!$.

Now we are able to calculate $M(\mathbf{Q})$ with the formula (19. 1). If we use once more the formulae concerning the L-series, we find that if $r = 5, 6$ or 7

$$M(\mathbf{Q}) = \frac{1}{r!}.$$

Thus $M(\mathbf{Q})$ equals the inverse of the number of integral rational automorphs, and it can be proved that the genius of $\mathbf{Q}$ contains one class only. So the proofs of theorem 7 and 8 are now completed.

§ 20. *On a surmise of* P. BRONKHORST.

In this paragraph we shall prove that, contrary to a surmise of BRONKHORST [2] subjoint "stelling" 1, the number of solutions of

$$\left.\begin{aligned}\sum_{i=1}^{9} x_i^2 &= n,\\ \sum_{i=1}^{9} x_i &= m,\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (20.1)$$

cannot be calculated with this theory.

Instead of calculating the measure of the genius of the quadratic form

$$9(y_1^2 + y_2^2 + \ldots + y_8^2) - (y_1 + y_2 + \ldots + y_8)^2 \quad \ldots \quad (20.2)$$

we determine "the number of solutions" of this equation as it is given by the sum of the singular series in a special case. Then we obtain instead of a rational integer, a fractional number. So we conclude that the genius of the quadratic form (20. 2) contains more then one class. Thus the exact number of solutions is not given in this special case by the sum of the singular series.

We consider

$$\sum_{i=1}^{9} x_i^2 = 1, \quad \sum_{i=1}^{9} x_i = 1.$$

Thus we are led to the equation

$$9(y_1^2 + \ldots + y_8^2) - (y_1 + \ldots + y_8)^2 = 8.$$

And we have the formulae

$$A_0(\mathbf{Q}, \Delta) = \frac{\pi^4 \cdot \Delta^3}{\Gamma(4) \cdot 3^7} \prod_p \alpha_p \quad \ldots\ldots \quad (20.3)$$

$$\text{If } p \neq 2, 3 \;:\; \alpha_p = (1 - p^{-4}) \sum_{d \mid p^\alpha} d^{-3}, \quad (p^\alpha \| \Delta).$$

$$\text{If } p = 3 \;:\; \alpha_3 = 2, \quad (\text{if } (\mathring{\Delta}, 3) = 1).$$

$$\text{If } p = 2 \;:\; \alpha_2 = 2(1 - 2^{-4}) \sum_{d \mid 2^\alpha} d^{-3}, \quad (2^{\alpha+1} \| \Delta).$$

If we introduce these results in (20. 3) we obtain for the number of solutions of (20. 1) if $(\Delta, 3) = 1$,

$$B_9(n, m) = \tfrac{1}{2} A_0(Q, \Delta) = \tfrac{1}{9} \sum_{d \mid N} d^3. \quad (\Delta = 2N). \quad . \quad . \quad (20.4)$$

At last we introduce $\Delta = 8$, $N = 4$,

$$B_9(n, m) = \tfrac{1}{9} \cdot 73.$$

And the number of solutions of the system (20. 1) with $n = m = 1$ equals 9, as it can be seen without trouble.

Geology. — *On linnaeite in the Flaad nickel ore deposit, Evje, South Norway.* By L. P. G. KONING. (Communicated by Prof. H. A. BROUWER.)

(Communicated at the meeting of February 22, 1947.)

Introduction.

The nickeliferous pyrrhotite deposits in gabbro near Evje in Sätersdalen, South Norway, contain mainly sulphides of Fe, Ni and Cu and oxydes of Fe and Ti as ore minerals. Two rock specimens among a collection from the Flaad nickel mine in this district — the largest of its kind in Norway —, which was presented to the Geological Institute of the University of Amsterdam during a geological trip through the Kristiansand area in the summer of 1939, revealed the development of the rather uncommon Co-Ni-Fe-mineral linnaeite in a gabbroic rock with an extraordinary concentration of magnetite, haematite and ilmenite. A short description is therefore given in this paper of the paragenetic relationship of this linnaeite, a mineral, which is more generally found in hydrothermal Co-Ni-ores of the arsenidic type. A couple of other ore samples from the same occurrence were examined for the purpose of comparison.

Geology and mineralogy of the Flaad nickel ore bodies [1]).

The Flaad nickel ore bodies occur along the periphery of an intrusive mass of gabbroic and noritic rocks, which are surrounded by older Archaeic gneisses and migmatitic granites. The ore deposits are probably also of Archaeic age. They are genetically related to, and even form part of the basic intrusiva. The original gabbro in the Flaad district has been uralitized and is called a gabbrodiorite (also uralite-gabbro or uralite-norite). In consequence of the uralitization process the nature of the original pyroxene has been obscured.

The nickeliferous pyrrhotite ore bodies preferably occur along the rims

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[1]) F. BEYSCHLAG, P. KRUSCH und J. H. L. VOGT, Die Lagerstätten der nutzbaren Mineralien und Gesteine, Bd. I, I, 318—324 (1914).

J. H. L. VOGT, Die Bildung von Erzlagerstätten durch Differentiationsprozesse in basischen Eruptivmagmata, Zt. f. prakt. Geol., 4—11, 125—143, 257—284 (1893). — Nickel in igneous rocks, Econ. Geol. **23**, 307—353 (1923).

The article of SMITH and S. MEYER, Flaatgrube, South Norway, Tidskrift för Kjemi og Bergwesen (1932), has not been accessible to the writer.

The writer wishes to thank Prof. Dr. J. WESTERVELD for putting the rock samples, which furnished the material for this study, at his disposal, and for his kind interest in this paper.

of the gabbro mass near its contact with the surrounding rocks, and only rarely in the central portion of the eruptive body. A few ore bodies have also been found in the surrounding rocks in the vicinity of the contacts. By their mode of occurrence most of the ore concentrations may be considered as marginal facies of the gabbroic rock and all gradations between pure gabbro and pure ore can be observed. At some places ore pockets may be found in contact with fragments of included portions of wall rock in the interior of the gabbro mass.

The Flaad ore deposits consist of at least three shoots, each with a thickness of 3—5 m, and dipping about 40°. These bodies consist of pyrrhotite-gabbrodiorite with local concentrations of pure sulphide ore. The ores are cut by veins of granite-aplite and granite-pegmatite, which might be acid differentiates of the marginal facies of the gabbro magma intimately connected with the liquid-magmatic segregation of the pyrrhotite ores.

Under the microscope the investigated samples appeared to be gabbroic rocks containing apatite, oxydic iron ores, hypersthene, biotite, hornblende (= green, uralitized pyroxene), labradorite, and sulphides. Apatite is one of the major constituents of the rock and its local enrichment in a strongly differentiated zone of the mass reminds of the concentrations of apatite in basic pegmatites near Ødegarden and Kragerø in the Bamle region in South Norway.

The investigated linnaeite-bearing ore samples are interesting examples of a combination of various liquid-magmatic ore minerals — which seperately are known to form important ore bodies in other parts of Norway — within one single rock specimen. The oxydes are known to be concentrated on an enormous scale in the Eggersund ilmenite-magnetite deposits in anorthositic norite; the Cu-Ni- and Fe-sulphides in many nickel-copper ore deposits in gabbro and norite; apatite, finally in pegmatitic phosphate deposits already referred to. In order of their relative abundance the ore minerals, apatite excluded, are: haematite, ilmenite, pyrrhotite, magnetite, chalcopyrite, pentlandite, pyrite and linnaeite.

Pyrite belongs to the first crystallized ore minerals, together with haematite, ilmenite and magnetite. Linnaeite, pyrrhotite and the other sulphides are later. In many cases felspar crystals can be seen surrounded by pyrrhotite or cut by small pyrrhotite veinlets. The silicate gangue minerals all seem to have been formed earlier than the ore minerals.

Microscopic description of ore specimens.

A number of ore specimens from the Flaad mine were subjected to microscopical examination. They comprise of three types of ore, the linnaeite-bearing specimens included: 1° massive pyrrhotite — pentlandite — chalcopyrite ore; 2° massive haematite — ilmenite — magnetite — pyrrhotite — linnaeite ore; 3° uralite-gabbro impregnated with ore minerals.

1. Massive pyrrhotite — pentlandite — chalcopyrite ore.

Pyrrhotite. The predominant sulphide ore is the rather highly reflecting, reddish brown and very strongly anisotropic pyrrhotite. In comparison with linnaeite in the second type of ore the colour seems to be yellowish. The very irregular grains in the polished sections studied have in general no crystal outlines. In some individuals a lamellar texture of darker and lighter bands, frequently observed in pyrrhotite, is to be seen. R. W. VAN DER VEEN [2]) supposed this texture to represent a transformation, perhaps a separation by unmixing in the solid state, or an allotropy, whereas H. SCHNEIDERHÖHN and P. RAMDOHR [3]) merely suggested unmixing in the case of this still little understood phenomenon.

Pentlandite. This mineral occurs as minute blades or feather-shaped masses in pyrrhotite, starting from cleavage planes or cracks within this host mineral (fig. 1), or as veinlets between pyrrhotite crystals (fig. 2).

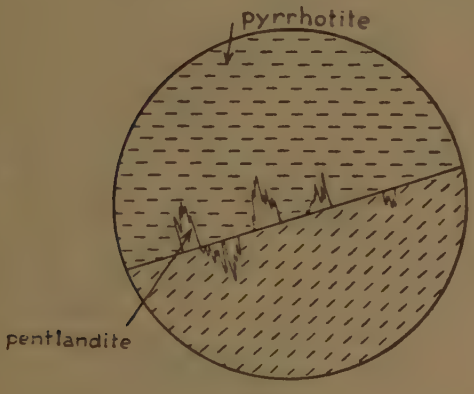

Fig. 1.

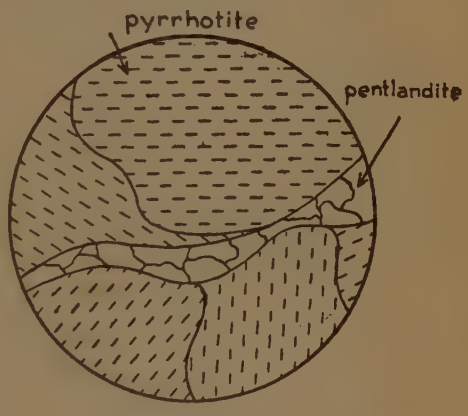

Fig. 2.

Compared with the adjacent pyrrhotite individuals the pentlandite of the veinlets is fine-grained and often shows cleavage along (111). The relationships between pyrrhotite and pentlandite in the ore specimens described here are the same as in other nickel-bearing pyrrhotite deposits (Sudbury, E. Griqualand and Pondoland in S. Africa, Petsamo region in Finland, etc.).

According to R. L. HEWITT [4]), who based his conclusions on experiments and descriptions of various other observers, pentlandite may be caused to unmix from pyrrhotite by slow cooling from 800° downwards.

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[2]) R. W. VAN DER VEEN, Mineragraphy and ore deposition, **49**, and figs. 53, 54 (1925).

[3]) H. SCHNEIDERHÖHN und P. RAMDOHR, Lehrbuch der Erzmikroskopie, 133—137, and fig. 58 (1931).

[4]) R. L. HEWITT, Experiments bearing on the relation of pyrrhotite to other sulphides, Econ. Geol., **23**, 305—338 (1938).

The unmixed pentlandite then segregates around the grain boundaries of the pyrrhotite. This kind of origin may have been responsible for part of the pentlandite in the veinlets, whilst another part may have been produced by crystallization from immiscible residuary liquids. Small pentlandite grains around pyrrhotite grain boundaries and feather-shaped pentlandite lenses within the pyrrhotite grains are to be regarded, probably in both instances, as the product of unmixing in the solid state.

Chalcopyrite. Chalcopyrite occurs as veinlets and grains in the pyrrhotite without distinct crystal outlines. It is distinctly younger than pyrrhotite and pentlandite and often replaces the pentlandite (fig. 3).

Haematite. This colourless to bluish, anisotropic, and strongly reflecting mineral is one of the most predominating constituents of the ores studied. It occurs as needle-shaped crystals often twinned on $(10\bar{1}1)$ (fig. 4). Hae-

Fig. 3.

Fig. 4.

matite is distinctly younger than magnetite, as is shown in fig. 7, where magnetite is seen in course of replacement by haematite.

Ilmenite. Ilmenite is present as rounded grains and lenses together with haematite. It shows strong anisotropy and reflection-pleochroism. In all titanic iron ores, in which both haematite and ilmenite are developed, these minerals are intimately intergrown. Lenses of haematite appear within the ilmenite grains, but also lenses of ilmenite within the haematite individuals (fig. 5).

Small inclusions of ilmenite are frequently developed with a distinctly regular orientation within haematite lenses, which lie intercalated between larger portions of ilmenite, while flakes of haematite, on the other hand, can often be observed in coarser parties of ilmenite.

RAMDOHR [5]) has given the following opinion on the mutual relations between Fe_2O_3 and $FeTiO_3$: at a temperature of about 700° Fe_2O_3 and $FeTiO_3$ are isomorphic and miscible in all proportions in the solid state. When the temperature decreases a discontinuity in the miscibility may occur. As a consequence of this discontinuity the original substance will separate into haematite containing a small quantity of ilmenite, and ilmenite

Fig. 5.

containing some haematite. By a further decrease of the mutual miscibilities of the two minerals a second generation of ilmenite blades will separate from the first generation of haematite crystals, and a similar, but reserve phenomenon will also affect the first segregations of ilmenite. The temperature, at which this second process takes place, may be about 550—600°.

In general ilmenite is younger than magnetite.

Magnetite. This isotropic, light brownish-gray mineral occurs as large individuals mostly showing some idiomorphy in its relation to haematite, pyrrhotite, and chalcopyrite. Sometimes magnetite seems to be younger than haematite. Many small, irregularly shaped particles of chalcopyrite and pyrrhotite are widely distributed in the larger magnetite crystals (figs. 7 and 8).

Pyrite. This very hard, isotropic, light yellow, and strongly reflecting mineral occurs as small grains, mostly surrounded by pyrrhotite and chalcopyrite. A tendency towards idiomorphy may be observed. In general pyrite is replaced by pyrrhotite and chalcopyrite.

2. Massive haematite — ilmenite — magnetite — pyrrhotite — linnaeite ore.

The polished sections of this group (two specimens) show haematite,

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[5]) P. RAMDOHR, Beobachtungen an Magnetit, Ilmenit, Eisenglanz und Ueberlegungen über das System FeO—Fe_2O_3—TiO_2. N. J. f. Min. Beil. Bd. 54, A, 320—379 (1926). — Beobachtungen an opaken Erzen, Archiv f. Lagerstättenforschung, Heft 34, 30 (1924).

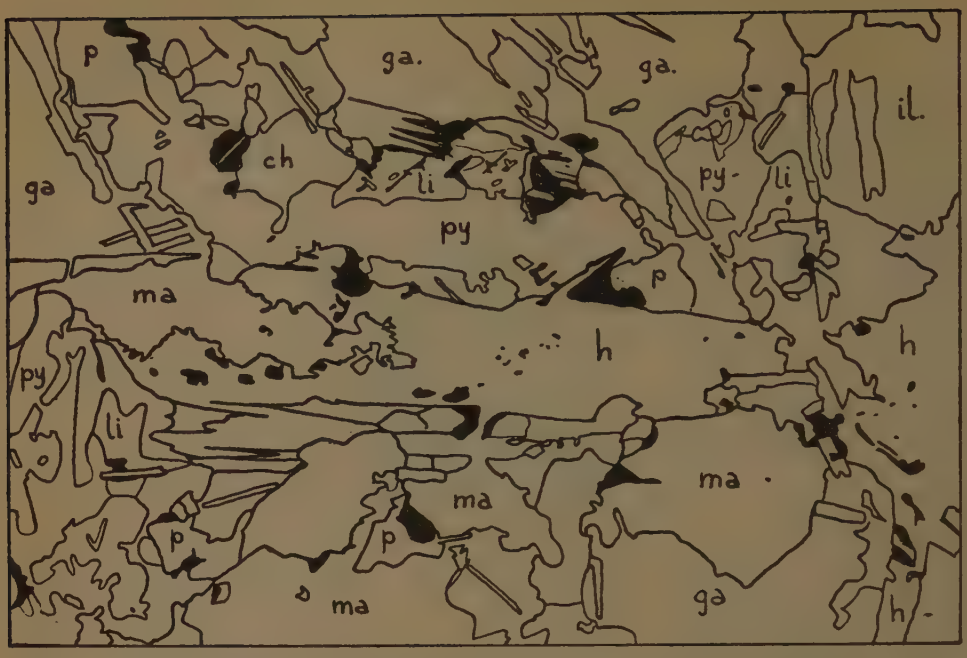

Fig. 7. Ilmenite-magnetite-haematite-apatite-segregation in gabbroic rock, Flaad nickel mine, Evje, S. Norway. Magnetite (*ma*), replaced by haematite (*h*) and ilmenite (*il*). Oxydes again replaced by pyrite (*p*), linnaeite (*li*), pyrrhotite (*py*), and chalcopyrite (*ch*). Magnification × ± 110.

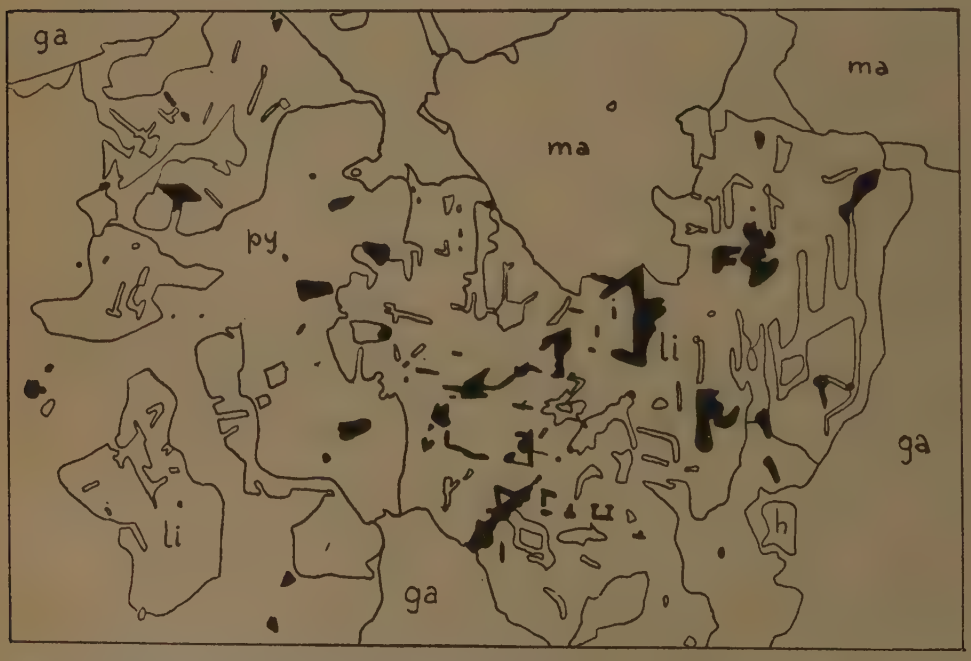

Fig. 8. Ilmenite-magnetite-haematite-apatite-segregation in gabbroic rock, Flaad nickel mine, Evje, S. Norway. Magnetite (*ma*), replaced by linnaeite (*li*) and pyrrhotite (*py*). Magnification × ± 150.

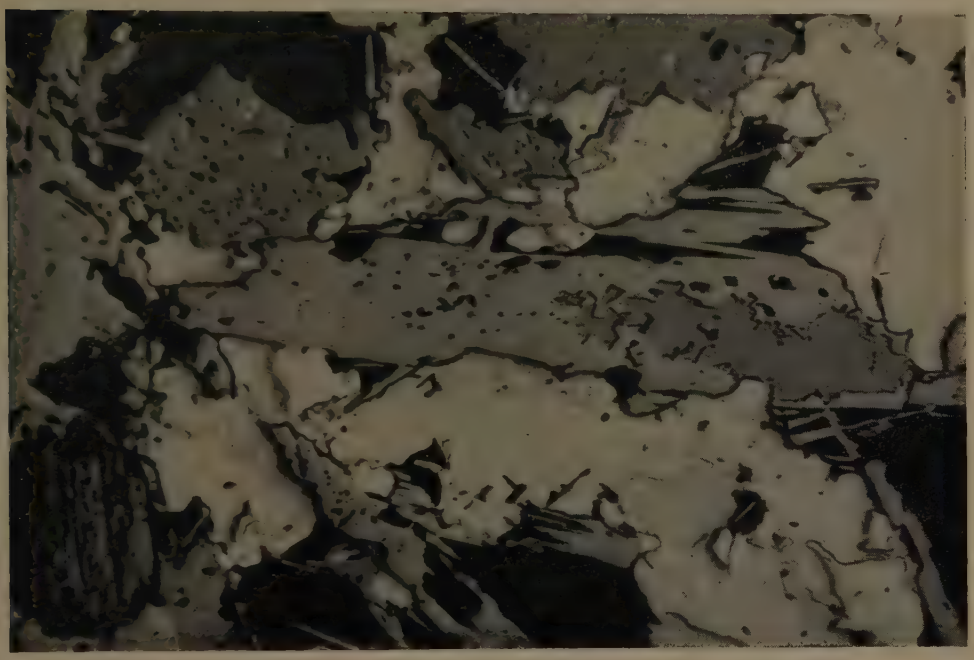

Fig. 7.

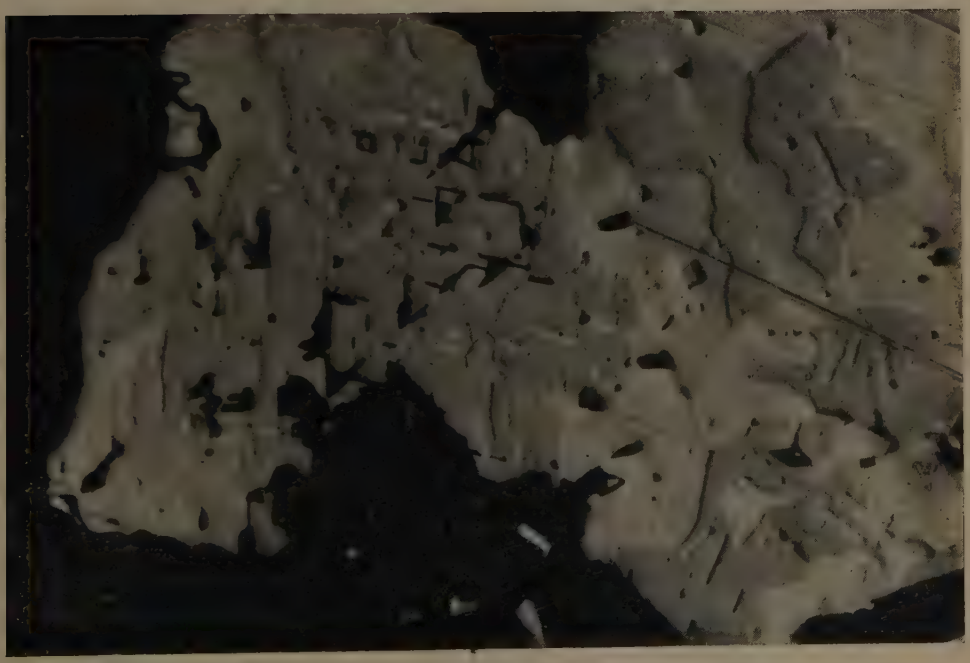

Fig. 8.

ilmenite, and magnetite, already described in the foregoing pages, as principal ore minerals. Pyrrhotite, pentlandite, linnaeite, pyrite, and chalcopyrite occur as minor constituents.

Linnaeite. This mineral occurs as a white to light reddish-cream component of the sulphide portions of the ore minerals. Its hardness is higher than that of pyrrhotite, but lower than that of pyrite. The mineral shows a very strong tendency towards idiomorphy (111), accompanied by a distinct cleavage along (100) (fig. 6). Many inclusions of chalcopyrite,

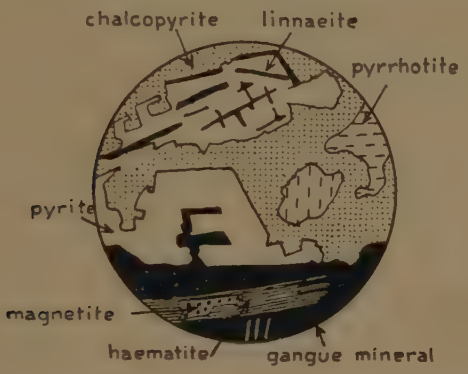

Fig. 6.

pyrrhotite and gangue minerals give the crystals a sieve-like structure. Replacement by these later sulphides proceeds along the cleavage planes (fig. 8).

In several ore deposits of the nickeliferous pyrrhotite sequence (Sudbury district in Ontario, Key West mine in Nevada) the minerals of the linnaeite group (polydymite, violarite, sychnodymite, carrolite) have been found.

For the polydymite occurring in ores of the Levack mine, Sudbury, A. WANDKE and R. HOFFMAN [6]) are of the opinion that this mineral has been formed from supergene solutions. The polydymite from the Vermilion mine is, according to C. F. TOLMAN and A. F. ROGERS [7]), probably due to the breaking down of pentlandite. W. LINDGREN and W. M. DAVY [8]) came to the conclusion that the so-called polydymite of the Sudbury district and the Key West mine in Nevada has resulted from secondary alteration of pentlandite by supergene waters.

There remains, however, no doubt that the mineral described in this paper as linnaeite, is a primary constituent of the ore.

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[6]) A. WANDKE and R. HOFFMAN, A study of the Sudbury ore deposits, Econ. Geol., **19**, 169—204 (1924).

[7]) C. F. TOLMAN and A. F. ROGERS, A study of the magmatic sulphide ores, Leland Stanford Univ. Publ., Stanford University, **33** (1916).

[8]) W. LINDGREN and W. M. DAVY, Nickel ores from Key West mine, Nevada, Econ. Geol., **19**, 309—319 (1924).

3. Uralite-gabbro, impregnated with ore minerals.

In polished sections this type of rock shows the same ore minerals, except linnaeite, as in the above-mentioned kinds of ore. Fe-Ti- and Fe-oxydes are the most important ore minerals, while the greater part of the rock consists of silicates.

Amsterdam, Geological Institute of the University.

Zoology. — *Some investigations on the embryonic development of Limnaea stagnalis L.* By ELSE ARENDSEN DE WOLFF-EXALTO. (From the Zoological Laboratory, University of Utrecht.) (Communicated by Prof. CHR. P. RAVEN.)

(Communicated at the meeting of February 22, 1947.)

I. *The influence of NaSCN on the development of Limnaea stagnalis L.*

In 1892/93 HERBST discovered that the vegetative differentiations of the sea urchin egg extend under the influence of the lithium-ion at the expense of the animal differentiations (ectoderm, cilia). This was confirmed by many other investigations. On the other hand, an extension of the animal differentiations at the expense of the vegetative differentiations is brought about by the action of NaSCN. According to RUNNSTRÖM, the development of the sea urchin eggs is controlled by the interaction of an animal and a vegetative gradient. Whereas Li exerts a depressing influence on the animal gradient, NaSCN, on the other hand, has the opposite effect. When he treated (1928) sea urchin eggs with a mixture, consisting of 80 parts of seawater and 20 parts of an isotonic solution of NaSCN, the influence-zone of the animal factors increased; in extreme cases the animal differentiations only developed. According to RANZI and TAMINI (1939, 1940; TAMINI 1941; RANZI 1942), the action of NaSCN is also in Amphibians opposite to that of lithium. As also with eggs of *Limnaea stagnalis* L. certain characteristic malformations have been obtained, indicating a suppression of the most animal differentiations under the influence of lithium (RAVEN 1942), it was only natural to investigate whether this material would, under the influence of NaSCN, show a reaction contrary to the one caused by Li.

The next pages show the results of this investigation.

Material and methods.

Egg-masses were obtained by stimulation of the snails with *Hydrocharis* in the manner described by RAVEN and BRETSCHNEIDER (1942).

In the beginning the eggs were freed of the surrounding jelly and, still enclosed in the egg-capsules, put into NaSCN-solution. Egg-capsules of the same egg-mass were cultivated in tap water as a control. However, this method of culture proved to be detrimental to the controls, so that often their development lagged behind. If, on the other hand, the eggs had been in a 0.1 % NaSCN-solution during some time (f.i. 24 hours), and were then put into tap water, they developed much better. Moreover, it proved to be much better to leave them in their jelly-coat, so that later the eggs were cultivated in the jelly.

At first, three different NaSCN-solutions were tried out, 1 %, 0.5 % and 0.1 % solutions. The first two solutions appeared to be too harmful, but the 0.1 % solution could be used.

Results of the experiments.

Under the influence of NaSCN two effects can appear:

1. An abnormal balloonlike swelling of a part or of the whole embryo, together with a deformation and dislocation of several organs.

2. A dark wine-red colouring especially in the foot, mouth region, eyes, mantle fold and opening, oesophagus, stomach and nephridial canal.

An extensive description of both effects may be given first.

1. The swelling is caused by the SCN'-ion [1]), by a disturbance in the water regulation. Much fluid is taken in and the embryo swells. This swelling can be more or less extensive depending on the circumstances. When it is slight, it often first appears above the head or in the foot. Different deformations may appear. In extreme cases the embryo becomes a large, transparent balloon, which fills the egg-capsule almost completely and enlarges it to two or three times its normal size. A definite division between the head and the foot does no more exist, but a line or groove runs over the balloon under the mouth. In most cases the organs are present in their normal forms and sizes, though often dislocated. As a rule, the following organs can be distinguished: pharynx, oesophagus, stomach, gut, albumen cells, nephridial canal, eyes and nerve system; the heart cannot always be seen; when present, its contractions are very irregular. In a few cases, a sort of dent in the abdominal sac could be seen, as a result of which the shell went to the inside and several organs bulged strongly to the outside. At the same time, the shell might show abnormalities.

Efforts were made to fix these swollen embryos. When removed from their capsules, however, they burst in consequence of the very high tension inside and shrink to normal size.

It is no use to fix the embryo within the capsule, because the albumen of the capsule-fluid becomes opaque and brittle. In the shrunken embryos no abnormalities of the organs were observed. Enlargement of the head, lateral displacement of the eyes, and other phenomena expected in connection with the supposed NaSCN-action, contrary to that of Li, could not be found. The preparations were drawn on scale by means of a drawing-prism and measured.

Before discussing the stages in which the swelling shows itself, first the scheme of the different phases of development at an average temperature of 20° may be given.

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[1]) The outer protoplasmic layer has the properties of a gel. $H^{\cdot}$ and OH' increase the swelling of a gel considerably, while the anions influence the process in accordance with the lyotrope series

SO_4''	CL'	Br'	NO_3'	J'	SCN'
decrease		the swelling			increase

Number of days:

0 days:	egg	8 days:	early "veliger"
1 day :	blastula	9 „ :	"veliger"
2 days:	gastrula	10 „ :	late "veliger"
3 „ :	very early trochophore	11 „ :	very young snail
4 „ :	early trochophore	12 „ :	young snail
5 „ :	trochophore	13 „ :	snail
6 „ :	late trochophore	14 „ :	snail at hatching.
7 „ :	very early "veliger" [2])		

It was found that if the developing embryo was left in the 0.1 % NaSCN-solution till at least the trochophore-stage, a swelling appeared. If the embryos stay in the solution till the late "veliger"-stage, and are then put into tap water, the already existing swelling can disappear again. At a later stage it does not disappear anymore. If the embryos are put into the solution at the "veliger"-stage (or later), no swelling appears, but if they are brought in before that stage it does appear. The disturbance in the osmotic regulation comes into effect between the stages of trochophore and very late "veliger", and cannot be suspended anymore later.

2. *Colouring.* When the embryo develops in a NaSCN-solution, at a certain moment a wine-red colour appears at special places. At first, the eyes show themselves as red circles and at the same time two red bands appear underneath the head, and the mantle fold and foot get a red colour. The black pigmentation of the foot and body continues normally. In later stages the colour of the eyes gets darker, the tentacles, the mantle fold, oesophagus and sometimes the nephridial canal become pink and the foot and mouth region red. This colour changes later into purple and dark-brown and at hatching it has almost disappeared. The stomach takes an orange-red colour, which stays till the end.

The moment at which the colour becomes visible is, for a normally developing embryo, the stage of "veliger". For the abnormal swollen embryos it begins in the same phase of development. The coloured embryos do not hatch, if they have stayed in the solution till the young snail stage. When the embryos are put into the solution at the stage of a very late "veliger", or later, no colouring appears but they do not hatch either. Brought in as a young snail or later they develop normally.

About this colouring the following can still be reported. If the coloured embryos are taken out of the solution and further cultivated in water, the red colour disappears. So the substance causing the colour is presumably a product of the metabolism, which product cannot exist without new supply of NaSCN. Which is the chemical composition of this substance

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[2]) As the velum of the fresh water Pulmonates remains rudimentary, this denomination of "veliger" for the stages following the trochophore is only used for the sake of brevity.

could not be discovered. It comes to existence under influence of the SCN'-ion; in a KSCN-solution the red colour appears in the same way.

It was only natural to attribute the red colour to the formation of the red $Fe(SCN)_3$ [3]). Therefore a study was made of the presence of free iron during embryonic development. For the method and a detailed description of the results, see part II. It may be stated here that no parallel could be found between the appearance of the red colour and the places where, during the development, iron was found.

II. *The presence of free iron in the embryos of Limnaea stagnalis during their development.*

Method.

To investigate the presence of iron in the several stages of development the method described by HIRSCH and BRETSCHNEIDER (1937) was used. This method is based on the Prussian blue method after TARTAKOWSKY. In this manner the masked iron is not detected; therefore the following details concern free iron only.

Beginning with the newly laid uncleaved egg, the embryo was tested with intervals of one day during the whole of its development.

Results.

a. *The uncleaved egg* (fig. 1) [4]).

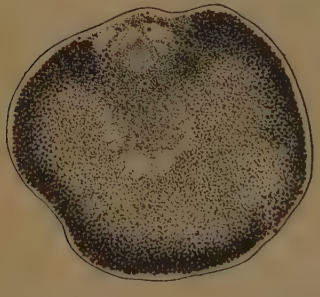

Fig. 1. Uncleaved egg of *Limnaea stagnalis*. Iron reaction.

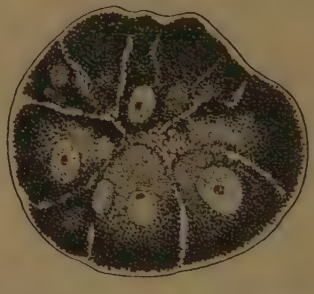

Fig. 2. Blastula. Iron reaction.

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[3]) NaSCN with a ferri-salt solution gives a blood-red colour

$$FeCL_3 + 3NaSCN \rightleftarrows 3NaCL + \underset{\text{red}}{Fe(SCN)_3}.$$

[4]) A determination of the amount of iron in newly laid egg-masses, executed under the direction of Dr. VONK at the Laboratory of comparative physiology of the University of Utrecht, gave the following results:

6,6 mgr. of ashes contain 0.06 mgr. Fe. The amount of ashes is 1.13 ‰ of the fresh weight and 3.64 % of the dry weight.

Therefore the amount of iron in a newly laid egg mass is:

$0.91\ \% \times 1.13\ ‰ = 0.0103\ ‰$ of the fresh weight, and

$0.91\ \% \times 3.64\ \% = 0.33\ ‰$ of the dry weight.

If one applies the iron reaction to newly laid eggs, one finds that they take a dark-blue colour instead of the original yellow one [5]). (The capsule fluid also takes this blue colour, which points to the presence of iron in this fluid too.) The iron proves to be bound to the β-granules, especially in the subcortical protoplasm, which covers the vegetative and equatorial sides of the egg (cf. RAVEN and BRETSCHNEIDER 1942; RAVEN 1945). The aster figure only is free of iron. In the granules the iron is laid down in several forms (fig. 3); in general, especially the outer layer of the granules is coloured.

Fig. 3. Localisation of iron reaction in the β-granules.

b. *The blastula and gastrula,* resp. 1 and 2 days old (fig. 2, fig. 4). At the blastula stage the small blastomeres at the animal pole are almost full of iron. The larger blastomeres at the vegetative pole also contain a large amount of iron granules, which, however, are limited to the ectoplasmic part of the cells. The nuclei are in the region which is free of iron. The nucleoli have a faint blue colour, which indicates the presence of iron.

The gastrula, which appears the next day, gives the same picture. The iron granules are found especially in the ectoplasm of the ectoderm and entoderm cells. However, the large mesoderm cells, which are visible now on either side of the archenteron, prove to be very rich in iron too.

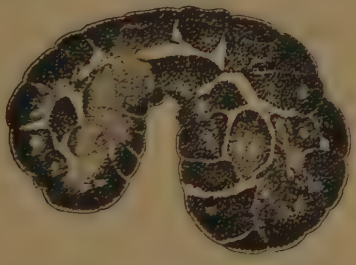

Fig. 4. Gastrula. Iron reaction.

c. During *the trochophore-stage* (3d till 7th day), the first distinct differentiation in the distribution of iron takes place, which differentiation proceeds during further development. The early trochophore still contains much iron, which, however, is concentrated in the mesoderm cells and the big cells of the ectoderm. In the course of further development, the large ciliary cells which are found in the mouth region, in the prototroch, apical plate and head-vesicle, keep their high concentration of iron just like the giant excretory cells of the protonephridium and the ciliary cells in pharynx and oesophagus. The cells of the shell gland contain iron like those of the

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5) BRETSCHNEIDER, using the same method to investigate the presence of iron in the gonads of *Limnaea stagnalis.* L., observed that in the beginning no iron can be shown in the young egg cells. During further growth of the egg, iron granules appear and the ripe eggs in the ovary are crowded with iron granules (personal communication).

entoderm, from which, however, the iron disappears gradually. The same process, though quicker, takes place in the small ectodermal cells.

From the albumen cells which, in the beginning, contain a large amount of iron, the iron granules withdraw quickly and at last can be found only along the cell walls and in local accumulations, forming half moons and triangles between the cells (fig. 5). The granules become smaller and stain less strongly.

Only at the early trochophore stage the nucleoli still contain iron.

Fig. 5. Albumen cells of trochophore stage. Iron reaction.

d. At the following stages of "*veliger*" (7th till 11th day), the concentration of free iron is still high in the ciliary cells around the mouth, of the velum, apical plate, head vesicle, pharynx (fig. 6) and oesophagus, and the giant excretory cells of the protonephridium.

Fig. 6. Ciliary cells in the roof of the pharynx. Iron reaction.

Between the albumen cells of the gut here and there some small groups of iron granules are present. From all other cells the iron has disappeared (fig. 7).

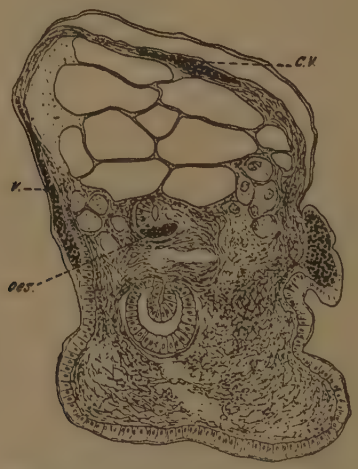

Fig. 7. "Veliger". Iron reaction localized in velum (*v*), head vesicle (*c.v.*) and ciliary cells of oesophagus (*oes*).

e. During the last days of development, ciliary cells containing iron are still found in the mouth region (11 days). Later on a small remainder of the head vesicle in the bend between head and mantle fold, and the few cells to which the velum is reduced, are the only places were iron is present. All other tissues are ironless.

At the moment of hatching no free iron can be found in the snail any more.

Discussion.

As has been stated before, a wine-red colouring of the embryo appears at special places under certain conditions after a treatment with NaSCN. A KSCN-solution has the same effect and, as no red colour is seen in embryos cultivated in other $Na^{\cdot}$- or $K^{\cdot}$-salt solutions, one may suppose that this colour is based on a chemical compound which is formed by (or under the influence of) the SCN'-ion. The composition of this compound is still unknown. As, without fresh supply of NaSCN, the red colour disappears from the embryo, the substance causing the colour may be a product of the metabolic processes.

At first the formation of $Fe(SCN)_3$ was supposed, especially as the Prussian blue reaction could be applied successfully to the eggs, which proves the presence of free iron.

So a research was made for the presence of iron in the different stages of development, with the following results:

The uncleaved egg contains a large amount of iron, which is concentrated in the β-granules. During development the even distribution of iron is lost: the amount of free iron decreases and it is limited to a few special cells. At the early trochophore stage, the large ectodermal cells and the giant excretory cells of the protonephridium show the largest concentration of free iron. The entoderm contains somewhat less, the small ectodermal cells still less iron. The nucleoplasm is free of iron, but the nucleoli show in the first stages a distinct blue colour. In the late trochophore the differential distribution has become more conspicuous. The greatest concentration of free iron is found in the large ciliated ectoderm cells and the protonephridium, somewhat less in the stomodaeum and the entoderm, still less in the small ectoderm cells.

In an early "veliger" stage (8 days old) the large ciliary cells of the velum, apical plate, and head vesicle, around the mouth, in the pharynx and the oesophagus contain a large quantity of iron granules; almost all other cells have become free of iron.

The last traces of free iron are found in a 12 day embryo in some cells of the neck region (remains of the head vesicle).

At the moment that the above mentioned red colour becomes visible, which is at the stage of "veliger", the iron is only concentrated in some velum cells, the head vesicle, and a few other cells. There is no iron in the

foot and the eyes, where the red colour is very strong. If the colouring were caused by a simple Fe-combination, then the places where free iron is present ought to be coloured in the first place. This was not the case.

Summary.

1. In eggs of *Limnaea stagnalis L.*, cultivated in a 0.1 % solution of NaSCN, two effects become visible: *swelling* and *colouring*.
2. The appearance of the colouring and the swelling depend on the moment at which the embryos are put in, and taken out of, the solution.
3. The swelling is due to osmotic disturbances; no changes in the structure of the embryo, indicating an influence on the determination process, could be observed.
4. The substance causing the colour participates in the metabolic processes; without new supply of NaSCN the colour disappears.
5. The substance will be formed under the influence of the SCN'-ion and does not consist of the simple formation of $Fe(SCN)_3$.
6. During the embryonic development of *Limnaea stagnalis L.* a decrease of the amount of free iron is found.
7. The iron is present in the β-granules.
8. In the newly laid egg, the iron is especially accumulated in the subcortical protoplasm; at later stages, it is found in the ectoplasmic parts of the cells.
9. In the course of development, it appears that only the cells with strong functional metabolic processes keep their iron.
   These cells are:
   a. the ciliated cells around the mouth.
   b. „ „ „ in the pharynx and the oesophagus.
   c. „ „ „ of the prototroch (velum).
   d. „ „ „ apical plate and the head vesicle.
   e. the protonephridium.
10. From all other tissues the iron granules disappear.

LITERATURE.

HERBST, C.: Z. Wiss. Zool. **55**, 446 (1892).
HIRSCH, G. C. & L. H. BRETSCHNEIDER: Protopl. **29**, 9 (1937).
LINDAHL, P. E.: Arch. Entw. Mech. **140**, 172 (1940).
————: Naturwiss. **29**, 672, (1941).
————: Protopl. **36**, 558 (1942).
RANZI, S.: Naturwiss. **30**, 329 (1942).
RANZI, S. & E. TAMINI: Naturwiss. **27**, 566 (1939).
————: Naturwiss. **28**, 458 (1940).
RAVEN, CHR. P.: Proc. Ned. Akad. v. Wetensch., Amsterdam, **45**, 856 (1942).
————: Arch. Neerl. Zool. **7**, 91 (1945).
————: Arch. Neerl. Zool. **7**, 353 (1946).
RAVEN, CHR. P. & L. H. BRETSCHNEIDER: Arch. Neerl. Zool. **6**, 255 (1942).
RÜNNSTRÖM, J.: Arch. Entw. Mech. **113**, 556 (1928).
TAMINI, E.: Monit. Zool. Ital. **52**, 81 (1941).

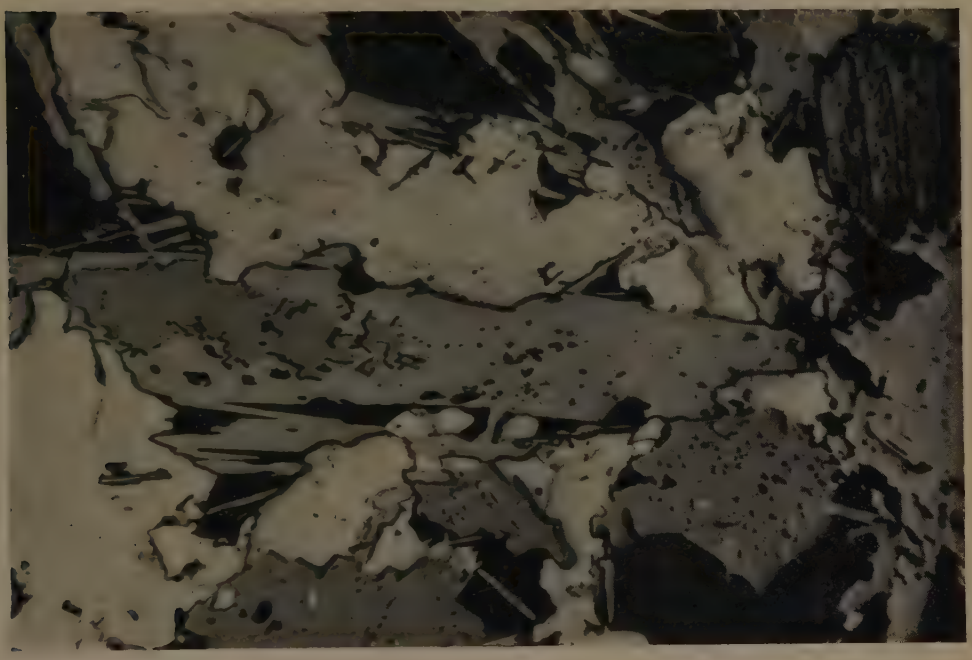

Fig. 7.

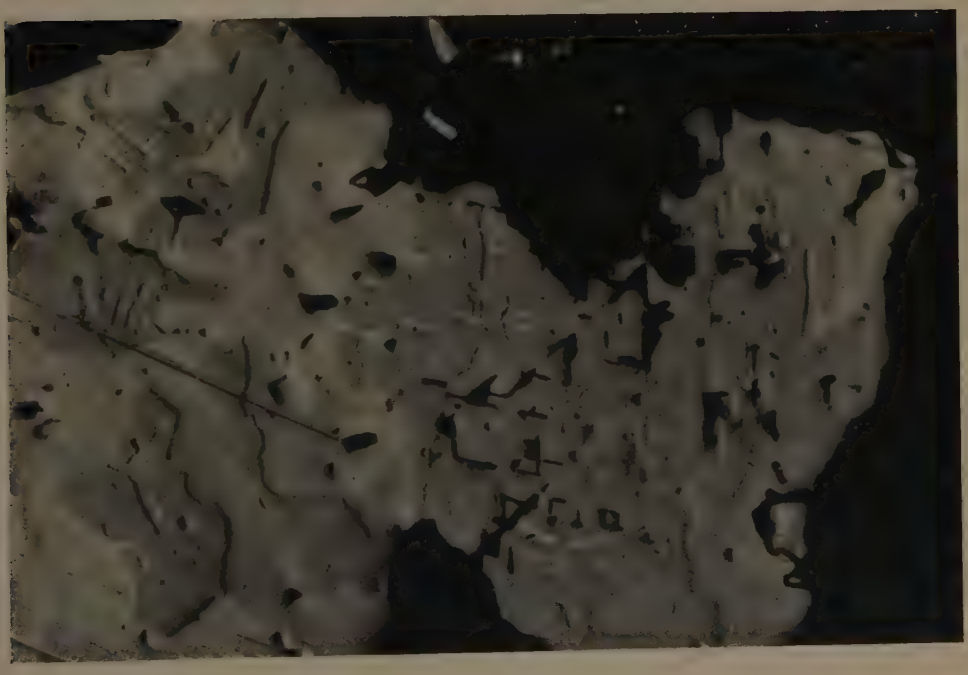

Fig. 8.

Physics. — *On the spreading of invertase solutions.* By E. GORTER and H. A. DIEU *). From the Laboratory of the Children's Clinic (Prof. Dr. E. GORTER) of the University of Leiden–Holland, and from the physical-chemistry Department of the University of Liège-Belgium (Prof. Dr. DESREUX).

(Communicated at the meeting of November 30, 1946.)

Introduction.

In the investigations on the purity of invertase solutions one of us [1]) has shown that it was easy by ethyl alcohol or acetone fractional precipitation to obtain solutions with constant specific activity (about 4500—5000 S.U. [2]). From many physico-chemical tests of purity (electrophoresis by the „Schlieren-Methode", micro-electrophoresis on collodion, diffusions by the Northrop's and Lam's methods, solubility by the Northrop's method) it was to be concluded that two constituents are present in these solutions; one of them possesses a molecular weight as low as 3.500, the other, a molecular weight as high as 100.000. Moreover, it seems that the substance with low molecular weight may be responsible for the invertase activity, although having a very small concentration (5% of the amount of protein material, generally 2 or 3 mg per cm^3). Thus, it remained to be seen how the separation between these two substances can be realised.

Undoubtedly various attempts could be made, like dialysis, electrodialysis, ultrafiltration, diffusion, adsorption.....

In fact many of these experiments were done as carefully as possible, and the results often were contradictory. In order to throw more light upon this question, we have thought that it perhaps would be possible to get some better indications by the spreading method, considering not only the large dissimilitude between the two components, but also the informations given by this method in the field of the protein-chemistry [3]).

Apparatus.

The apparatus used is a Langmuir balance already described in the litterature [4]). It is obvious as it was again very recently noted by H. B. BULL [5]), that the complication due to surface active-contaminants from the air must be avoided. We have found very helpful to cover the spread films by a glass plate (intermediate space must be as small as possible).

Experiments.

The method of spreading consists in carefully blowing a small quantity

*) Travelling fellow of the "Accords culturels belgo-néerlandais".

1) H. A. DIEU, Etudes sur l'invertase. Bul. Soc. Chim. Biol. 28, 543 (1946).

2) For the signification of these numbers, see ref. 1).

3) E. GORTER, Rapport au VIIIième Congrès de Chimie Biologique de Liège—Liège 1946.

4) E. GORTER, Chem. Weekbl. **31**, 587 (1934).

5) H. B. BULL, J. A. C. S., **67**, 4 (1945).

of invertase solution on to the buffer surface by means of a micro-pipette. Spreading generally is rapid (1 minute), sometimes it is slow (in correlation with the salt-concentration of the liquid in the trough, and also with the hydrogen ion-concentration of the same liquid. The pH of all buffer solutions was controlled by colorimetric and electrometric measurements with a quin-hydrone electrode (the "salt error" is here negligible). The readings cover the pH range of 1—5,6 at room temperature. For the solutions of pH 1—3,5, we use diluted HCl, for pH > 3,0, 1/300 M acetate buffers are used.

1e) *Spreading at pH 1.*

We have first studied the influence of the purification of invertase solutions upon their spreading at pH 1. In fig. 1 are shown some pressure-

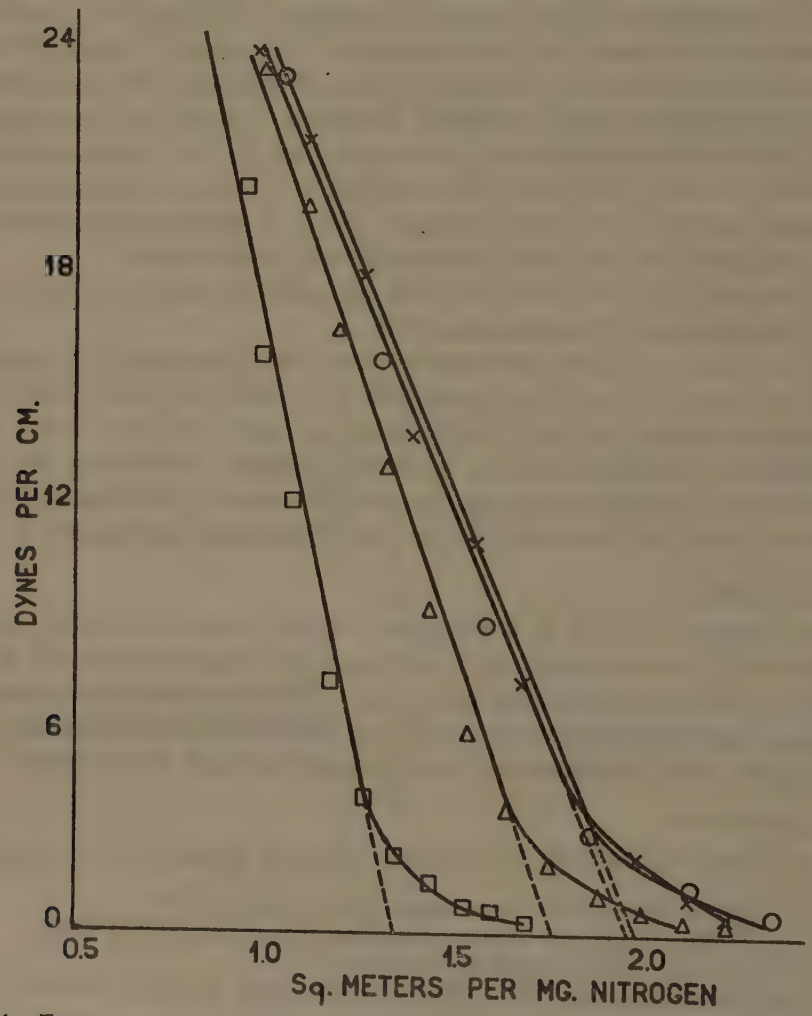

Fig. 1. Force-area-curve for invertase solutions of variable purity at pH 1, after 1 minute.
1 = S. Act.: 4420 ○ 3 = S. Act.: 1880 △
2 = S. Act.: 2630 × 4 = S. Act.: 226 □

area curves at several stages of purity. These curves contain a linear part as is the case of most proteins. Thus, it is easy to extrapolate the curves in the usual way to zero-pressure. Usually this area, when divided by the number of molecules in the film is supposed to yield the area per molecule of spread material. This is the customary procedure, but BATEMAN and CHAMBERS [6]) believe that it has no physical justification and that a more logical method is to determine the coefficient of compressibility β of the spread film. This coefficient is given by the relation:

$$\beta = -\frac{1}{A} \times \left(\frac{\partial A}{\partial F}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where A is the area in sq. meters per milligram of nitrogen and F, the free surface energy drop produced by the film molecules, in dynes per centimeter. In fact, it was found necessary to compare the magnitude of the spreading with the quantity of total nitrogen instead of the quantity of proteins. We have also applied this method to the study of invertase films.

In figure 2 is shown the compressibility coefficient plotted against the area of the spread film at several stages of purity. Also, it must be noted that the applicability of equation (1) is only significative of a film at moderate or high pressure. (When protein films are compressed to a moderate extent, inhomogeneous parts occupy a negligible surface area). From these curves the sharp minima of the compressibility coefficient can easily be drawn, and therefore, the areas of these minima. They correspond to film pressures which represent the maximum compressions to which the several invertase films can be subjected and remain uncollapsed. The results of such an interpolation are shown in table 1.

TABLE 1.

Specif. Activ.	β min.	$\bar{A}$: m²/mg N_2	Pres : dyn/cm
4420	0.024	1.44	14.0
2630	0.025	1.47	12.6
1880	0.018	1.49	9.1
226	0.017	1.13	10.2

From these results, it may be concluded that by purification the films become more stable. Naturally these values reported in this table, have been based on the hypothesis that, during the compression dF, the film composition is unchanged, what is not always true. Moreover, the film pressures and the corresponding film area of egg albumin and β lactoglobulin on 35 % $(NH)_2 SO_4$ solutions have been determined at the point of minimum compressibility [7]). At this point the egg-albumin has a film pressure of 14,6 dyn/cm and the β lactoglobulin, a film pressure of 16,7

[6]) J. B. BATEMAN and L. A. CHAMBERS, J. Chem. Phys. 7, 244 (1939).
[7]) H. B. BULL, J. A. C. S. 67, 4, 8, 10 (1945).

dyn/cm. Our solutions of high purity have a film pressure of 14 dyn/cm. Therefore it seems possible to compare the packing of these films. Accor-

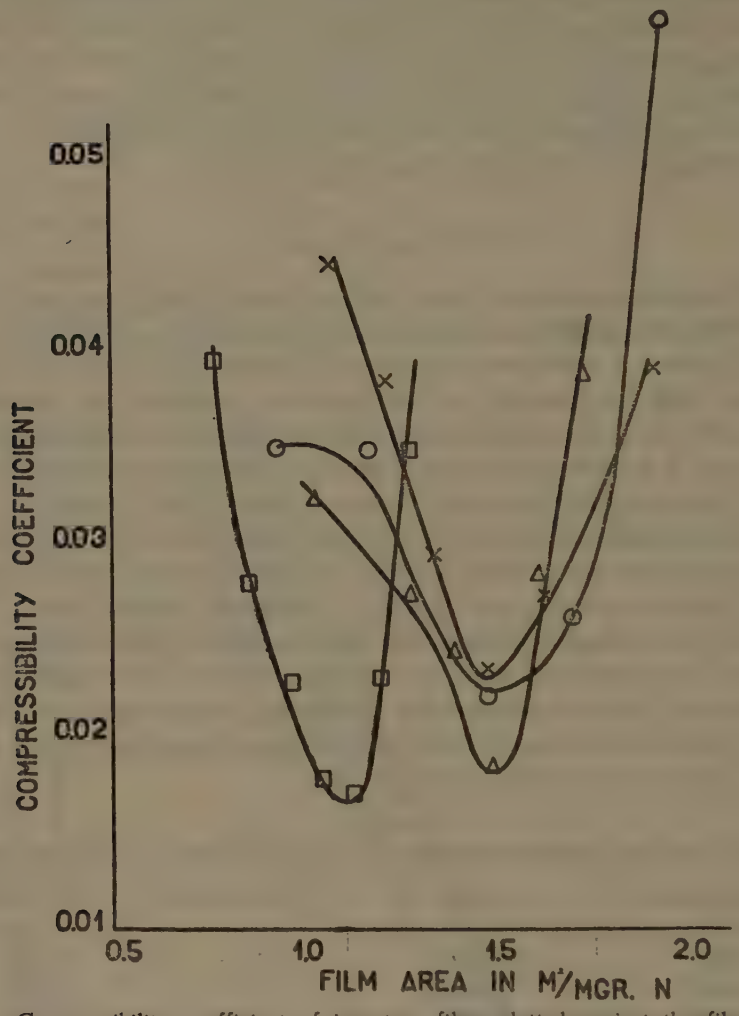

Fig. 2. Compressibility coefficient of invertase films plotted against the film area at pH 1 after 1 minute.

1 = S. Act.: 4420 ○ 3 = S. Act.: 1880 △
2 = S. Act.: 2630 × 4 = S. Act.: 226 □

ding to the sensibility of the balance, at last, it appears impossible to calculate the molecular weight of these films, because the molecular weight is only significant at a pressure below one dyne per centimeter pressure (gaseous films).

2e. *Influence of pH on the spreading.*

The influence of the hydrogen ion content on the spreading of proteins is well known, and sometimes brings many informations on their structure of their physico-chemical properties. The data obtained from the invertase

films confirm very well these hypothesis, as shown in fig 3. The percentage of spreading for several solutions is plotted against the pH. When

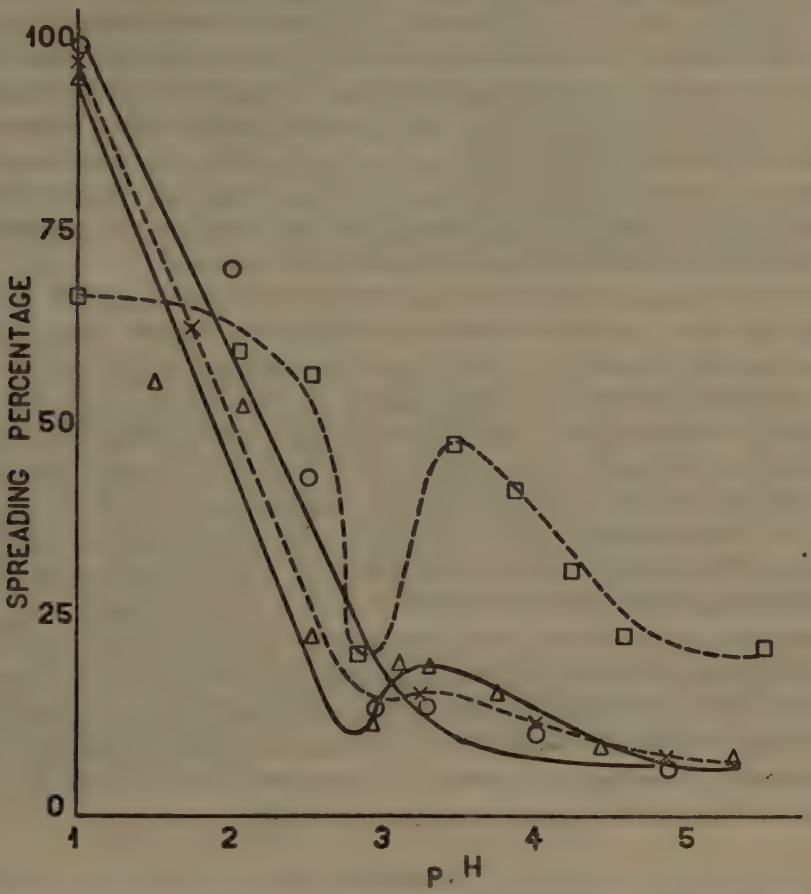

Fig. 3. Influence of pH on the spreading (in %) of solutions of variable purity.
1 = S. Act.: 4420 ○ 3 = S. Act.: 1880 △
2 = S. Act.: 2630 × 4 = S. Act.: 226 □

examining the spreading of the fourth solution (weak sp.-activity) we find to exist a typical W-shaped curve and the spreading is not too large at pH 1. With the increasing purification, this W-shaped curve changes. The top of the curves decreases rapidly and finally disappears. Moreover, the spreading at pH 1 increases up to the same value. Strikingly similar results have been obtained recently in this clinic by one of us [8]) in collaboration with G. A. ELINGS, with seroglycoid protein of Hewitt. In this research, the changes, associated with the progressive elimination of crystalline albumin, in the form of the spreading curve, are essentially similar to the changes observed in the progressive purification by acetone of our

[8]) E. GORTER and G. A. ELINGS, Proc. Kon. Ned. Akad. v. Wetensch., A'dam, **49**, 891 (1946).

invertase solutions. They summarize their studies in the following manner: „In order to find out the difference in spreading tendency between the two proteins, one has to examine this spreading at various pH. It appears that there is a very great difference in so far that crystal albumin shows the properties of a protein, in which only a few free COO^- and NH_3 groups are present, whereas the spreading of seroglycoid with a low minimum at a pH 3,0 and only very little spreading tendency at a higher pH above 3,0 behaves like a protein in which the NH_3^+ groups have disappeared and replaced by acid groups. It is possible that the seroglycoid is more soluble than the crystalbumin fraction owing to the presence in this conjugated protein molecule of several OH-groups."

Hence it appears that the inventase solutions of constant specific activity (4420 Sp. Un.) contain essentially a mucoprotein. It has been possible to estimate in the solution in the trough, the invertase activity for several pH. At pH 1, invertase is entirely spread on the surface; at pH 2,90, invertase is dissolved in the bulk of the solution. Unfortunately the characterisation of mucoproteins is a problem for which at the present time, there are few effective approaches as has been recently shown by K. MEIJER [9]). For the determination of the polysaccharide content SØRENSEN-HAUGAARD's [10]) colorimetric method was used. This method has been criticized by several authors, but it is difficult to find an actually better one. The standardizations were done with galactose. In the table 2 data relating to activity, nitrogen content and polysaccharide estimations per cm^3 of invertase solutions are shown. Further informations useful for the elucidation of the polysaccharide composition is derived from the estimation of the glucosamine, according to HEWITT [11]). It appears that the ratios glucosamine : nitrogen-total nitrogen is so great as 1 : 10.

TABLE 2.

Activity	Nitrog. mg/cm^3	Polys. *)	Sp. Act.	Act./Polys.	Nitrog./Polys.
2630	1.00	52.0	2630	50	0.020
2260	0.52	43.2	4420	52	0.012

*) The numbers shown in this column have a relative value. That is due to the fact that the absorption spectrum and the colour intensity of the orcinol-reaction products change with the monosaccharides responsible for the coloration.

Moreover, our preparations of greatest activity do not contain phosphorus. As has been pointed out by SEVAG [12]), an other method of polysaccharide-protein separation consists in allowing $CHCl_3$ to act upon this

[9]) K. MEIJER, Advances in Protein Chemistry, Vol. II, p. 256 N. Y. 1944.
[10]) M. SØRENSEN and G. HAUGAARD, Bioch. Zeitschr. 260, 247 (1933).
[11]) HEWITT, Bioch. J. 32, 1554 (1938).
[12]) SEVAG, Biochem. Z., 273, 419 (1934).

mixture. This separation is very interesting to isolate polysaccharides, but it is not possible to avoid the degradation of the protein. Recently, in the field of invertase researches, very active solutions were obtained [13]) by adsorption methods. In contrast, our solutions gave no good results by this process.

Summing up the results of various investigations, it would seem fairly safe to draw the following conclusions: our solutions of constant activity consist chiefly of a polysaccharide and protein mixture.

The duality is a fact, but it has not yet been possible to confirm that the invertase is certainly a low molecular protein as indicated by our diffusion data.

Summary.

From invertase solutions with progressive purity it is found that the curves indicating the spreading at different pH give results similar to these obtained with both proteins isolated from the serum following HEWITT'S technique. For solutions of greatest activity the spreading decreases rapidly up to pH 3,0, and then at higher pH above 3,0, the solutions show very little tendency to spread. Orcinol and glucosamine estimations indicate that these solutions contain a mucoprotein. It seems that invertase is this substance.

It appears also that much can be learned about the characterisation of mucoproteins through a study of spreading. This method could be sometimes used for the separation of protein-mixtures with spreading-properties, which greatly differ.

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[13]) C. S. HUDSON, N. ADAMS, J. A. C. S. 65, 1366 (1943).

4. *Attempt at constructing an asymptotic solution.* — Along with the solution giving the first stages of the motion it would be of importance to have an asymptotic solution, valid for large values of the time. Its domain of applicability might be restricted to small values of s, as the quantities principally sought are the expression for V and the temperature in the immediate neighbourhood of the front of the advancing cloud. For this purpose it has been tried to use a development of Φ of the type:

$$\Phi = (1-\sigma)\,\Phi_0 + \sigma^2\,\Phi_2 + \sigma^3\,\Phi_3 + \ldots \quad . \quad . \quad . \quad . \quad (24)$$

where the Φ_n represent functions of τ. This series already satisfies the boundary condition (14a) for $\sigma = 0$. From the results obtained in the preceding section we have:

$$(S')^2 = \tfrac{1}{9}\,V_0^2\,(1 - \tfrac{8}{9}\,\sigma + \tfrac{82}{45}\,\sigma^2 - 0{,}5492\,\sigma^3 + \ldots) \quad . \quad . \quad (24a)$$

Inserting these expressions into (15), working out the differentiations and equating coefficients of corresponding powers of σ on both sides, an infinite set of equations is obtained for the functions Φ_n, of which the first and second are:

$$\Phi_0'' = -\,\Phi_0^{-5/3}\left(\frac{392}{45} - 10\,\frac{\Phi_2}{\Phi_0}\right) \quad . \quad . \quad . \quad . \quad . \quad (25a)$$

$$\Phi_0'' = +\,\Phi_0^{-5/3}\left(\frac{90328}{3645} - \frac{160}{3}\,\frac{\Phi_2}{\Phi_0} - 30\,\frac{\Phi_3}{\Phi_0}\right) \quad . \quad . \quad . \quad (25b)$$

Every subsequent equation introduces a further function Φ_n. It thus appears possible, at least in principle, to express all functions Φ_n in terms of Φ_0, but it does not seem promising to introduce the expressions obtained in this way into the series (24), in order to investigate its convergence and to see whether it can be made to satisfy the boundary condition along the path of the compression wave.

From eqs. (25a) and (25b) it is seen that there must exist a linear relation with constant coefficients between the three functions Φ_0, Φ_2 and Φ_3. Should one conclude from this fact that these functions would be proportional to each other, then it appears probable that the subsequent equations of the set will lead to the same result for the other functions Φ_n, so that the expression for Φ would become of the form $\Phi_0(\tau) \cdot F(\sigma)$. By direct substitution it can be verified that it is possible to satisfy the differential equation (15) by means of a function of this type, which also

can be made to satisfy the boundary condition (14a) for $\sigma = 0$. It is impossible, however, at the same time to satisfy the second boundary condition.

It may be mentioned nevertheless that a rough approximation is given by:

$$\Phi = (1 + \tfrac{4}{3}\tau)^{3/4}(1 + \tfrac{8}{9}\sigma)^{-9/8},$$

which gives:

$$a S = \tfrac{9}{8}(1 + \tfrac{4}{3}\tau)^{2/3} - \tfrac{9}{8}.$$

This expression for Φ satisfies both boundary conditions, but not the differential equation. The corresponding expression for φ is found to be:

$$a\varphi = \tfrac{27}{2}(1 + \tfrac{4}{3}\tau)^{2/3} - 9(1 + \tfrac{4}{3}\tau)^{3/4}(1 + \tfrac{8}{9}\sigma)^{-1/8} - \tfrac{9}{2},$$

from which:

$$V = V_0\{4(1 + \tfrac{4}{3}\tau)^{-1/3} - 3(1 + \tfrac{4}{3}\tau)^{-1/4}\},$$

or developing:

$$V = V_0\{1 - 0{,}778\,\tau + 0{,}75\,\tau^2 - 0{,}81\,\tau^3 \ldots\}.$$

The correct expression is, according to (23):

$$V = V_0\{1 - 0{,}778\,\tau + 0{,}65\,\tau^2 - 0{,}61\,\tau^3 \ldots\}.$$ [4])

When these formulae are used with the same data as considered at the end of section 3 we find: $\tau = 0{,}70$; $a V_0^2 = 1{,}10$; $a = 3{,}4 \cdot 10^{-17}$; $\bar{\varrho}/\varrho_0 = 1{,}5 \cdot 10^{17}$; $t = 3{,}4 \cdot 10^8$ sec; $\Phi = 1{,}64$; $RT = 0{,}66 \cdot 10^{16}$; $T = 79.000.000°$.

5. *Derivation of a solution from the equations for the isentropic expansion of a homogeneous mass of gas.* — As a variant of the problem investigated until now, it is useful to consider a simpler example, referring to a problem where the advancing cloud or piston first during a considerable period of time is moved forward with a *constant* velocity, through the application of an appropriate force. The shock wave generated in this case runs ahead with constant velocity of propagation and has a constant intensity; hence all the gas through which this shock wave passes will suffer the same change of state, so that the compressed gas between shock wave and piston everywhere has the same density, pressure, temperature and velocity. If now from a given instant onward the piston is left to itself, its motion will be retarded by the effect of the pressure of the

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[4]) From the expression for Φ we obtain:

$$\tau = \tfrac{3}{4}\Phi^{4/3}(1 + \tfrac{8}{9}\sigma)^{3/2} - \tfrac{3}{4}.$$

This formula for τ satisfies the boundary conditions given for t as a function of Φ and s in footnote [3]). When it is substituted into the differential equation for t mentioned in that note, in which $\theta = (1 + \frac{8}{9}\sigma)^{-1}$, the left hand side gives: $-88/45 = -1{,}96$; the right hand side: $-248/135 = -1{,}84$ [omitting in both cases a factor $27\,\Phi^2(1 + \frac{8}{9}\sigma)^{5/2}/aV_0^3$]. As the boundary conditions for this equation refer to fixed lines, it may be possible to use the equation for the calculation of a correction.

compressed gas in front of it. Again expansion phenomena will appear, but so long as the first of the expansion waves after its reflection at the shock wave has not come back to the piston, we can calculate with the equations for the motion of a homogeneous gas, the physical state of which is described by one variable only, for instance by its density, from which the pressure and the temperature will follow unambiguously [5]).

The equations of motion for the gas in this case can be put into the EULERIAN form, given in the paper quoted in footnote [1]) [§ 2, eqs. (1) and (2)], to which we now can add POISSON's equation: $p/\varrho^{5/3} = constant$, with the same value of the constant throughout the whole mass of gas between shock wave and piston. By introducing the velocity of sound c the equations can be transformed into:

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} &= -3c\frac{\partial u}{\partial x} \\ \frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} &= -\tfrac{1}{3}c\frac{\partial c}{\partial x} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (26)$$

A general solution of these equations can be found, which expresses that from every point of the path of the front of the cloud or piston (the

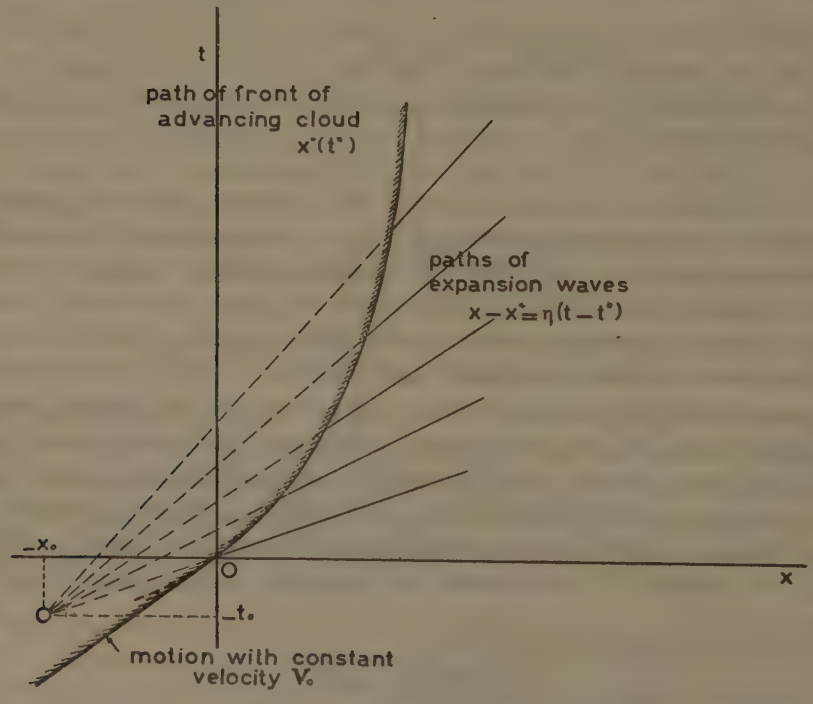

Fig. 3.

[5]) The case in which the piston, after having been moved for a certain period with a constant velocity, is suddenly brought to complete rest, has been treated in the paper mentioned in footnote [2]), where particular attention has been given to the phenomena accompanying the reflection of expansion waves when overtaking the shock wave.

coordinates of which we shall indicate by x^*, t^*), an expansion wave propagates itself into the gas, in such a way that each wave is characterized by constant values of density, pressure, temperature, velocity of sound c and velocity of motion u. Consequently the velocity of propagation of the wave in space, which is given by $u + c$, likewise is constant, so that in an x, t-diagram (compare fig. 3) the path of every wave is represented by a straight line, starting from the point x^*, t^* where the wave originated. This solution is represented by the formulae:

$$u = \tfrac{3}{4}\eta - \eta^*; \quad c = \tfrac{1}{4}\eta + \eta^* \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

where η is the parameter giving the slope of the path of the wave in the diagram, in such a way that:

$$x - x^* = \eta\,(t - t^*) = (u + c)\,(t - t^*) \quad . \quad . \quad . \quad . \quad . \quad (27a)$$

while η^* is a constant.

When we assume that the state 1 from which the gas expanded was the state found after the passage of the shock wave, we have:

$$\left.\begin{aligned} u_1 = V_0; \quad \varrho_1 = 4\,\varrho_0; \quad p_1 = \tfrac{4}{3}\,\varrho_0\,V_0^2 \\ R\,T_1 = \tfrac{1}{3}\,V_0^2; \quad c_1 = (\sqrt{5}/3)\,V_0 = 0{,}745\,V_0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (28)$$

from which we derive:

$$\eta_1 = \tfrac{1}{3}\,(3 + \sqrt{5})\,V_0 = 1{,}745\,V_0; \quad \eta^* = \tfrac{1}{4}\,(\sqrt{5} - 1)\,V_0 = 0{,}309\,V_0 \quad (29)$$

At any later instant u and c follow from (27); while density, pressure and temperature are obtained from:

$$\varrho = \varrho_1\,(c/c_1)^3; \quad p = p_1\,(c/c_1)^5; \quad R\,T = \tfrac{3}{5}\,c^2. \quad . \quad . \quad . \quad (30)$$

In order to find the motion of the piston we observe that the velocity of the gas immediately adjoining the piston must be equal to that of the piston itself; hence for every expansion wave:

$$u = V = dx^*/dt^* \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

The motion of the piston is determined by the equation:

$$d^2x^*/dt^{*2} = -\,p/\varrho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Inserting the value of p given by (30), with:

$$c = (u + 4\,\eta^*)/3 = (dx^*/dt^* + 4\,\eta^*)/3; \quad p_1/c_1^{\,5} = \frac{324}{5^{5/2}}\,\frac{\varrho_0}{V_0^3},$$

we obtain a differential equation for x^*, the solution of which is:

$$x^* + x_0 = \frac{2.5^{5/8}}{3^{3/4}}\left(\frac{\bar{\varrho}\,V_0^3}{\varrho_0}\right)^{1/4}(t^* + t_0)^{3/4} - 4\,\eta^*\,(t^* + t_0) \quad . \quad . \quad (33)$$

where x_0, t_0 are integration constants. The numerical factor in the first term on the right hand side has the value 2,399. It appears from this equation that ultimately the piston will move with the constant velocity $4\,\eta^* = 1{,}236\ V_0$ in the *negative* direction. This negative final velocity,

however, will only be obtained when we let the gas expand to vanishing densities, that is, below the original density which it had before the passage of the shock wave [6]).

From (33) we obtain the velocity of the piston:

$$V = 1{,}800\,(\bar{\varrho}\,V_0^3/\varrho_0)^{1/4}\,(t^* + t_0)^{-1/4} - 1{,}236\,V_0 \quad . \quad . \quad . \quad (34)$$

It follows that the slope η of the wave starting from the piston at the instant t^* is given by:

$$\eta = 2{,}399\,(\bar{\varrho}\,V_0^3/\varrho_0)^{1/4}\,(t^* + t_0)^{-1/4} - 1{,}236\,V_0 \quad . \quad . \quad . \quad (35)$$

Along any such wave: $u = V(t^*)$, while:

$$c = 0{,}600\,(\bar{\varrho}\,V_0^3/\varrho_0)^{1/4}\,(t^* + t_0)^{-1/4} \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

It can be proved that all the paths of waves represented by (27a) with x^* and t^* related by (33), when produced backwards, pass through a single point, having the coordinates $-x_0$, $-t_0$; this point has been indicated in fig. 3. Hence we can bring the equation for the path of a wave into the simple form:

$$x + x_0 = \eta\,(t + t_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

with the same values x_0, t_0 for all waves.

We shall fix the constant t_0 in such a way that $V = V_0$ for $t^* = 0$ (in fig. 3 it has been assumed that at the same instant $x^* = 0$); this gives:

$$1{,}800\,(\bar{\varrho}\,V_0^3/\varrho_0)^{1/4}\,t_0^{-1/4} = 2{,}236\,V_0 \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Formula (34) can then be written:

$$V = 2{,}236\,V_0\,t_0^{1/4}/(t^* + t_0)^{1/4} - 1{,}236\,V_0 \quad . \quad . \quad . \quad . \quad (34a)$$

from which the retardation is found to be:

$$dV/dt = -0{,}559\,V_0\,t_0^{1/4}/(t^* + t_0)^{5/4} \quad . \quad . \quad . \quad . \quad . \quad (34b)$$

At the same time the expression for c becomes:

$$c = 0{,}745\,V_0\,t_0^{1/4}/(t^* + t_0)^{1/4} \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

The form of the solution in this case has the advantage that it immediately gives the distribution of various quantities, such as the temperature, in function of x and t. From (27) and (37) it follows that:

$$c = (x + x_0)/4\,(t + t_0) + 0{,}309\,V_0 \quad . \quad . \quad . \quad . \quad . \quad (40)$$

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[6]) It will be seen from eq. (34a) below that the motion changes direction when $[(t^* + t_0)/t_0]^{1/4} = 1{,}81$; we then have: $c = 0{,}412\,V_0 = c_1/1{,}81$; hence $\varrho = \varrho_1/5{,}9 = 0{,}68\,\varrho_0$. — The pressure and the temperature at this instant still are considerable.

The question whether such a reversal of the motion, which means that the cloud rebounds from the interstellar gas upon which it has struck, may occur also in the case treated in sections 3 and 4, can only be decided when one should succeed in giving the asymptotic form of the solution.

[insertion of x^*, t^* for x and t respectively reduces this expression again to (36)]; the temperature then is obtained from the third formula (30).

Although this solution refers to rather artificial initial conditions, we may attempt to adapt it to the case of the moving cloud of gas. It will be seen that when the values of V_0, ϱ_0 and $\bar{\varrho}$ should be given, the value of t_0 can be obtained from (38) so that for any given instant t^* the value of V and the temperature etc. of the gas can be calculated. On the other hand when we start from the observational data for the velocity and the retardation, we again come to the conclusion that there is one more unknown than we have relations. In order to make it possible to calculate backward from the observational data it is necessary to introduce an additional assumption, just as proved to be the case with the solution obtained in section 3. When also here we assume that the value of V_0 was 1,5 times the value of V at the epoch of observation (1934), the following results are obtained:

$$t_0 = 4{,}08.10^8; \quad t^* = 3{,}71.10^8 \; (= 11{,}7 \text{ years});$$

$$\bar{\varrho}/\varrho_0 = 1{,}74.10^{17}; \quad c = 1{,}142.10^8; \quad RT = 0{,}78.10^6,$$

from which $T = 94.000.000°$.

It is remarkable that the values of t^*, $\bar{\varrho}/\varrho_0$ and T derived in this way are not very different from those obtained in section 3. There is found, however, a difference in the temperature gradient in the neighbourhood of the advancing cloud or piston. In the case treated here it was assumed that after the passage of the shock wave the whole mass of gas had a constant temperature determined by $RT_1 = \frac{1}{3} V_0^2$. In the expansion process every subsequent expansion wave brings a lower temperature. When from a given point x^*, t^* of the path of the front of the advancing piston we move outward in the positive direction of x, we reach waves which have started from this front at earlier instants. As these waves carry with them higher temperatures, $\partial T/\partial x$ is positive in this case. Its value is found most easily in the following way [compare (40)]:

$$\partial (RT)/\partial x = \tfrac{6}{5} c \, (\partial c/\partial x) = 0{,}3 \, c/(t^* + t_0),$$

which at the instant of observation ($t^* = 3{,}71 \cdot 10^8$) gives: $\partial(RT)/\partial x = 0{,}044$.

In the case treated originally, the shock wave, although starting with the same intensity and initially producing a temperature of the same height as mentioned above, immediately is retarded, as will be seen from the expression for S given by (21). This has the consequence that layers of gas at further distances from the advancing front, after the shock wave has passed over them, start with a compression temperature lower than that of the first layer (the one immediately adjacent to the front). Notwithstanding the circumstance that the expansion itself is more efficient the nearer we are to the advancing front, the temperature is now found to decrease when we move outward. This result is obtained from the formula

$RT = 3\,S'^{2}/\Phi^{2/3}$ mentioned at the end of section 3; for small values of σ we have: $S'^{2} = \frac{1}{9} V_0^2\,(1 - \frac{8}{9}\,\sigma)$; $\Phi = \Phi_0\,(1 - \sigma)$, so that:

$$RT = \tfrac{1}{3}\,V_0^2\,\Phi_0^{-2/3}\,(1 - \tfrac{2}{9}\,\sigma).$$

Hence:

$$\partial\,(RT)/\partial x = (a/\Phi)\,.\,\partial\,(RT)/\partial\sigma = -\tfrac{2}{27}\,a\,V_0^2\,\Phi_0^{-5/3}.$$

At the instant of observation this expression gives $\partial(RT)/\partial x = -0{,}031$. The order of magnitude is the same, but the sign is different.

Although the negative value is obtained for a case which starts from less artificial initial conditions than does the other case, not much weight can be given to the result, as the distribution of the temperature in the compressed gas may depend upon the precise distribution of the density in the interstellar gas before the advancing cloud struck upon it, which distribution is unknown to us.

(To be continued.)

Mathematics. — *Richtlijnen der intuïtionistische wiskunde.* By L. E. J. BROUWER.

(Communicated at the meeting of March 29, 1947.)

Sinds haar ontstaan heeft de intuïtionistische denkoriënteering in de eerste plaats getast naar een nieuwe routine van scheppend wiskundig werk, in de tweede plaats gezocht naar een zoo adequaat mogelijke formuleering voor de richtlijnen dezer routine.

Van genoemde doelstellingen schijnt de eerste thans vrijwel bereikt. Waarschijnlijk is zij ook volledíger bereikbaar dan de tweede, aangezien immers voor de intuïtionistische denkwijze door de wiskundige taal geen andere rol kan worden gespeeld dan die van een bij alle practische doelmatigheid niettemin tegen misverstanden nimmer volledig vrijwarend hulpmiddel, om mathematische constructies of constructiemethoden te memorizeeren of aan anderen te suggereeren.

Dit vooropgesteld, moge voor bedoelde richtlijnen hieronder de formuleering worden gegeven, die mij op het oogenblik de meest adequate tòeschijnt. In sommige opzichten wijkt zij van vroegere formuleeringen af.

De intuïtionistische wiskunde dan is een essentieel taallooze gedachtenconstructie. Zij ontstaat door zelfontvouwing van de tweeheidsabstractie als wiskundige oerintuïtie. Deze zelfontvouwing veroorlooft in eerste instantie de samenvatting niet slechts van eindige rijen van wiskundige systemen, doch ook van *wetmatig* onbegrensd voortschrijdende rijen van inductief voorgevormde wiskundige systemen. Doch in tweede instantie evenzeer de samenvatting van *in volledige of aan (eventueel wisselende) beperking onderhevige vrijheid* onbegrensd voortschrijdende rijen van wiskundige systemen. Ten slotte mogen bij dezen opbouw der wiskunde in ieder stadium eigenschappen, die voor reeds verkregen mathematische denkbaarheden ondersteld kunnen worden, als nieuwe mathematische denkbaarheden worden toegevoegd onder den naam van *soorten*. Reeds verkregen mathematische denkbaarheden, waarvoor zulk een eigenschap geldt, worden dan als *elementen* der betrokken soort gequalificeerd.

De soort, die meer dan andere de intuïtionistische wiskunde fertilizeert, is de *spreiding* (*Menge*), met haar elementen o.a. gedefinieerd in deze Proceedings Vol. XXIII, p. 950, 951. Wegens de taalloosheid der wiskunde behoort in genoemde definitie bij het woord *teeken* (*Zeichen*), in het bijzonder ook bij het woord *cijfergroep* (*Ziffernkomplex*), gedacht te worden aan *gedachtenteekens*, bestaande in reeds verkregen mathematische denkbaarheden.

Mathematics. — *On sets of integers.* (Second communication.) By J. G. VAN DER CORPUT.

(Communicated at the meeting of March 29, 1947.)

§ 2. *Generalisations.*

Let us put the following question: If two couples A, B and C, D satisfy the inequalities

$$A(m)+B(m)\geqq\gamma m;\ C(m)+D(m)\geqq\delta m \qquad (m=1,\ldots,g) \quad (17)$$

and if each of the sets A, B, C and D contains zero, is it then possible to deduce similar inequalities involving $A+C$, $A+D$, $B+C$ and $B+D$? In fact, that is possible. If $\gamma+\delta\leqq 1$ we find

$$(A+C)(m)+(A+D)(m)+(B+C)(m)+(B+D)(m)\geqq 2(\gamma+\delta)m. \quad (18)$$

In particular, if $\gamma\leqq\frac{1}{2}$ and A and B both contain zero, we deduce from

$$A(m)+B(m)\geqq\gamma m \qquad (m=1,\ldots,g) \quad . \quad . \quad . \quad . \quad (19)$$

the inequalities

$$(A+A)(m)+(A+B)(m)+(A+B)(m)+(B+B)(m)\geqq 4\gamma m \quad (m=1,\ldots,g).$$

I do not know whether the inequalities (19) imply

$$(A+A)(m)+(B+B)(m)\geqq 2\gamma m \qquad (m=1,\ldots,g).$$

The above result may be deduced by the following argument, due to F. J. DYSON:

Lemma 4. *If a couple A, B is transformed by the corresponding e-transformation into A^*, B^*, then we have for $m=1,\ldots,g$ and for every set U consisting of integers $\geqq 0$*

$$(U+A^*)(m)-(U+A)(m)\leqq(U+B)(m-e)-(U+B^*)(m-e),$$

thus also

$$(U+A^*)(m)-(U+A)(m)\leqq(U+B)(m)-(U+B^*)(m).$$

For a proof of this lemma it is sufficient to show that every integer $h\leqq g$, which occurs in $U+A^*$ and not in $U+A$, has the property that $h-e$ occurs in $U+B$ and not in $U+B^*$. The number h in question has the form $u+a^*$, where a^* occurs in A^* and not in A. Hence $a^*=e+b$, where the positive integer b is contained in B, so that $h-e=u+b$ occurs in $U+B$. If $h-e$ was an element of $U+B^*$, then $h-e$ would be of the form u^*+b^*, where u^* would occur in U and b^* in B^*. Then $a=e+b^*$ would be an element A. In fact, if not, b^* is positive (because e is an element of A) and this positive integer b^* would

have been cancelled in B by the e-transformation and therefore not occur in B^*. This gives a contradiction, since $h = u^* + a$ does not occur in $U + A$.

Theorem 5. *Let the systems A and B both contain zero and satisfy*

$$A(m) + B(m) \geqq \gamma m \qquad (m = 1, \ldots, g), \quad . \quad . \quad . \quad . \quad (20)$$

where $\gamma \leqq 1$. Then a set Z exists, containing zero and satisfying the inequalities

$$Z(m) \geqq \gamma m \qquad (m = 1, \ldots, g), \quad . \quad . \quad . \quad . \quad . \quad (21)$$

with the property that for every system U of integers $\geqq 0$ and for $m = 1, \ldots, g$ the inequalities

$$(U + A)(m) + (U + B)(m) \geqq (U + Z)(m) + U(m) \quad . \quad . \quad (22)$$

and

$$(U + S)(m) \geqq (U + Z)(m). \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

are valid; here $S = A + B$.

The right hand sides of (20) and (21) may be replaced by $\gamma(m + 1) - 1$.

If U consists only of the number zero, (23) gives the theorem of Mann.

The proof proceeds as follows:

If B contains no positive elements $\leqq g$, then the set $Z = A$ possesses the required properties. We may therefore suppose that B contains at least one natural number $\leqq g$ and further we may suppose the theorem proved, if $B(g)$ is replaced by a smaller number. We transform the couple A, B by the corresponding e-transformation into a couple A^*, B^*, satisfying the inequalities

$$A^*(m) + B^*(m) \geqq \gamma m \qquad (m = 1, \ldots, g).$$

In virtue of $B^*(g) < B(g)$ there exists by induction a set Z of integers $\geqq 0$ with (21), containing zero, such that for $m = 1, \ldots, g$

$$(U + A^*)(m) + (U + B^*)(m) \geqq (U + Z)(m) + U(m)$$

and

$$(U + S^*)(m) \geqq (U + Z)(m), \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where $S^* = A^* + B^*$. The preceding lemma gives (22). From lemma 3 it follows that S^* is a subset of S, so that (23) follows from (24).

Now it is easy to show that the inequalities (17) imply (18), if each of the sets A, B, C and D contains zero. In fact, by the above theorem, two sets Z and Y exist, with

$$Z(m) \geqq \gamma m \text{ and } Y(m) \geqq \delta m \qquad (m = 1, \ldots, g),$$

such that we obtain for these values of m

$$(A + C)(m) + (B + C)(m) \geqq (C + Z)(m) + C(m)$$

and

$$(A + D)(m) + (B + D)(m) \geqq (D + Z)(m) + D(m).$$

and that the left hand side of (18) is at least equal to

$$(C+Z)(m)+(D+Z)(m)+C(m)+D(m)$$
$$\geqq (Y+Z)(m)+Z(m)+C(m)+D(m) \geqq (Y+Z)(m)+\gamma m+\delta m.$$

From the theorem of MANN it follows that $(Y+Z)(m)$ is at least equal to $(\gamma+\delta)m$, which proves (18).

Above I gave the argument of DYSON, simplified and generalised. The proof, given by MANN of the $\alpha+\beta$-hypothesis does not apply elementary but more complicated transformations, used also by ARTIN and SCHERK. These transformations enable us to prove in a simple manner the following theorem:

Theorem 6. *If A and B both contain zero and*

$$A(m) \geqq \alpha(m+1) \qquad (m=1,\ldots,g), \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where g denotes a natural number, not belonging to $A+B$, then

$$(A+B)(g) \geqq \alpha(g+1)+B(g). \quad . \quad . \quad . \quad . \quad . \quad (26)$$

To illustrate the sharpness of this proposition I consider the two systems A and B, both consisting of the numbers 0, 1, 2, 6, 7, 8, 12, so that $A+B$ is formed by the integers

$$0, 1, 2, 3, 4, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 18, 19, 20, 24.$$

Here

$$A(m) \geqq \tfrac{1}{3}(m+1) \qquad (m=1,\ldots,11),$$

so that we obtain by means of the above theorem

$$(A+B)(11) \geqq \tfrac{1}{3}.12+5=9.$$

It is impossible to find a better result, since the left hand side just equals 9. The theorem is even so sharp, that it is not allowed to replace in (25) the number $\alpha(m+1)$ by αm and at the same time in (26) the number $\alpha(g+1)$ by αg. In fact, in our example we have

$$A(m) \geqq \tfrac{2}{5} m \qquad (m=1,\ldots,11),$$

but

$$\tfrac{2}{5}.11+5 > (A+B)(11).$$

Inequality (26) is not necessarily true for an element g of $A+B$, for in our example the inequalities (25) are valid with $\alpha=\frac{1}{3}$ for $m=1,\ldots,12$, but

$$\tfrac{1}{3}.13+B(g)=\tfrac{13}{3}+6$$

is greater than $(A+B)(12)=10$.

In the proof I may suppose that B contains at least one positive element $\leqq g$, since otherwise the assertion is evident in virtue of $B(g)=0$.

Be e an arbitrary element $<g$ of B. I denote by S the system $A+B$ and by $\overline{S}$ the system of integers $\geqq e$ and $<g$ not belonging to S. I consider

the elements $\bar{s}$ of $\bar{S}$ to which corresponds at least one element $\bar{s}_1$ of $\bar{S}$ so that $\bar{s} + \bar{s}_1 - g - e$ denotes an element of A; the set formed by these integers $\bar{s}$ will be denoted by S'. If $\bar{s}$ runs through the set S', then $b' = g + e - \bar{s}$ runs through a set which I call B'. In virtue of $e \leqq \bar{s} < g$ each element of B' is $> e$ and $\leqq g$. The integer $\bar{s}$ not belonging to S is not equal to zero. Hence there is an $(1, 1)$-correspondence between the positive elements $\leqq g$ of B' and S', thus $B'(g) = S'(g)$.

Each element b' of B' can be written in the form $\bar{s}_1 - a$, where $\bar{s}_1$ is an element of $\bar{S}$ and a an element of A. If b' would belong to B, then $\bar{s}_1 = a + b'$ would belong to $A + B = S$, which is excluded. Consequently B and B' are disjunct.

If S does not contain $g - 1$, it is possible to choose the element $e < g$ of B, such that $g - 1$ is an element of S'. In fact, be m the greatest element $< g$ of S. Since $g - 1$ does not belong to S, we have $m \leqq g - 2$. The integer m belonging to $S = A + B$ can be written in the form $m = a + e$, where a is an element of A and e is an element of B. The integers $g - 1$ and $m + 1$ do not belong to S and are both $\geqq e$ and $< g$; further $(g - 1) + (m + 1) - g - e = m - e$ is equal to the element a of A, consequently $g - 1$ and $m + 1$ are both elements of S'.

Denote by B^* the set, formed by the integers belonging to at least one of the sets B and B'; further by S^* the set formed by the integers, belonging to at least one of the sets S and S'. In order to show that $A + B^*$ and S^* contain the same integers $\leqq g$, it is sufficient to prove, that each element $\leqq g$ of $A + B^*$ occurs in S^* and that each element $\leqq g$ of S^* occurs in $A + B^*$.

Be s an element $\leqq g$ of $A + B^*$. If s is an element of $A + B = S$, then it is certainly an element of S^*. In the remaining case s is an element $\leqq g$ of $A + B'$. If g belongs to $A + B'$, then $g = a + b'$, where $b' = g + e - \bar{s}$, hence $\bar{s} = a + e$, contrary to the hypothesis that $\bar{s}$ does not belong to S. Consequently s denotes an element $< g$ of $A + B'$ and $s = a + b' = a + g + e - \bar{s}$, where $\bar{s}$ is an element of S'. Thus $s \geqq b' \geqq e$ and s and $\bar{s}$ denote two integers both $\geqq e$ and $< g$ not belonging to S, with the property that $s + \bar{s} - g - e = a$ is an element of A. By convention s is an element of S', consequently also of S^*.

Each element of $S = A + B$ occurs in $A + B^*$. To each element $\bar{s}_1$ of S' corresponds at least one element $\bar{s}$ of S' with the property that $\bar{s} + \bar{s}_1 - g - e = a$ denotes an element of A; then $g + e - \bar{s} = b'$ belongs to B', consequently $\bar{s}_1 = a + b'$ to $A + B'$ and therefore also to $A + B^*$. Hence each element of S^* belongs to $A + B^*$.

The proof now runs easily. I may assume the theorem already proved if g is replaced by a smaller natural number, not belonging to S. I distinguish two cases:

1. Be $g-1$ an element of S. I put $q=0$, if S contains all natural numbers $\leqq g-1$; otherwise q be the greatest natural number $\leqq g-2$, not belonging to S. The set B contains $B(g)-B(q)$ integers $>q$ and $\leqq g$ and none of these numbers has the form $g-a$, where a denotes an element of A. The number of integers $>q$ and $\leqq g$ of the form $g-a$ is $1+A(g-q-1)$, hence

$$g-q \geqq B(g)-B(q)+1+A(g-q-1). \quad . \quad . \quad . \quad . \quad (27)$$

Since S contains the numbers $g-1, \ldots, q+1$, we have

$$S(g)-S(q)=g-q-1.$$

If $q=0$, we obtain by (27), applied with $A(g-1)=A(g)$,

$$S(g)=g-1 \geqq A(g)+B(g) \geqq B(g)+a(g+1),$$

so that then the proof is established.

If $q>0$, I deduce from (27)

$$S(g)-S(q)=g-q-1 \geqq B(g)-B(q)+a(g-q).$$

Since the theorem is assumed true, if g is replaced by the smaller integer q not belonging to S, I obtain

$$S(q) \geqq B(q)+a(q+1).$$

Adding we obtain the required result.

2. Let us suppose that $g-1$ is not an element of S. I choose the element e of B, so that $g-1$ is an element of S'. Then $g-1$ belongs to S^*, hence also to $A+B^*$. The number g belongs neither to S nor to S', hence neither to S^* nor to $A+B^*$. I may therefore apply the result, found in 1. with B^* instead of B. In this manner I find, since S^* and $A+B^*$ contain the same numbers $\leqq g$,

$$S^*(g) \geqq B^*(g)+a(g+1). \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

The disjunct sets B and B' form B^* and the disjunct sets S and S' form S^*, consequently

$$B(g)+B'(g)=B^*(g) \text{ and } S(g)+S'(g)=S^*(g),$$

so that $B'(g)=S'(g)$ implies

$$S(g)-S^*(g)=B(g)-B^*(g)$$

and the assertion follows from (28).

§ 3. *Weights.*

In § 1 we considered the number of positive integers not exceeding a certain bound g and occurring in given systems. In this section I give to every natural number $m \leqq g$ a positive weight $f(m)$, so that $A(m)$ does not denote the number of positive elements $\leqq m$ of A, but the sum

$$A(m)=\sum_{a \leqq m} f(a),$$

extended over all positive integers $\leqq m$, occurring in A. Just as in the preceding section zero is never counted. It will appear, that the preceding argument may be applied almost unchanged, if we impose to the weights $f(m)$ the conditions

$$\left.\begin{aligned} &f(m)>0 \qquad (m=1,\ldots,g)\\ &f(m+1)\geqq f(m) \qquad (m=1,\ldots,g-1)\\ &f^2(m+1)\geqq f(m)\,f(m+2) \quad (m=1,\ldots,g-2). \end{aligned}\right\} \quad . \; . \; (29)$$

We shall find

Theorem 7. *If A and B both contain zero, if the weights $f(m)$ satisfy the conditions (29) and if*

$$A(m)+B(m)\geqq\gamma\sum_{h=1}^{m}f(h) \qquad (m=1,\ldots,g), \quad . \; . \; . \; (30)$$

where $\gamma\leqq 1$, then

$$(A+B)(m)\geqq\gamma\sum_{h=1}^{m}f(h) \qquad (m=1,\ldots,g). \quad . \; . \; . \; (31)$$

If we choose $f(m)=m$, then $A(m)$ denotes the sum of the positive elements $\leqq m$ of A. In the first example, given in § 1, we considered the systems

$$A \ldots 0, 1, 2, 6, 7, 12, 19$$
$$B \ldots 0, 1, 2, 6, 7, 8, 12, 20$$
$$A+B \ldots 0, 1, 2, 3, 4, 6, 7, 8, 9, 10, 12, 13, 14, 15, 18, 19, 20, 21, 22, 24, 25, 26, 27, 31, 32, 39.$$

Here we have for $m=1, \ldots, 12$

$$A(m)+B(m)\geqq\tfrac{2}{6}(1+\ldots+m),$$

hence

$$(A+B)(m)\geqq\tfrac{2}{6}(1+\ldots+m),$$

in particular

$$(A+B)(12)\geqq\tfrac{2}{6}\times 6\times 13=31.2.$$

This result is rather bad, for the value of $(A+B)(12)$ is equal to 62.

Above we gave a positive weight $f(m)$ only to the positive integers $m\leqq g$, but it is possible to provide each number m with a positive weight $f(m)$, such that

$$f(m+1)\geqq f(m) \text{ and } f^2(m+1)\geqq f(m)\,f(m+2). \quad . \; . \; (32)$$

for each positive integer m. In fact, if $g\geqq 2$ and if $f(m)$ satisfies the conditions (29), then I put

$$f(m+1)=\frac{f^2(m)}{f(m-1)} \quad \text{for } m\geqq g.$$

This function possesses the required properties, since we obtain for $m \geqq g$

$$\frac{f(m+1)}{f(m)} = \frac{f(m)}{f(m-1)} = \ldots = \frac{f(g)}{f(g-1)} \geqq 1$$

and further for $m \geqq g-1$

$$f^2(m+1) = f(m)\, f(m+2).$$

In order to be able to prove theorem 7 by means of the principle of mathematical induction, it is necessary to generalise this theorem, wherefore I use the following lemma:

Lemma 5. *If every natural number m possesses a positive weight $f(m)$ with $f^2(m+1) \geqq f(m)\, f(m+2)$ and if the n systems $A_1, \ldots, A_n$ $(n \geqq 1)$ satisfy the inequalities*

$$A_1(m) + \ldots + A_n(m) \geqq \gamma \sum_{h=1}^{m} f(h) \qquad (m = 1, \ldots, g), \quad . \quad . \quad (33)$$

then we obtain for the same values of m and for each integer $r \geqq 0$

$$\sum_{a_1 \leqq m} f(a_1 + r) + \ldots + \sum_{a_n \leqq m} f(a_n + r) \geqq \gamma \sum_{h=1}^{m} f(h + r); \quad . \quad . \quad (34)$$

the sum $\sum_{a_h \leqq m}$ is extended over all positive elements $a_h \leqq m$ of A_h.

Proof. In the special case $r = 0$ the formula (34) becomes (33), so that then there is nothing to be proved. If the special case $r = 1$ of lemme 5 has been proved and if we replace $f(m)$ by the function $f(m+1)$ (this function satisfies also the imposed conditions), then we obtain (34) with $r = 2$. In this manner we find the assertion successively for $r = 2, 3, \ldots$, so that we have only to give the proof for $r = 1$.

Since the assertion is evident for $\gamma \leqq 0$, I may suppose $\gamma > 0$. Then $A_1(1) + \ldots + A_n(1) > 0$, so that at least one of the sets $A_1, \ldots, A_n$ contains the number 1. Suppose $1 = t_1 < t_2 < \ldots < t_u$ are the different integers $\leqq m$, belonging to at least one of the sets $A_1, \ldots, A_n$. Put $t_{u+1} = m + 1$. Inequality (33), applied successively with $t_2 - 1, \ldots, t_{u+1} - 1$ instead of m, gives

$$\lambda_1 f(t_1) \geqq \gamma \sum_{h=1}^{t_2-1} f(h);$$

$$\lambda_1 f(t_1) + \lambda_2 f(t_2) \geqq \gamma \sum_{h=1}^{t_3-1} f(h);$$

. . . . . .

$$\lambda_1 f(t_1) + \ldots + \lambda_u f(t_u) \geqq \gamma \sum_{h=1}^{t_{u+1}-1} f(h);$$

here λ_h denotes the number of systems $A_1, \ldots, A_n$, containing t_h.

The left hand side of (34) can be written in the form

$$\lambda_1 f(t_1+1)+\ldots+\lambda_u f(t_u+1). \quad \ldots\ldots \quad (35)$$

In order to obtain for this sum an appropriate lower bound, I multiply the sides of the above inequalities respectively by

$$\frac{f(t_1+1)}{f(t_1)}-\frac{f(t_2+1)}{f(t_2)};\ \frac{f(t_2+1)}{f(t_2)}-\frac{f(t_3+1)}{f(t_3)};\ldots;$$

$$\frac{f(t_{u-1}+1)}{f(t_{u-1})}-\frac{f(t_u+1)}{f(t_u)};\ \frac{f(t_u+1)}{f(t_u)}.$$

All these factors are $\geqq 0$ in virtue of

$$\frac{f(h+1)}{f(h)} \geqq \frac{f(h+2)}{f(h+1)} \quad \ldots\ldots\ldots \quad (36)$$

Adding we find, since $t_1 = 1$,

$$\lambda_1 f(t_1+1)+\ldots+\lambda_u f(t_u+1)$$

$$\geqq \gamma \frac{f(t_1+1)}{f(t_1)} \sum_{h=t_1}^{t_2-1} f(h) + \gamma \frac{f(t_2+1)}{f(t_2)} \sum_{h=t_2}^{t_3-1} f(h) + \ldots + \gamma \frac{f(t_u+1)}{f(t_u)} \sum_{h=t_u}^{t_{u+1}-1} f(h).$$

In the first sum we have $\dfrac{f(t_1+1)}{f(t_1)}\, f(h) \geqq f(h+1)$ in virtue of (36); similarly in the second sum $\dfrac{f(t_2+1)}{f(t_2)}\, f(h) \geqq f(h+1)$, and so on. Hence (35) is at least equal to

$$\gamma \sum_{h=t_1}^{t_2-1} f(h+1)+\ldots+\gamma \sum_{t_u}^{t_{u+1}-1} f(h+1) = \gamma \sum_{h=1}^{m} f(h+1),$$

which proves the lemma.

As it will appear, it is sufficient for the proof of theorem 7 to repeat the argument of § 1, but now applied on the case the weights 1 are replaced by arbitrary weights $f(m)$ with the properties (32). Instead of lemma 1 we obtain

Lemma 6. *If A contains zero and*

$$A(m)+B(m) \geqq \gamma \sum_{h=1}^{m} f(h) \qquad (m=1,\ldots,g),$$

where $\gamma \leqq 1$, then each elementary transformation transforms the couple A, B into a couple A', B' with the property

$$A'(m)+B'(m) \geqq \gamma \sum_{h=1}^{m} f(h) \qquad (m=1,\ldots,g).$$

For the proof it is recommendable to deduce a more general result, viz.: under the conditions of lemma 6 the inequalities

$$\sum_{a' \leqq m} f(a'+r) + \sum_{b' \leqq m} f(b'+r) \geqq \gamma \sum_{h=1}^{m} f(h+r) \qquad (m = 1, \ldots, g) \quad (37)$$

are true for every integer $r \geqq 0$; the first sum is extended over all positive elements $a' \leqq m$ of A' and the second over all positive elements $b' \leqq m$ of B'.

The special case $r = 0$ gives lemma 6.

Let us suppose that this assertion is not true. Be k the smallest positive integer $\leqq g$, for which the lemma is not valid, more precisely: the positive integer $k \leqq g$ possesses the following properties:

1. It is possible to find a system A containing zero and a system B satisfying the inequalities

$$A(m) + B(m) \geqq \gamma \sum_{h=1}^{m} f(h) \qquad (m = 1, \ldots, k), \quad . \quad . \quad . \quad (38)$$

such that a convenient elementary transformation transforms A, B into a couple A', B', for which at least one of the inequalities

$$\sum_{a' \leqq m} f(a'+r) + \sum_{b' \leqq m} f(b'+r) \geqq \gamma \sum_{h=1}^{m} f(h+r) \qquad (m = 1, \ldots, k) \quad (39)$$

is not true for a conveniently chosen integer $r \geqq 0$.

2. Be l an arbitrary positive integer $< k$. If C contains zero and if

$$C(m) + D(m) \geqq \gamma \sum_{h=1}^{m} f(h) \qquad (m = 1, \ldots, l),$$

then each elementary transformation transforms C, D into a couple C', D' such that

$$\sum_{c' \leqq m} f(c'+t) + \sum_{d' \leqq m} f(d'+t) \geqq \gamma \sum_{h=1}^{m} f(h+t) \qquad (m = 1, \ldots, l), \quad (40)$$

where t denotes an arbitrary integer $\geqq 0$.

The special case $l = k-1$, $t = r$, $C = A$ and $D = B$ gives that the inequalities (39) are true for $m = 1, \ldots, k-1$. Consequently (39) is not valid for $m = k$, hence

$$\sum_{a' \leqq k} f(a'+r) + \sum_{b' \leqq k} f(b'+r) < \gamma \sum_{h=1}^{k} f(h+r). \quad . \quad . \quad . \quad (41)$$

From lemma 5, applied with $n = 2$, $A_1 = A$ and $A_2 = B$, it follows for $m = 1, \ldots, g$, that

$$\sum_{a \leqq m} f(a+r) + \sum_{b \leqq m} f(b+r) \geqq \gamma \sum_{h=1}^{m} f(h+r) \quad (m = 1, \ldots, g),$$

so that we obtain, by putting $m = k$,

$$\sum_{a \leqq k} f(a+r) + \sum_{b \leqq k} f(b+r) \geqq \gamma \sum_{h=1}^{k} f(h+r). \quad . \quad . \quad . \quad (42)$$

The elementary transformation, which transforms the couple A, B into A', B', cancels in B a positive element b_0. The case $b_0 > k$ is excluded. In fact, in that case we would not have cancelled in B any element $\leqq k$, nor added to A any element $\leqq k$, so that

$$\sum_{a' \leqq k} f(a' + r) + \sum_{b' \leqq k} f(b' + r) = \sum_{a \leqq k} f(a + r) + \sum_{b \leqq k} f(b + r),$$

contrary to (41) and (42). Similarly the integer $e + b_0$, possibly added to A, is certainly greater than k, for otherwise we would have added to A an integer $e + b_0 \leqq k$ with weight $f(e + b_0 + r) \geqq f(b_0 + r)$, whereas we would have cancelled in B an integer $b_0 \leqq k$ with weight $f(b_0 + r)$ and this is also contrary to (41) and (42). Consequently $b_0 \leqq k < e + b_0$.

Since e is the smallest element of A, such that $e + b_0$ does not belong to A, each element $a < e$ of A, consequently also each element $a \leqq k - b_0$ of A, possesses the property, that $a + b_0$ belongs to A. Thus b_0 is an element of A and

$$\sum_{b_0 < a \leqq k} f(a + r) \geqq \sum_{a \leqq k - b_0} f(a + r + b_0),$$

hence

$$\sum_{a \leqq k} f(a + r) \geqq \sum_{a \leqq b_0 - 1} f(a + r) + f(b_0 + r) + \sum_{a \leqq k - b_0} f(a + r + b_0).$$

The inequalities (39) are valid for $m = 1, \ldots, k - 1$, whence

$$\sum_{a' \leqq b_0 - 1} f(a' + r) + \sum_{b' \leqq b_0 - 1} f(b' + r) \geqq \gamma \sum_{h=1}^{b_0 - 1} f(h + r),$$

since this inequality is also valid for $b_0 = 1$. The proof is established, if we show

$$\sum_{a \leqq k - b_0} f(a + r + b_0) \geqq \gamma \sum_{h=1}^{k - b_0} f(h + r + b_0). \quad . \quad . \quad . \quad (43)$$

In fact, then we find, since A and A' consist of the same integers $\leqq k$,

$$\begin{aligned} \sum_{a' \leqq k} f(a' + r) + \sum_{b' \leqq k} f(b' + r) &\geqq \sum_{a \leqq k} f(a + r) + \sum_{b' \leqq b_0 - 1} f(b' + r) \\ &\geqq \gamma \sum_{h=1}^{b_0 - 1} f(h + r) + f(b_0 + r) + \gamma \sum_{h=1}^{k - b_0} f(h + r + b_0) \\ &\geqq \gamma \sum_{h=1}^{k} f(h + r), \end{aligned}$$

contrary to (41).

Lemma 5, applied with $n = 2$, $A_1 = A$ and $A_2 = B$, furnishes for $m = 1, \ldots, g$ and for each integer $t \geqq 0$

$$\sum_{a \leqq m} f(a + t) + \sum_{b \leqq m} f(b + t) \geqq \gamma \sum_{h=1}^{m} f(h + t).$$

I apply this inequality with $t = r + b_0$ for $m = 1, 2, \ldots, k - b_0$. In this

manner I find, that there exists at least one subset D of B with the properties

$$\sum_{a \leqq m} f(a+t) + \sum_{d \leqq m} f(d+t) \equiv \gamma \sum_{h=1}^{m} f(h+t) \qquad (m = 1, \ldots, k - b_0); \quad (44)$$

in fact $D = B$ possesses the required property. I choose the subset D of B with these properties, such that the number of elements of D is as small as possible. It is sufficient to prove that D does not contain any positive elements, for then $\sum_{d \leqq m} = 0$ and (43) follows from (44).

Let us suppose for a moment, that D contains at least one positive element. Just as in the proof of lemma 1, the couple A, D can be transformed into a couple C', D' by an elementary transformation, which cancels in D one positive number and does not add to A an integer $\leqq k - b_0$, so that $C'(m) = A(m)$ for $m = 1, \ldots, k - b_0$. Since the inequalities (44) are true for $m = 1, \ldots, k - b_0$, and the couple A, D is transformed into C', D' by an elementary transformation, we obtain by induction for $m = 1, \ldots, k - b_0$

$$\sum_{c' \leqq m} f(c'+t) + \sum_{d' \leqq m} f(d'+t) \equiv \gamma \sum_{h=1}^{m} f(h+t),$$

hence

$$\sum_{a \leqq m} f(a+t) + \sum_{d' \leqq m} f(d'+t) \equiv \gamma \sum_{h=1}^{m} f(h+t) \qquad (m = 1, \ldots, k - b_0).$$

These inequalities are impossible since D' is a subset of B and contains less elements than D. This proves the lemma.

Since an e-transformation is composed of a finite number of elementary transformations, lemma 6 gives immediately:

Lemma 7. *If the set A, containing zero, satisfies*

$$A(m) + B(m) \equiv \gamma \sum_{h=1}^{m} f(h) \qquad (m = 1, \ldots, g),$$

where $\gamma \leqq 1$, and if the couple A, B is transformed by the corresponding e-transformation into a couple A^*, B^*, then

$$A^*(m) + B^*(m) \equiv \gamma \sum_{h=1}^{m} f(h) \qquad (m = 1, \ldots, g). \quad . \quad . \quad . \quad (45)$$

The proof of theorem 7 runs as follows:

I may assume that B contains at least one positive element and I may suppose the theorem proved, if B is replaced by a system containing less elements. By an e-transformation we can transform the couple A, B into a couple A^*,B^*, which satisfies by lemma 7 the inequalities (45). Since B^* contains less elements than B, we have by induction

$$(A^* + B^*)(m) \equiv \gamma \sum_{h=1}^{m} f(h) \qquad (m = 1, \ldots, g),$$

wherefrom the assumption follows by lemma 3.

Mathematics. — *Sur les systèmes d'équations aux dérivées partielles, qui, comme les systèmes normaux, comportent autant d'équations que de fonctions inconnues* (quatre communication). By A. FINZI. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of March 29, 1947.)

§ X.

Equations de l'applicabilité des surfaces.

Nous allons utiliser les résultats obtenus pour étudier un système non normal remarquable; les équations de l'applicabilité des surfaces.

Nous retrouverons un théorème connu de géométrie différentielle.

Cette étude a déjà été faite, par une autre méthode, par HADAMARD [1]).

Les équations dont nous voulons parler ont, comme on sait, la forme

$$\begin{aligned} &\left(\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial y}{\partial u}\right)^2+\left(\frac{\partial z}{\partial u}\right)^2=E(u,v),\\ &\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}+\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}+\frac{\partial z}{\partial u}\frac{\partial z}{\partial v}=F(u,v),\\ &\left(\frac{\partial x}{\partial v}\right)^2+\left(\frac{\partial y}{\partial v}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2=G(u,v), \end{aligned} \tag{18}$$

x, y, z étant les coordonnées cartésiennes d'un point quelconque de la surface, qu'il s'agit de déterminer, et u et v sont les coordonnées de Gauss sur la même surface, dont le ds^2 est:

$$E du^2+2F du\,dv+G dv^2.$$

La caractéristique du JACOBIEN du système par rapport à $\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}$ est égale à deux, et on ne peut pas espérer réduire le système à la forme normale par un changement de variables puisque, comme il est facile de vérifier, il a un caractère invariantif.

A cause de ce caractère nous pouvons supposer dans notre note, que la courbe portant les données initiales est $u=0$.

Appliquant aux équations (18) la méthode exposée dans les § 1 et 2, on arrive au système d'ordre $3\cdot 1-1=2$:

$$\begin{aligned} &\left(\frac{\partial^2 x}{\partial u\,\partial v}\right)^2-\frac{\partial^2 x}{\partial u^2}\frac{\partial^2 x}{\partial v^2}+\left(\frac{\partial^2 y}{\partial u\,\partial v}\right)^2-\frac{\partial^2 y}{\partial u^2}\frac{\partial^2 y}{\partial v^2}+\left(\frac{\partial^2 z}{\partial u\,\partial v}\right)^2-\frac{\partial^2 z}{\partial u^2}\frac{\partial^2 z}{\partial v^2}=\\ &\qquad=\frac{1}{2}\left(\frac{\partial^2 E}{\partial v^2}-2\frac{\partial^2 F}{\partial u\,\partial v}+\frac{\partial^2 G}{\partial u^2}\right),\\ &\frac{\partial x}{\partial u}\frac{\partial^2 x}{\partial u^2}+\frac{\partial y}{\partial u}\frac{\partial^2 y}{\partial u^2}+\frac{\partial z}{\partial u}\frac{\partial^2 z}{\partial u^2}=\frac{1}{2}\frac{\partial E}{\partial u}, \end{aligned} \tag{19}$$

$$\frac{\partial x}{\partial u}\frac{\partial^2 x}{\partial u\,\partial v}+\frac{\partial x}{\partial v}\frac{\partial^2 x}{\partial u^2}+\frac{\partial y}{\partial u}\frac{\partial^2 y}{\partial u\,\partial v}+\frac{\partial y}{\partial v}\frac{\partial^2 y}{\partial u^2}+\frac{\partial z}{\partial u}\frac{\partial^2 z}{\partial u\,\partial v}+\frac{\partial z}{\partial v}\frac{\partial^2 z}{\partial u^2}=\frac{\partial F}{\partial u}.$$

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[1]) Bull. Soc. Math. France (1906), 48.

Les équations (19) forment un système normal puisque le déterminant des coefficients des dérivées secondes de x, y, z par rapport à u

$$\begin{vmatrix} \frac{\partial^2 x}{\partial v^2} & \frac{\partial^2 y}{\partial v^2} & \frac{\partial^2 z}{\partial v^2} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} \tag{20}$$

n'est pas identiquement nul.

Le système (18) admet donc une solution et une seule quand on se donne les trois fonctions $x(0, v)$, $y(0, v)$, $z(0, v)$ satisfaisant à la condition nécessaire constituée par la troisième équation (18).

Cela revient à dire que la forme de la surface est déterminée quand on se donne la courbe $u = 0$.

On retrouve donc le théorème connu: *Si sur une surface flexible on maintient une courbe rigide, la surface ne peut pas être déformée.*

Le cas d'exception se présente quand les valeurs initiales rendent nul le déterminant (20), où les dérivées par rapport à u, conformément à ce qui a été dit au § IX, peuvent s'exprimer en fonction des valeurs initiales elles-mêmes.

Quand le déterminant (20) s'annule on dit que la courbe $u = 0$ est asymptotique; dans ce cas les valeurs initiales doivent satisfaire, non seulement à une, mais à deux conditions. D'autre part, de telles valeurs ne déterminant plus la forme de la surface, il existe des déformations qui laissent rigide une asymptotique [2]).

§ XI.

Extension au cas le plus général des résultats de la présente recherche.

Les résultats des paragraphes précédents peuvent être étendus au cas où le déterminant Ω, d'ordre n, a la caractéristique $n - t$. Nous montrerons qu'il existe dans une telle hypothèse t combinaisons entre les dérivées d'ordre $h(n - t)$ des équations du système donné, dans lesquelles les coefficients des inconnues de l'ordre le plus élevé $h(n - t + 1)$ sont nuls.

De la matrice constituée par les $n - t$ premières lignes de Ω et par une des t suivantes, par exemple par la $n - t + 1$-ème, nous pouvons extraire un déterminant d'ordre $n - t + 1$, en associant la colonne j-ième ($j = 1, 2, \ldots, t$) aux $n - t$ dernières.

Le développement de ce déterminant montre qu'il s'agit d'une forme homogène en p de degré $h(n - t + 1)$ à coefficients identiquement nuls.

En supprimant les éléments appartenant à une quelconque des $n - t + 1$

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[2]) Cf. BIANCHI, Lezioni di geometria differenziale [Pisa 1894], pag. 199.

lignes de la même matrice, on tire ensuite des $n-t$ dernières colonnes un déterminant d'ordre $n-t$, qui est, à son tour, un polynome homogène en p de degré $h(n-t)$: désignons par

$$\Delta^{i}_{r_0 r_1 \ldots r_m} \quad (r_0 + r_1 + \ldots + r_m = h(n-t))$$

le coefficient de $p_0^{r_0} p_1^{r_1} \ldots p_m^{r_m}$ dans le déterminant obtenu en supprimant les éléments de la i-ième ligne $(i = 1, 2, \ldots, n-t+1)$.

Le coefficient de $p_0^{s_0} p_1^{s_1} \ldots p_m^{s_m}$ $(s_0 + s_1 + \ldots + s_m = (n-t+1)h)$ dans le déterminant d'ordre $n-t+1$, considéré plus haut, est donné, comme on voit en développant suivant les éléments de la j-ième colonne de Ω, par

$$\Sigma E^{ij}_{h_0 h_1 \ldots h_m} \Delta^{i}_{r_0 r_1 \ldots r_m},$$

la sommation étant étendue à toutes les valeurs des h et des r pour lesquelles

$$h_0 + r_0 = s_0; \; h_1 + r_1 = s_1; \; h_m + r_m = s_m, \tag{7'}$$

alors que l'indice i varie de 1 à $n-t+1$.

On aura donc

$$\Sigma E^{ij}_{h_0 h_1 \ldots h_m} \Delta^{i}_{r_0 r_1 \ldots r_m} = 0 \qquad (j = 1, 2, \ldots, t). \tag{21}$$

Formons maintenant la combinaison

$$\Phi_1^{(1)} = \Sigma \Delta^{i}_{r_0 r_1 \ldots r_m} \frac{\partial^{h(n-t)} F_i}{\partial x^{0 r_0} \partial x^{1 r_1} \ldots \partial x^{m r_m}}.$$

Dans une telle combinaison le coefficient d'une dérivée de l'ordre le plus élevé $\dfrac{\partial^{h(n-t+1)} \varphi_j}{\partial x^{0 s_0} \partial x^{1 s_1} \ldots \partial x^{m s_m}}$ est donné — on le reconnaît comme au paragraphe I — par

$$\Sigma E^{ij}_{h_0 h_1 \ldots h_m} \Delta^{i}_{r_0 r_1 \ldots r_m},$$

la sommation étant étendue aux valeurs des h et des r qui vérifient (7'). Si $j = 1, 2, \ldots, t$, une telle expression est nulle à cause de (21): donc les coefficients des dérivées d'ordre $h(n-t+1)$ des $\varphi_1, \varphi_2, \ldots, \varphi_t$ sont nuls.

Si ensuite $j > t$ l'expression est également nulle puisqu'elle coïncide avec le coefficient de $p_0^{s_0} p_1^{s_1} \ldots p_m^{s_m}$ dans un déterminant qui a deux lignes identiques et qui est par suite identiquement nul.

Les dérivées de l'ordre le plus élevé de toutes les inconnues φ sont donc éliminées.

En associant aux $n-t$ premières lignes, au lieu de la $(n-t+1)$-ième, chacune de celles qui restent, on obtient en somme $t-1$ autres combinaisons

$$\Phi_1^{(2)} = 0; \; \Phi_1^{(3)} = 0; \ldots; \; \Phi_1^{(t)} = 0,$$

jouissant de la même propriété.

Nous pouvons maintenant construire un second système d'ordre $(n-t+1)h-1=h^{(1)}$

$$\frac{\partial^{h^{(1)}-h} F_i}{\partial x^{0^{h^{(1)}-h}}} = 0 \qquad (i=t+1, t+2, \ldots, n),$$

$$\Phi_1^{(i)} = 0 \qquad (i=1, 2, \ldots, t).$$

Si ce second système est normal ou si, en répétant un certain nombre de fois les opérations indiquées, on parvient à construire un système normal, on peut établir pour le cas actuel des considérations analogues à celles des paragraphes II, III, V.

Dans le cas contraire il devra exister entre les F, les Φ et leurs dérivées un certain nombre de relations identiques, d'une manière analogue au cas $t=1$.

Pour considérer un cas typique supposons qu'il existe entre les F, les Φ et leurs dérivées jusqu'à l'ordre h^*, t relations d'identité

$$\Psi_r(F_1, F_2, \ldots, F_n, \ldots, \Phi, \ldots, x)=0 \qquad (r=1, 2, \ldots, t).$$

Supposons que la matrice formée à l'aide des $n-t$ dernières lignes de Ω ait la caractéristique $n-t$. Nous pouvons construire t expressions différentielles en φ d'ordre $h, Z_1, Z_2, \ldots, Z_t$, telles que le système

$$\begin{aligned} &F_i^* = F_i + Z_i = 0 \qquad (i=1, 2, \ldots, t) \\ &F_i = 0 \qquad\qquad\quad (t=t+1, t+2, \ldots, n) \end{aligned} \tag{13'}$$

soit normal.

Supposons encore que le système en Z_i d'ordre h^*

$$\Psi_r(F_1^*, F_2^*, \ldots, F_t^*, F_{t+1}, \ldots, F_n, \ldots, \Phi, \ldots, x)=0 \qquad (r=1, 2, \ldots, t)$$

soit normal.

Ce système admet certainement la solution nulle. Quand la solution du système normal (13') rend initialement égaux à zéro les Z et leurs h^*-1 premières dérivées, on aura donc partout

$$Z_i = 0 \qquad (i=1, 2, \ldots, t) \tag{14'}$$

et la solution du système normal (12') vérifiera aussi le système donné.

§ XII.

Equations de la gravitation d'Einstein.

Les considérations du paragraphe précédent trouvent une application dans l'étude des solutions des *équations de la gravitation d'Einstein*. Ce sont dix équations du second ordre

$$E_{ik} \equiv G_{ik} - \tfrac{1}{2} G g_{ik} = -\varkappa T_{ik} \tag{22}$$

dans les dix inconnues g_{ik}, coefficients de la forme quadratique, qui définit la métrique de l'espace-temps.

Les G_{ik} sont des expressions formées avec les g_{ik} et leurs dérivées premières et secondes, linéaires par rapport à ces dernières; elles constituent le tenseur de RICCI–EINSTEIN; la combinaison

$$\Sigma g^{ik} G_{ik} \equiv G$$

(où les g^{ik} sont les réciproques des g_{ik} dans le déterminant $\| g_{ik} \|$) en constitue l'invariant linéaire; $\varkappa$ est la constante universelle de la gravitation et les T_{ik} sont les composantes du tenseur énergétique, lesquelles, dans le cas le plus simple, doivent être considérées comme des fonctions données de la place.

Les $E_{ik} = G_{ik} - \frac{1}{2} G g_{ik}$ satisfont quatre relations différentielles identiques du premier ordre

$$\sum_{k=0}^{3} E_{ik}^{|k} = 0 \qquad (i = 0, 1, 2, 3), \tag{23}$$

en désignant par le symbole $|^k$ la dérivation contravariante.

A cause de ces relations le système (22) n'est pas normal et la caractéristique du déterminant Ω est égale, en général, à $10 - 4 = 6$.

On déduit facilement des (23) que dans les quatre combinaisons

$$\sum_{0}^{3} {}_k p^k (E_{ik} + \varkappa T_{ik}) = 0 \qquad (i = 0, 1, 2, 3) \tag{24}$$

les dérivées secondes par rapport à $z^0(x^0, x^1, x^2, x^3)$ sont éliminées [3]), de façon que ces combinaisions constituent certainement, pour les valeurs initiales, quatre conditions nécessaires pour la résolubilité des (22). Il faut maintenant distinguer deux cas: $T_{ik} \neq 0$ et $T_{ik} = 0$ $(i,\ k = 0, 1, 2, 3)$.

Dans le premier cas on peut déduire des (22) le système normal du second ordre

$$\begin{aligned} \Sigma_k \frac{\partial}{\partial x^0} (E_{ik} + \varkappa T_{ik})^{|k} &= 0 \qquad (i = 0, 1, 2, 3) \\ G_{ik} + \varkappa \tau_{ik} &= 0 \qquad (i, k = 1, 2, 3) \end{aligned} \tag{25}$$

(avec $\tau_{ik} = (T_{ik} - \frac{1}{2} g_{ik} T)$) de façon que les (22) rentrent (et précisement avec $l = 2$), dans le premier des deux types fondamentaux dans lesquels se partagent les systèmes non normaux.

On peut démontrer que la solution des (25) vérifie aussi les équations de la gravitation (22) lorsque les valeurs initiales des g_{ik} et leurs premières dérivées satisfont à huit condition convenables: savoir les (24) et, en outre les

$$\sum_{k=0}^{3} T_{ik}^{|k} = 0 \qquad (i = 0, 1, 2, 3). \tag{26}$$

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[3]) Cfr. LEVI-CIVITA, [Rendiconti dell'Accad. dei Lincei **11** (1930), 113—✳].

C'était d'ailleurs à prévoir, d'après la règle donnée au § VIII, que la connaissance des valeurs initiales des inconnues et de leurs premières dérivées était suffisante pour déterminer la solution des (22).

Lorsque au contraire $T_{ik} = 0$, les équations de la gravitation rentrent dans le second des deux types fondamentaux, puisque (23) constituent précisément quatre conditions différentielles identiques entre les (22). On pourrait démontrer que dans ce cas les quatre conditions nécessaires (24) sont aussi suffisantes pour la résolubilité des équations (22).

Naturellement, la solution du système n'est plus univoquement déterminée par la connaissance des valeurs initiales.

Mathematics. — *On the theory of linear integral equations.* VII. By A. C. ZAANEN. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of March 29, 1947.)

§ 1. *Introduction.*

In a series of six papers which were published in these Proceedings [1]), and will be denoted by I—VI in what follows, we have discussed the theory of linear integral equations with "symmetrisable" kernels. In another series, of which the present paper is the first one, we shall treat the case of systems of linear integral equations with "symmetrisable matrix-kernels". We shall prove, in fact, that the theory of completely continuous, symmetrisable transformations in a HILBERT space, expounded in I, is sufficiently general so as to have this case as one of its applications.

Before entering into details, we shortly recall some definitions, given already in II: Let $a_p, b_p\,(p=1,\ldots,m)$ be real numbers, such that $a_p<b_p\,(p=1,\ldots,m)$. Then $\Delta=[a_1,b_1;\ldots;a_m,b_m]$ is an interval in m-dimensional Euclidean space. The point $(x_1,\ldots,x_m)$ in this space will be denoted by x, and the set of all complex-valued functions $f(x)$, for which $|f(x)|^2$ is summable (in the sense of LEBESGUE) over Δ, by $L_2^{(m)}(\Delta)$ or $L_2(\Delta)$ or shortly by L_2. The interval $[a_1,b_1;\ldots;a_m,b_m;a_1,b_1;\ldots;a_m,b_m]$ in $2m$-dimensional Euclidean space will be denoted by $\Delta\times\Delta$, and the set of all complex-valued functions $f(x,y)$ $(x,y\in\Delta)$, for which $|f(x,y)|^2$ is summable over $\Delta\times\Delta$, by $L_2^{(2m)}(\Delta)$. Let now $n\geqq 1$, $K_{ij}(x,y)\in L_2^{(2m)}(\Delta)$ $(i,j=1,\ldots,n)$, $f^i(x)$ and $g^i(x)\in L_2^{(m)}(\Delta)$ $(i=1,\ldots,n)$ [2]), λ a complex parameter. We consider the system of linear integral equations

$$\Sigma_{j=1}^{n}\int_{\Delta}K_{ij}(x,y)\,f^j(y)\,dy-\lambda f^i(x)=g^i(x)\qquad(i=1,\ldots,n),\qquad(1)$$

where $K_{ij}(x,y)$ and $g^i(x)$ are supposed to be given. We shall say that the system (1) has the "matrix-kernel" $\|K_{ij}(x,y)\|$. If the conjugate complex number of any complex number a is denoted by $\bar{a}$, the adjoint matrix-kernel $\|K^*_{ij}(x,y)\|$ of $\|K_{ij}(x,y)\|$ is defined by

$$\|K^*_{ij}(x,y)\|=\|\overline{K_{ji}(y,x)}\|.$$

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1) A. C. ZAANEN, On the theory of linear integral equations, Proc. Kon. Akad. v. Wetensch., Amsterdam, **49**, I 194—204, II 205—212, III 292—301, IV 409—423, V 571—585, VI 608—621 (1946). (Indagationes Mathematicae VIII 91—101, 102—109, 161—170, 264—278, 352—366, 367—380).

2) The letters i and j in expressions such as $f^i(x)$ or $g^j(x)$ are no exponents but always indices.

Several cases will be treated:

1°. The matrix-kernel $\|K_{ij}\|$ is Hermitian, that is $\|K_{ij}\| = \|K_{ij}^*\|$, in other words, $K_{ij}(x,y) = \overline{K_{ji}(y,x)}$ almost everywhere in $\Delta \times \Delta$. This is the analogue of the Hermitian case treated for $n=1$ in II.

2°. The matrix-kernel $\|K_{ij}(x,y)\|$ is the matrix-product of the matrix-kernel $\|A_{ij}(x,y)\|$ and the matrix $\|h_{ij}(y)\|$, where the following conditions are satisfied:

a. All $h_{ij}(y)$ are bounded and measurable in Δ,

b. $h_{ij}(y) = \overline{h_{ji}(y)}$ and $\sum_{i,j=1}^{n} h_{ij}(y)\,\bar{a}_i\,a_j \geqq 0$ for every system of complex numbers a_i and every $y \in \Delta$; the matrix $\|h_{ij}(y)\|$ is therefore Hermitian and of positive type,

c. All $A_{ij}(x,y) \in L_2^{(2m)}(\Delta)$,

d. $\|h_{ij}(x)\| \cdot \|K_{ij}(x,y)\| = \|K_{ij}^*(x,y)\| \cdot \|h_{ij}(y)\|$, in other words $\|h_{ij}(x)\| \cdot \|A_{ij}(x,y)\| \cdot \|h_{ij}(y)\| = \|h_{ij}(x)\| \cdot \|A_{ij}^*(x,y)\| \cdot \|h_{ij}(y)\|$.

This condition is automatically satisfied if $\|A_{ij}(x,y)\|$ is Hermitian.

This case is the analogue of that which we treated for $n=1$ in III. For general $n \geqq 1$ it was considered by J. ERNEST WILKINS in 1944 [3]), who, however, besides the already mentioned conditions, supposed all $h_{ij}(y)$ to be continuous in Δ, and all $A_{ij}(x,y)$ to be bounded in $\Delta \times \Delta$ with their discontinuities "regularly distributed". We shall free ourselves from these limitations, and, moreover, prove a great deal more.

3°. The matrix-kernel $\|K_{ij}(x,y)\|$ (all $K_{ij}(x,y) \in L_2^{(2m)}(\Delta)$) has the property that there exists an Hermitian matrix-kernel $\|H_{ij}(x,y)\|$ (all $H_{ij}(x,y) \in L_2^{(2m)}(\Delta)$) of positive type such that the matrix-product

$$\|P_{ij}\| = \|H_{ij}\| \cdot \|K_{ij}\|$$

is Hermitian. This is the analogue of the case of the Marty-kernel treated in IV. Several specializations, corresponding with the cases treated in V and VI, will also be considered.

In all these cases we shall show that the characteristic values of (1), if any exist, are real, that under very general conditions this system has at least one characteristic value $\lambda \neq 0$, and that the characteristic values possess a remarkable maximum-minimum property. Furthermore we shall prove some expansion theorems for functions of the form $\int_{\Delta} K_{ij}(x,y)\,f(y)\,dy$, as well as for the elements of the matrix-kernel $\|K_{ij}(x,y)\|$ themselves.

In the present paper some general theorems will be proved, and the case of an Hermitian matrix-kernel will be treated. A knowledge of the contents of the papers I and II is assumed.

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[3]) J. ERNEST WILKINS, Definitely self-conjugate adjoint integral equations, Duke Mathematical Journal 11, 155—166 (1944).

§ 2. *Some general theorems.*

Let R be a complete (not necessarily separable) HILBERT space. If the elements of R are denoted by $f, g, \ldots$, the scalar product of f and g by (f, g), and if λ is an arbitrary complex number, it is well-known that the set R^n of all elements $\{f\} = \{f^1, \ldots, f^n\}$, when the fundamental operations and the scalar product in it are defined by

$$\{f\} + \{g\} = \{f^1 + g^1, \ldots, f^n + g^n\}, \lambda\{f\} = \{\lambda f^1, \ldots, \lambda f^n\}, (\{f\}, \{g\}) = \sum_{i=1}^{n} (f^i, g^i),$$

is also a complete HILBERT space.

We suppose that K_{ij} $(i, j = 1, \ldots n)$ are linear transformations, defined for every $f \in R$. Then, by

$$g^i = \sum_{j=1}^{n} K_{ij} f^j \qquad (i = 1, \ldots, n), \quad . \quad . \quad . \quad . \quad . \quad (2)$$

a linear transformation $\{g\} = K\{f\}$ in R^n is determined.

Theorem 1. *K is bounded in R^n if and only if all K_{ij} are bounded in R.*

Proof. 1°. Let $\|K_{ij}\| \leqq M$ $(i, j = 1, \ldots, n)$. Then

$$\|\{g\}\|^2 = \sum_{i=1}^{n} \|g^i\|^2 = \sum_{i=1}^{n} \|\sum_{j=1}^{n} K_{ij} f^j\|^2 \leqq$$
$$\sum_{i=1}^{n} (\sum_{j=1}^{n} M \|f^j\|)^2 \leqq \sum_{i=1}^{n} (Mn \|\{f\}\|)^2 = M^2 n^3 \|\{f\}\|^2,$$

hence $\|K\| \leqq M n^{3/2}$.

2°. Let $\|K_{ij}\| \leqq M$, and suppose that for one (at least) of the transformations K_{ij} there exists an element $f_0 \in R$ such that $\|K_{ij} f_0\| > M \|f_0\|$. We define $\{f_0\} = \{f_0^1, \ldots, f_0^n\}$ by $f_0^j = f_0$ and $f_0^k = 0$ for $k \neq j$, hence $\|\{f_0\}\|^2 = \sum_{i=1}^{n} \|f_0^i\|^2 = \|f_0\|^2$. Then, writing $\{g_0\} = K\{f_0\}$, we have

$$\|K\{f_0\}\|^2 = \sum_{i=1}^{n} \|g_0^i\|^2 \geqq \|g_0^i\|^2 = \|K_{ij} f_0^j\|^2 =$$
$$\|K_{ij} f_0\|^2 > M^2 \|f_0\|^2 = M^2 \|\{f_0\}\|^2,$$

in contradiction with $\|K\| \leqq M$. Hence $\|K_{ij}\| \leqq M$ $(i, j = 1, \ldots, n)$.

Theorem 2. *$K \neq O$ (the nulltransformation) in R_n if and only if one at least of the transformations $K_{ij} \neq O$ in R.*

Proof. It is trivial that $K \neq O$ implies $K_{ij} \neq O$ for one at least of the transformations K_{ij}. If, conversely, $K_{ij} \neq O$ for a certain set of indices i, j, we define $\{f\} = \{f^1, \ldots, f^n\}$ by $f^j = f$, where $K_{ij} f \neq 0$, and $f^k = 0$ for $k \neq j$. Then, when $\{g\} = K\{f\}$, we see that $g^i = K_{ij} f \neq 0$, hence $K \neq O$.

As well-known, the bounded linear transformation T, defined for all elements of a HILBERT space, is said to be completely continuous when every bounded infinite set of elements contains a sequence f_k such that the sequence $T f_k$ converges.

Theorem 3. *K is completely continuous in R^n if and only if all K_{ij} are completely continuous in R.*

Proof. 1°. Let $K_{ij}\,(i,j=1,\ldots,n)$ be completely continuous. Let, furthermore, $\{f\}$ be an infinite set of elements in R^n, such that $\|\{f\}\|^2=\sum_{i=1}^n\|f^i\|^2\leqq M$ for all $\{f\}$. Then this set contains a sequence $\{f_l\}$ $(l=1,2,\ldots)$ such that $K_{11}f_l^1$ converges; the sequence $\{f_l\}$ in its turn contains a subsequence $\{f_m\}$ $(m=l_1,l_2,\ldots)$ such that $K_{12}f_m^2$ converges, and so on. In this way we obtain a subsequence $\{f_r\}$ such that

$$\lim_{r\to\infty}K_{ij}f_r^j=g^{ij}.$$

Writing $\sum_{j=1}^n g^{ij}=g^i$ and $\{g\}=\{g^1,\ldots,g^n\}$ we have $\lim_{r\to\infty}K\{f_r\}=\{g\}$, which shows that K is completely continuous.

2°. Let K be completely continuous. Let, furthermore, the infinite set of elements $f\in R$ be given, where $\|f\|^2\leqq M$ for all f. Defining the corresponding set of elements $\{f\}=\{f^1,\ldots,f^n\}\in R^n$ by $f^j=f$ and $f^k=0$ for $k\neq j$, we have $\|\{f\}\|^2=\|f^j\|^2=\|f\|^2\leqq M$ for all f. The set $\{f\}$ contains therefore a sequence $\{f_r\}$ such that $\{g_r\}=K\{f_r\}$ converges in R^n. Then $g_r^i=K_{ij}f_r^j=K_{ij}f_r$ converges in R, which shows that $K_{ij}\,(i,j=1,\ldots,n)$ is completely continuous.

We shall consider only bounded linear transformations K of the form (2) in R^n. Then the adjoint transformation K^* exists, and a simple computation shows that K^* is of the same form, and that

$$(K^*)_{ij}=(K_{ji})^*.$$

It follows that K is self-adjoint if and only if $K_{ij}=(K_{ji})^*$ $(i,j=1,\ldots,n)$. We observe that if the bounded linear transformations K and L in R^n are of the form (2), the same holds for LK, and

$$(LK)_{ij}=\sum_{k=1}^n L_{ik}K_{kj}.$$

To be able to mention some concrete applications of the foregoing theorems, we recall the well-known fact that the set $L_2^{(m)}(\triangle)$ of functions, defined in § 1, is a HILBERT space if the fundamental operations are defined in the usual way, and the scalar product of f and g as

$$(f,g)=\int_\triangle f(x)\,\overline{g(x)}\,dx,$$

so that

$$\|f\|=\left(\int_\triangle|f|^2\,dx\right)^{1/2}.$$

Convergence of $\sum_{k=1}^\infty f_k(x)$ to $f(x)$ in this space means that $\lim_{r\to\infty}\|f-\sum_{k=1}^r f_k\|=0$; to avoid any misunderstandings, we shall say that $\sum_{k=1}^\infty f_k(x)$ converges in mean to $f(x)$, and write

$$f(x)\backsim\sum_{k=1}^\infty f_k(x),$$

reserving the term convergence for ordinary point-wise convergence.

In the same way the set $L_2^{(2m)}(\triangle)$ may be considered as a HILBERT space with scalar product

$$(f, g)_{2m} = \int_{\triangle\times\triangle} f(x, y)\, \overline{g(x, y)}\, dx\, dy.$$

The following lemma's we suppose to be known:

Lemma 1. *If, for $x \in \triangle$, $A(x)$ is a complex-valued, measurable function, the transformation A, defined by*

$$Af = A(x) f(x),$$

is a bounded linear transformation in $L_2^{(m)}(\triangle)$ if and only if $A(x)$ is bounded on $\triangle$. In this case the adjoint A^* is given by

$$A^* f = \overline{A(x)}\, f(x);$$

we have $A \neq O$ (the nulltransformation) if and only if $A(x) \neq 0$ on a set of positive measure; A is self-adjoint if and only if $A(x)$ has only real values (almost everywhere on $\triangle$). Supposing A to be bounded and self-adjoint, it is of positive type if and only if $A(x)$ is non-negative (almost everywhere on $\triangle$).

Lemma 2. *If, for $(x, y) \in \triangle\times\triangle$, $A(x, y)$ is a complex-valued, measurable function, such that*

$$\int_{\triangle\times\triangle} |A(x, y)|^2\, dx\, dy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

is finite (in other words, $A(x, y) \in L_2^{(2m)}(\triangle)$) the linear "integral transformation" A in $L_2^{(m)}(\triangle)$, defined by

$$Af = \int_{\triangle} A(x, y) f(y)\, dy,$$

is completely continuous. The adjoint A^* is given by

$$A^* f = \int_{\triangle} \overline{A(y, x)}\, f(y)\, dy.$$

We have $A \neq O$ if and only if the integral (3) does not vanish, that is, if and only if $A(x, y) \neq 0$ on a set of positive measure in $\triangle\times\triangle$. A is self-adjoint if and only if the "kernel" $A(x, y)$ is Hermitian, that is, if and only if $A(x, y) = \overline{A(y, x)}$ almost everywhere in $\triangle\times\triangle$. Supposing A to be self-adjoint, it is of positive type if and only if

$$\int_{\triangle\times\triangle} A(x, y)\, \overline{f(x)}\, f(y)\, dx\, dy \geqq 0$$

for arbitrary $f(x) \in L_2^{(m)}(\triangle)$.

The following theorems in the HILBERT space $[L_2^{(m)}(\triangle)]^n$ of all elements $\{f\} = \{f^1(x), \ldots, f^n(x)\}$, where $f^i(x) \in L_2^{(m)}(\triangle)$ $(i = 1, \ldots, n)$, are now immediate consequences of the foregoing theorems and lemma's:

Theorem 4. *If, for $x \in \triangle$, the functions $A_{ij}(x)$ $(i, j = 1, \ldots, n)$ are complex-valued, measurable functions, the transformation A, defined by $\{g\} = A\{f\}$, where*

$$g^i(x) = \Sigma_{j=1}^n A_{ij}(x) f^j(x) \qquad (i = 1, \ldots, n),$$

is a bounded linear transformation in $[L_2^{(m)}(\triangle)]^n$ if and only if all functions $A_{ij}(x)$ are bounded on $\triangle$. In this case the adjoint $\{h\} = A^*\{f\}$ is given by

$$h^i(x) = \Sigma_{j=1}^n \overline{A_{ji}(x)} f^j(x) \qquad (i = 1, \ldots, n);$$

we have $A \neq O$ if and only if one at least of the functions $A_{ij}(x) \neq 0$ on a set of positive measure; A is self-adjoint if and only if the matrix $\|A_{ij}(x)\|$ is Hermitian, that is, if and only if $A_{ij}(x) = \overline{A_{ji}(x)}$ for almost every $x \in \triangle$, and, supposing A to be bounded and self-adjoint, it is of positive type if and only if

$$\Sigma_{i,j=1}^n A_{ij}(x) \overline{f^i(x)} f^j(x) \geqq 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

for arbitrary $\{f\} \in [L_2]^n$ and for almost every $x \in \triangle$.

Proof. It is only the last part of the theorem that needs a proof. The statement that the bounded self-adjoint transformation is of positive type means by definition that

$$\Sigma_{i,j=1}^n \int_{\triangle} A_{ij}(x) \overline{f^i(x)} f^j(x)\, dx = (A\{f\}, \{f\}) \geqq 0$$

for every $\{f\} \in [L_2]^n$; this however is equivalent with (4).

Remark. The condition (4) is certainly satisfied if

$$\Sigma_{i,j=1}^n A_{ij}(x) \overline{a_i}\, a_j \geqq 0$$

for every system of complex numbers $a_1, \ldots, a_n$ and every $x \in \triangle$.

Theorem 5. *If, for $(x, y) \in \triangle \times \triangle$, the functions $A_{ij}(x, y)$ $(i, j = 1, \ldots n)$ are complex-valued and measurable, and if the integrals*

$$\int_{\triangle \times \triangle} |A_{ij}(x, y)|^2\, dx\, dy \qquad (i, j = 1, \ldots, n) \quad . \quad . \quad . \quad . \quad (5)$$

are finite (in other words, if $A_{ij}(x, y) \in L_2^{(2m)}(\triangle)$), the linear "integral transformation" A in $[L_2^{(m)}(\triangle)]^n$, defined by $\{g\} = A\{f\}$, where

$$g^i(x) = \Sigma_{j=1}^n \int_{\triangle} A_{ij}(x, y) f^j(y)\, dy \qquad (i = 1, \ldots, n),$$

is completely continuous. The adjoint $\{h\} = A^*\{f\}$ *is given by*

$$h^i(x) = \Sigma_{j=1}^n \int_\Delta \overline{A_{ji}(y,x)}\, f^j(y)\, dy \qquad (i = 1, \ldots, n).$$

We have $A \neq O$ *if and only if one at least of the integrals* (5) *does not vanish, that is, if and only if one at least of the functions* $A_{ij}(x,y) \neq 0$ *on a set of positive measure in* $\Delta \times \Delta$. *A is self-adjoint if and only if the "matrix-kernel"* $\|A_{ij}(x,y)\|$ *is Hermitian, that is, if and only if* $A_{ij}(x,y) = \overline{A_{ji}(y,x)}$ *almost everywhere in* $\Delta \times \Delta$. *Supposing A to be self-adjoint, it is of positive type if and only if*

$$\Sigma_{i,j=1}^n \int_{\Delta\times\Delta} A_{ij}(x,y)\, \overline{f^i(x)}\, f^j(y)\, dx\, dy \geqq 0$$

for arbitrary $\{f\} \in [L_2]^n$.

Remark. We shall say that the transformation A in Theorem 5 is "determined" by the matrix-kernel $\|A_{ij}(x,y)\|$.

§ 3. *Equivalence of the integral transformation with matrix-kernel with an ordinary integral transformation.*

The method of considering

$$g^i(x) = \Sigma_{j=1}^n \int_\Delta A_{ij}(x,y)\, f^j(y)\, dy \qquad (i = 1, \ldots, n) \quad . \quad . \quad (6)$$

as a linear transformation in $[L_2^{(m)}(\Delta)]^n$ is the abstract analogue of a well-known device, used already by I. FREDHOLM in his fundamental paper on Integral Equations [4]). Supposing, for a moment, that Δ is the linear interval $0 \leqq x \leqq 1$ (hence $m = 1$), we define the functions $f(x)$ and $g(x)$ in $0 \leqq x \leqq n$ by

$f(0) = f^1(0)$, $f(x) = f^i(x-i+1)$ for $i-1 < x \leqq i$ $\qquad (i = 1, \ldots, n)$,

$g(0) = g^1(0)$, $g(x) = g^i(x-i+1)$ for $i-1 < x \leqq i$ $\qquad (i = 1, \ldots, n)$,

and the kernel $A(x,y)$ in $0 \leqq x, y \leqq n$ by

$$A(x,y) = A_{ij}(x-i+1, y-j+1)$$
$$\text{for } i-1 < x \leqq i, \quad j-1 < y \leqq j \qquad (i,j = 1, \ldots, n),$$
$$A(x,y) = 0 \text{ in all other points.}$$

Denoting now the interval $0 \leqq x \leqq n$ by $n\Delta$, the integral transformation (6) with a matrix-kernel is equivalent with the ordinary integral transformation

$$g(x) = \int_{n\Delta} A(x,y)\, f(y)\, dy$$

in $L_2(n\Delta)$.

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[4]) I. FREDHOLM, Sur une classe d'équations fonctionnelles, Acta Math, 27, 365—390 (1903).

For $m > 1$ we may, clearly, proceed in a similar way.

Furthermore, it is evident that the kernel $A(x, y)$ in $n\Delta$ is Hermitian, that is, $A(x, y) = \overline{A(y, x)}$ almost everywhere in $n\Delta \times n\Delta$, if and only if the matrix $\|A_{ij}(x, y)\|$ is Hermitian.

Both methods of considering (6) have their advantages in the theory of systems of linear integral equations. The simpler of the two, described in the present paragraph, will suffice in the case of a system with Hermitian matrix-kernel, but, in the more complicated cases, the other method will be essential to obtain quick results.

§ 4. *System of linear integral equations with Hermitian matrix-kernel.* We consider the system of linear integral equations

$$\Sigma_{j=1}^{n} \int_{\Delta} K_{ij}(x, y) f^j(y)\, dy - \lambda f^i(x) = g^i(x) \qquad (i = 1, \ldots, n), \quad . \quad (1)$$

where $K_{ij}(x, y) \in L_2^{(2m)}(\Delta)$ $(i, j = 1, \ldots, n)$, $\{f\}$ and $\{g\} \in [L_2^{(m)}(\Delta)]^n$, the matrix-kernel $\|K_{ij}(x, y)\|$ is Hermitian, and λ is a complex parameter. As well-known, every value of λ for which the system (1), with $g^i(x) \equiv 0$ $(i = 1, \ldots, n)$, has a solution $\{f\}$ which does not vanish identically, is called a characteristic value. The corresponding solution $\{f\}$ we shall call a characteristic functionset.

The equivalence of (1) with the ordinary linear integral equation

$$\int_{n\Delta} K(x, y) f(y)\, dy - \lambda f(x) = g(x),$$

where the Hermitian kernel $K(x, y)$ and the functions $f(x)$ and $g(x)$ are defined as described in § 3, enables us, on account of $K(x, y) \in L_2^{(2m)}(n\Delta)$, $f(x)$ and $g(x) \in L_2^{(m)}(n\Delta)$, to announce the following theorems:

Theorem 6. *Every characteric value $\lambda \neq 0$ of (1) has finite multiplicity. The total number of different characteristic values λ_k is finite or enumerable, and in the latter case $\lim_{k \to \infty} \lambda_k = 0$.*

Proof. Follows from II, Theorem 1.

Theorem 7. *The characteristic values of (1) are real, and characteristic functionsets, belonging to different characteristic values, are orthogonal, that is, if $\varphi_k^i(x)$ belongs to the characteristic value λ_k, $\varphi_l^i(x)$ belongs to λ_l, and $\lambda_k \neq \lambda_l$, then*

$$\Sigma_{i=1}^{n} \int_{\Delta} \varphi_k^i(x)\, \overline{\varphi_l^i(x)}\, dx = 0.$$

Proof. Follows from II, Theorem 2.

Theorem 8. *If $\lambda \neq 0$ is a characteristic value of (1), this system of equations has, for a given $\{g\} \in [L_2]^n$, a solution $\{f\} \in [L_2]^n$ for those*

and only those $\{g\} \in [L_2]^n$ *that are orthogonal to all characteristic functionsets, belonging to the characteristic value* λ. *If* $\lambda \neq 0$ *is no characteristic value of* (1), *the system* (1) *has a uniquely determined solution for every* $\{g\} \in [L_2]^n$.

Proof. Follows from II, Theorem 3.

Theorem 9. *If*

$$\| K_{ij}(x, y) \|_{2m}^2 = \int_{\Delta \times \Delta} | K_{ij}(x, y) |^2 \, dx \, dy \neq 0$$

for at least one of the kernels $K_{ij}(x, y)$, *then the system* (1) *has a characteristic value* $\neq 0$.

Proof. Since

$$\int_{n\Delta \times n\Delta} | K(x, y) |^2 \, dx \, dy = \Sigma_{i,j=1}^{n} \int_{\Delta \times \Delta} | K_{ij}(x, y) |^2 \, dx \, dy,$$

our theorem is a consequence from II, Theorem 4.

Let now $\lambda_k \, (|\lambda_1| \geqq |\lambda_2| \geqq \ldots)$ be the sequence of all characteristic values $\neq 0$, each of them occurring in this sequence as many times as denoted by its multiplicity, and $\varphi_k^i(x)$ $(i = 1, \ldots, n)$ a corresponding orthonormal sequence of characteristic functionsets. We have therefore

$$\Sigma_{l=1}^{n} \int_{\Delta} \varphi_k^l(x) \, \overline{\varphi_l^l(x)} \, dx = \begin{cases} 1 & \text{for } k = l, \\ 0 & \text{for } k \neq l. \end{cases}$$

Theorem 10. *If*

$$a_k^j = \int_{\Delta} f(x) \, \overline{\varphi_k^j(x)} \, dx \qquad (j = 1, \ldots, n;\ k = 1, 2, \ldots),$$

then

$$\int_{\Delta} K_{ij}(x, y) \, f(y) \, dy \backsim \Sigma_k \lambda_k \, a_k^j \, \varphi_k^i(x) \quad \textit{(Expansion Theorem)} \qquad (7)$$

$$\int_{\Delta \times \Delta} K_{ij}(x, y) \, \overline{f(x)} \, f(y) \, dx \, dy = \Sigma_k \lambda_k \, \overline{a_k^i} \, a_k^j \quad . \quad . \quad . \quad . \qquad (8)$$

for any $f(x) \in L_2^{(m)}(\Delta)$.

Proof. Supposing, for a moment, that Δ is the linear interval $[0, 1]$, we define $g(x)$ in $n\Delta = [0, n]$ by

$$g(x) = f(x - j + 1) \quad \text{for} \quad j - 1 < x \leqq j,$$
$$g(x) = 0 \qquad\qquad\quad \text{elsewhere,}$$

while $\varphi_k(x)$ is defined in $n\Delta$ by

$$\varphi_k(0) = 0, \quad \varphi_k(x) = \varphi_k^i(x - i + 1) \text{ for } i - 1 < x \leqq i \qquad (i = 1, \ldots, n).$$

Then, if $a_k = \int_{n\Delta} g(x)\,\overline{\varphi_k(x)}\,dx$, we have by II, Theorem 5

$$\int_{n\Delta} K(x, y)\, g(y)\, dy \backsim \Sigma_k \lambda_k a_k \varphi_k(x),$$

from which, since by our definition of $g(x)$

$$a_k = \int_{\Delta} f(x)\,\overline{\varphi_k^l(x)}\,dx = a_k^l,$$

the relation (7) follows. (8) is an easy consequence of (7).

If Δ is not the interval $[0, 1]$, we proceed in a similar way.

Theorem 11. *Let λ_{k_l} $(l = 1, 2, \ldots)$ be the subsequence of all positive characteristic values, where $\lambda_{k_1} \geqq \lambda_{k_2} \geqq \ldots$, let the functionsets $p_1^i(x), \ldots, p_{l-1}^i(x)$ $(i = 1, \ldots, n;$ all $p_r^i(x) \in L_2^{(m)}(\Delta))$ be arbitrarily given, and let*

$$\mu_l = \text{upper bound } \Sigma_{i,j=1}^n \int_{\Delta\times\Delta} K_{ij}(x, y)\,\overline{f(x)}\, f(y)\,dx\,dy \Big/ \Sigma_{i=1}^n \int_{\Delta} |f^i|^2\,dx$$

for all sets $\{f\} = \{f^1, \ldots, f^n\} \in [L_2]^n$ satisfying $\Sigma_{i=1}^m \int_{\Delta} |f^i|^2\,dx \neq 0$ and

$$\Sigma_{i=1}^n \int_{\Delta} f^i\,\overline{p_1^i}\,dx = \ldots = \Sigma_{i=1}^n \int_{\Delta} f^i\,\overline{p_{l-1}^i}\,dx = 0.$$

The number μ_l depends on $p_1^i(x), \ldots, p_{l-1}^i(x)$. Letting now these functions run through the whole space $L_2^{(m)}(\Delta)$, we have

$$\lambda_{k_l} = \min \mu_l.$$

A similar statement holds for the negative characteristic values.

Proof. Follows from II, Theorem 6.

Theorem 12. *Let $\lambda \neq 0$, and let $\{g\} \in [L_2]^n$ be orthogonal to all characteristic functionsets of (1), belonging to the characteristic value λ (If λ is no characteristic value, $\{g\}$ is therefore arbitrary in $[L_2]^n$). Then the solution of (1) is given by*

$$f^i(x) \backsim -\frac{g^i(x)}{\lambda} - \Sigma_k' \frac{\lambda_k}{\lambda(\lambda - \lambda_k)}\, a_k \varphi_k^i(x) \qquad (i = 1, \ldots, n),$$

where $a_k = \Sigma_{i=1}^n \int_{\Delta} g^i(x)\,\overline{\varphi_k^i(x)}\,dx$ for $\lambda_k \neq \lambda$, and where Σ_k' denotes that for those values of k for which $\lambda_k = \lambda$ the coefficient of $\varphi_k^i(x)$ may have any arbitrary value (but the same value for $i = 1, \ldots, n$).

Proof. Follows from II, Theorem 7.

Theorem 13. *We have*

$$\Sigma_{j=1}^n \int_{\Delta} |K_{ij}(x, y)|^2\,dy = \Sigma_k \lambda_k^2 |\varphi_k^i(x)|^2 \quad . \quad . \quad . \quad . \quad (9)$$

for $i=1,\ldots,n$, *and for almost every* $x\in\Delta$:

$$\Sigma_{i,j=1}^{n}\int_{\Delta\times\Delta}|K_{ij}(x,y)|^2\,dx\,dy=\Sigma_k\lambda_k^2; \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$K_{ij}(x,y)\backsim\Sigma_k\lambda_k\varphi_k^i(x)\overline{\varphi_k^j(y)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Proof. Follows from II, Theorem 8.

Similar theorems hold for the equation with iterated kernel. The p-th iterated matrix-kernel $\|K_{ij}^{(p)}(x,y)\|$ is defined by

$$K_{ij}^{(1)}(x,y)=K_{ij}(x,y),$$

$$K_{ij}^{(p)}(x,y)=\Sigma_{l=1}^{n}\int_{\Delta}K_{il}(x,z)\,K_{lj}^{(p-1)}(z,y)\,dz.$$

It is not difficult to prove that $K_{ij}^{(p)}(x,y)\in L_2^{(2m)}(\Delta)$, that $\|K_{ij}^{(p)}(x,y)\|$ is also Hermitian, that $\lambda_k^p\,(k=1,2,\ldots)$ is the sequence of all characteristic values $\neq 0$ of $\|K_{ij}^{(p)}(x,y)\|$, and that $\varphi_k^i(x)$ is a corresponding sequence of characteristic functionsets.

Theorem 14. $\Sigma_{i=1}^{n}\int_{\Delta}K_{ii}^{(p)}(x,x)\,dx=\Sigma_k\lambda_k^p$ *for* $p\geqq 2$.

Proof. Follows from II, Theorem 9.

Remark. If the Hermitian matrix-kernel $\|K_{ij}(x,y)\|$ satisfies the conditions that

$$\int_{\Delta}|K_{ij}(x,y)|^2\,dy \text{ is finite for } i,j=1,\ldots,n, \text{ and for every } x\in\Delta,$$

and

$$\lim_{x_2\to x_1}\int_{\Delta}|K_{ij}(x_2,y)-K_{ij}(x_1,y)|^2\,dy=0 \qquad (i,j=1,\ldots,n),$$

it follows from well-known results in the theory of an ordinary linear integral equation with Hermitian kernel that several of the mentioned theorems can be improved. The convergence in mean in the Theorems 10 and 12, and the convergence almost everywhere in Theorem 13, (9), may be replaced by uniform convergence. Moreover, for $p\geqq 2$, the series $\Sigma_k\lambda_k^p\varphi_k^i(x)\overline{\varphi_k^j(y)}$ converges uniformly in $\Delta\times\Delta$ to $K_{ij}^{(p)}(x,y)$.

For later purposes we finally mention the following

Theorem 15. *If in the space* $[L_2^{(m)}(\Delta)]^n$ *the bounded self-adjoint transformation* H *of positive type is determined by the matrix-kernel* $\|H_{ij}(x,y)\|$, *where* $H_{ij}(x,y)$ $(i,j=1,\ldots,n)$ *is continuous in* $\Delta\times\Delta$,

then there exists an Hermitian matrix-kernel $\|H_{ij}^{(1/2)}(x,y)\|$ *of positive type for which*

$$\int_{\Delta} |H_{ij}^{(1/2)}(x,y)|^2\,dy \qquad (i,j=1,\ldots,n)$$

is bounded, such that the uniquely determined, bounded self-adjoint transformation $H^{1/2}$ *of positive type* ($H^{1/2}$ *is therefore the "square root" of* H) *is determined by the matrix-kernel* $\|H_{ij}^{1/2}(x,y)\|$.

Proof. Follows from II, Theorem 10.

Mathematics. — *Ueber Definitionen von Perron-Integralen.* I. By J. RIDDER. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of March 29, 1947.)

Die Lektüre von Kapitel 8 des Buches: *Integration* von EDWARD JAMES MC SHANE (Princeton Univ. Press) führte uns zu den hier folgenden Betrachtungen über Definitionen von Perron-Integralen.

KAPITEL I.

§ 1. **Definition.** $f(x)$ sei eine im abgeschlossenen Intervall $[a, b]$ definierte Funktion. Eine in $[a, b]$ stetige Funktion $\psi_l(x)$ $[\psi_r(x)]$ heisst *eine zu $f(x)$ in $[a, b]$ adjungierte linksseitige [rechtsseitige] Oberfunktion,* wenn: a) $\psi_l(a) = 0$ $[\psi_r(a) = 0]$ ist; b) für jedes x in $[a, b]$, die Punkte einer höchstens abzählbar unendlichen Teilmenge ausgenommen, die linke untere Derivierte von $\psi_l(x)$, $D_- \psi_l(x)$, $\neq -\infty$ und $\geqq f(x)$ ist [die rechte untere Derivierte von $\psi_r(x)$, $D_+ \psi_r(x)$, $\neq -\infty$ und $\geqq f(x)$ ist].

Definition. Ebenso soll eine in $[a, b]$ stetige Funktion $\varphi_l(x)$ $[\varphi_r(x)]$ *eine zu $f(x)$ in $[a, b]$ adjungierte linksseitige [rechtsseitige] Unterfunktion* sein, wenn: a) $\varphi_l(a) = 0$ $[\varphi_r(a) = 0]$ ist; b) für jedes x in $[a, b]$, die Punkte einer höchstens abzählbar unendlichen Teilmenge ausgenommen, die linke obere Derivierte von $\varphi_l(x)$, $D^- \varphi_l(x)$, $\neq +\infty$ und $\leqq f(x)$ ist [die rechte obere Derivierte von $\varphi_r(x)$, $D^+ \varphi_r(x)$, $\neq +\infty$ und $\leqq f(x)$ ist].

Nun lautet die MC SHANEsche Integraldefinition wie folgt:

Definition I. Die in $[a, b]$ definierte Funktion $f(x)$ ist *P_1-integrierbar über $[a, b]$,* wenn es zu willkürlich positivem ε ein Quadrupel $[\psi_l(x), \psi_r(x), \varphi_l(x), \varphi_r(x)]$ von Ober- und Unterfunktionen gibt, in $[a, b]$ zu $f(x)$ adjungiert, deren je zwei in jedem Punkte von $[a, b]$ eine Differenz haben, welche, absolut genommen, kleiner als ε ist.

Nach MC SHANE ist die Differenz einer jeden Oberfunktion und einer jeden Unterfunktion monoton nicht-abnehmend in $[a, b]$ [1]. Ist $f(x)$ P_1-integrierbar über $[a, b]$, so gibt es eine Funktion $F_1(x)$, welche gleich der unteren Schranke aller $\psi_l(x)$, der unteren Schranke aller $\psi_r(x)$, der oberen Schranke aller $\varphi_l(x)$ und der oberen Schranke aller $\varphi_r(x)$ ist. Wir definieren nun:

$$(P_1)\int_a^b f(x)\,dx = F_1(b).$$ [2]

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[1] Siehe MC SHANE, *Integration* (Princeton Un. Press 1944), S. 313 (Nr. 57.3).

[2] Siehe MC SHANE, loc. cit. 1), S. 315 [Nr. 57.7].

§ 2. **Definition.** $f(x)$ sei definiert in $[a, b]$. Eine in $[a, b]$ stetige Funktion $\psi(x)$ heisst *eine zu $f(x)$ in $[a, b]$ adjungierte Perron-Oberfunktion*, wenn: a) $\psi(a) = 0$ ist; b) für jedes x in $[a, b]$, die Punkte einer höchstens abzählbar unendlichen Menge ausgenommen, die untere Derivierte von $\psi(x)$, $\underline{D}\psi(x)$, $\neq -\infty$ und $\geqq f(x)$ ist.

Definition. Ebenso soll eine in $[a, b]$ stetige Funktion $\varphi(x)$ *eine zu $f(x)$ in $[a, b]$ adjungierte Perron-Unterfunktion* sein, wenn: a) $\varphi(a) = 0$ ist; b) für jedes x in $[a, b]$, die Punkte einer höchstens abzählbar unendlichen Menge ausgenommen, die obere Derivierte von $\varphi(x)$, $\overline{D}\varphi(x)$, $\neq +\infty$ und $\leqq f(x)$ ist.

Definition II (nach PERRON und BAUER). Die in $[a, b]$ definierte Funktion $f(x)$ ist *$\mathfrak{P}^*$-integrierbar* (integrierbar im Perronschen Sinne) *über $[a, b]$*, wenn die untere Schranke aller (existierend anzunehmenden) Perron-Oberfunktionen $\psi(x)$ und die obere Schranke aller (ebenfalls existierend anzunehmenden) Perron-Unterfunktionen $\varphi(x)$ in jedem Punkte von $[a, b]$ zusammenfallen. Ist $F(x)$ der gemeinsame Wert beider Schranken, so sei das Perron-Integral

$$(\mathfrak{P}^*)\int_a^b f(x)\,dx = F(b).$$

Satz 1. *Die Definition I von* MC SHANE *ist aequivalent mit der Definition II nach* PERRON *und* BAUER [3]).

Beweis. Aus Integrierbarkeit gemäss Definition II folgt sofort Integrierbarkeit gemäss Definition I; die Integralwerte sind dieselben.

Nehmen wir, umgekehrt, Integrierbarkeit von $f(x)$ über $[a, b]$ gemäss Definition I an. Ist $(P_1)\int_a^x f(x)\,dx = F_1(x)$ in $[a, b]$, so gibt es zu willkürlich positivem ε eine linksseitige Oberfunktion $\psi_l(x)$, eine rechtsseitige Oberfunktion $\psi_r(x)$, eine linksseitige Unterfunktion $\varphi_l(x)$ und eine rechtsseitige Unterfunktion $\varphi_r(x)$ mit

$$0 \leqq \psi_l(x) - F_1(x) < \varepsilon/4,\ 0 \leqq \psi_r(x) - F_1(x) < \varepsilon/4,$$
$$0 \leqq F_1(x) - \varphi_l(x) < \varepsilon/4,\ 0 \leqq F_1(x) - \varphi_r(x) < \varepsilon/4$$

in jedem Punkte x von $[a, b]$.

Die Differenzen $\sigma_1(x) \equiv \psi_l(x) - F_1(x)$, $\sigma_2(x) \equiv \psi_r(x) - F_1(x)$, $\sigma_3(x) \equiv F_1(x) - \varphi_l(x)$ und $\sigma_4(x) \equiv F_1(x) - \varphi_r(x)$ sind monoton nichtabnehmend [4]). Daraus und aus den Definitionen folgt, dass

$$F_1(x) + \sigma_1(x) + \sigma_2(x)$$

eine zu $f(x)$ adjungierte Perron-Oberfunktion, $F_1(x) - \sigma_3(x) - \sigma_4(x)$ eine zu $f(x)$ adjungierte Perron-Unterfunktion ist. In jedem Punkte von $[a, b]$ ist ihre Differenz $< \varepsilon$, und weicht eine jede von ihnen um weniger

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[3]) Ueber diesen bemerkt MC SHANE, loc. cit. 1), S. 316: "the proof is far from simple"; der hier folgende einfache Beweis kann damit wohl nicht gemeint sein.

[4]) Siehe MC SHANE, loc. cit. 1), S. 317 (Remark).

als $\frac{\varepsilon}{2}$ von $F_1(x)$ ab. Somit ist $f(x)$ auch integrierbar über $[a, b]$ gemäss Definition II, und ist der Wert des Perron-Bauerschen Integrals über $[a, b]$ ebenfalls $F_1(b)$.

KAPITEL II.

§ 3. **Definition III.** Die in $[a, b]$ definierte Funktion $f(x)$ ist P_2-*integrierbar über* $[a, b]$, wenn es zu willkürlich positivem ε ein Paar $[\psi_r(x), \varphi_l(x)]$ von rechtsseitiger Oberfunktion und linksseitiger Unterfunktion gibt, beide in $[a, b]$ zu $f(x)$ adjungiert, deren Differenz in jedem Punkte von $[a, b]$, absolut genommen, kleiner als ε ist.

Die Differenz $\psi_r(x) - \varphi_l(x)$ ist monoton nicht-abnehmend in $[a, b]$ [1]). Ist $f(x)$ P_2-integrierbar über $[a, b]$, so gibt es eine Funktion $F_2(x)$, welche gleich der unteren Schranke aller $\psi_r(x)$ und der oberen Schranke aller $\varphi_l(x)$ ist. Wir definieren nun:

$$(P_2)\int_a^b f(x)\,dx = F_2(b).$$

Definition. Eine in $[a, b]$ definierte, endlichwertige Funktion $\Psi(x)$ ist *unterhalb totalstetig auf der perfekten Teilmenge P von* $[a, b]$, wenn die Summe der Funktionsdifferenzen: $\sum_{(J)} \{\Psi(b_j) - \Psi(a_j)\}$ in den zu P gehörenden Endpunkten $a_j < b_j$ von endlich vielen, nicht übereinander greifenden Intervallen immer einen unteren Limes $\geqq 0$ hat, sobald die Längensumme dieser Intervalle gegen Null konvergiert.

Definition. Eine in $[a, b]$ definierte, endlichwertige Funktion $\Phi(x)$ ist *oberhalb totalstetig auf der perfekten Teilmenge P von* $[a, b]$, wenn die Summe der Funktionsdifferenzen: $\sum_{(J)} \{\Phi(b_j) - \Phi(a_j)\}$ in den zu P gehörenden Endpunkten $a_j < b_j$ von endlich vielen, nicht übereinander greifenden Intervallen immer einen oberen Limes $\leqq 0$ hat, sobald die Längensumme dieser Intervalle gegen Null konvergiert.

Definition. Eine in $[a, b]$ definierte, endlichwertige Funktion $\Psi(x)$ ist *in* $[a, b]$ *unterhalb totalstetig im verallgemeinerten Sinne* [abgekürzt: UTV], wenn $[a, b]$ mit Ausnahme von höchstens abzählbar unendlich vielen Punkten sich überdecken lässt durch abzählbar viele perfekte Mengen $\{P_k\}$, derartig, dass $\Psi(x)$ unterhalb totalstetig ist auf jeder Menge P_k.

Definition. Eine in $[a, b]$ definierte, endlichwertige Funktion $\Phi(x)$ ist *in* $[a, b]$ *oberhalb totalstetig im verallgemeinerten Sinne* [abgekürzt: OTV], wenn $[a, b]$ mit Ausnahme von höchstens abzählbar unendlich vielen Punkten sich überdecken lässt durch abzählbar viele perfekte Mengen $\{P_k\}$, derartig, dass $\Phi(x)$ oberhalb totalstetig ist auf jeder Menge P_k.

Definition. Eine in $[a, b]$ definierte, endlichwertige Funktion $F(x)$ ist *in* $[a, b]$ *totalstetig im verallgemeinerten Sinne* [abgekürzt: TV], wenn $[a, b]$ mit Ausnahme von höchstens abzählbar unendlich vielen Punkten

sich überdecken lässt durch abzählbar viele perfekte Mengen $\{P_k\}$, derartig, dass $F(x)$ totalstetig ist auf jeder Menge P_k.

Eine Funktion, welche in $[a, b]$ sowohl UTV wie OTV ist, ist daselbst auch TV; und umgekehrt.

Satz 2. *In $[a, b]$ zu einer Funktion $f(x)$ adjungierte rechts- oder linksseitige Oberfunktionen sind daselbst UTV; in $[a, b]$ zu $f(x)$ adjungierte rechts- oder linksseitige Unterfunktionen sind daselbst OTV.*

Beweis. Es wird genügen zu zeigen, dass jede rechtsseitige Oberfunktion $\psi_r(x)$ in $[a, b]$ UTV ist.

Dies folgt sofort aus dem folgenden

Lemma. Wenn die in $[a, b]$ stetige Funktion $F(x)$ in den Punkten einer Teilmenge E von $[a, b]$ eine rechte untere Derivierte $\neq -\infty$ hat, so lässt sich E, ausgenommen in den Punkten einer (ev. leeren) abzählbaren Teilmenge, überdecken durch abzählbar viele perfekte Mengen $\{P_j\}$, auf deren jeder $F(x)$ unterhalb totalstetig ist [5]).

Beweis. E ist Summe von abzählbar vielen Mengen $\{E_n^i\}$, wobei i eine ganze, n eine natürliche Zahl ist, welche in abgeschlossenen Intervallen $\left[\frac{i}{n}, \frac{i+1}{n}\right]$ enthalten sind, und die Eigenschaft haben, dass für jeden Punkt x eines E_n^i

$$\text{aus } 0 < x' - x \leqq \frac{1}{n} \text{ folgt } F(x') - F(x) \geqq -n \cdot (x' - x). \quad . \quad (1)$$

Es sei H_n^i die abgeschlossene Hülle von E_n^i. Dann folgt aus (1) und der Stetigkeit von $F(x)$, dass für jedes Paar (x, x') von Punkten von H_n^i mit $x < x'$

$$F(x') - F(x) \geqq -n \cdot (x - x')$$

ist. Somit ist $F(x)$ unterhalb totalstetig auf dem perfekten Kern von H_n^i.

Das Lemma folgt daraus sofort.

Satz 3. *Ist $f(x)$ integrierbar über $[a, b]$ gemäss Definition III, so ist sie es auch nach der allgemeinen Denjoyschen Integraldefinition; und die Integralwerte sind einander gleich:*

$$(P_2)\int_a^b f(x)\,dx = (D)\int_a^b f(x)\,dx.$$

Beweis. Aus der deskriptiven Form der Denjoyschen Integraldefinition folgt, dass es genügt zu zeigen: 1°. $(P_2)\int_a^x f(x)dx \equiv F_2(x)$ ist in $[a, b]$ TV, 2°. $F_2(x)$ hat in fast allen Punkten von $[a, b]$ eine approximative Ableitung gleich $f(x)$.

Beweis von 1°: Es gibt eine Folge von rechtsseitigen Oberfunktionen $\{\psi_r^{(k)}(x)\}$ und eine Folge von linksseitigen Unterfunktionen $\{\varphi_l^{(k)}(x)\}$

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[5]) Ein etwas allgemeineres Lemma findet man bei J. RIDDER, Fund. math. 22 (1934), S. 157, 158.

$(k = 1, 2, \ldots)$, adjungiert zu $f(x)$ in $[a, b]$, für die

$$\lim_{k \to \infty} \psi_r^{(k)}(b) = \lim_{k \to \infty} \varphi_l^{(k)}(b) = F_2(b)$$

ist.

Aus Satz 2 folgt, dass $[a, b]$ sich überdecken lässt durch eine abzählbare Menge H und abzählbar viele perfekte Teilmengen (P_j), derartig, dass $\psi_r^{(1)}(x)$ unterhalb- und $\varphi_l^{(1)}(x)$ oberhalb totalstetig ist auf einer jeden der Mengen (P_j).

Zu jedem $k = 1, 2, 3, \ldots$ ist die Differenz $\psi_r^{(k)}(x) - \varphi_l^{(1)}(x)$ in $[a, b]$ nicht-abnehmend; somit auch $\psi_r^{(k;j)}(x) - \varphi_l^{(1;j)}(x)$, wobei $\psi_r^{(k;j)}(x)$ und $\varphi_l^{(1;j)}(x)$ in den Punkten der perfekten Menge P_j mit $\psi_r^{(k)}(x)$ bzw. mit $\varphi_l^{(1)}(x)$ zusammenfallen und sich linear ändern in den zu P_j komplementären abgeschlossenen Teilintervallen von $[a, b]$. Da $\varphi_l^{(1)}(x)$ auf der Menge P_j oberhalb totalstetig ist, wird $\varphi_l^{(1;j)}(x)$ in $[a, b]$ von beschränkter Variation sein; somit auch $\psi_r^{(k;j)}(x)$ als Summe von $\psi_r^{(k;j)}(x) - \varphi_l^{(1;j)}(x)$ und $\varphi_l^{(1;j)}(x)$. Ausserdem ist $\psi_r^{(k;j)}(x)$ stetig in $[a, b]$.

Da $\psi_r^{(k)}(x)$ in $[a, b]$ UTV ist, gilt dasselbe von $\psi_r^{(k;j)}(x)$; $[a, b]$ lässt sich somit bis auf eine abzählbare Menge überdecken durch abzählbar viele perfekte Mengen $\{M_p^{(k;j)}\}$ $(p = 1, 2, \ldots)$, auf deren jeder $\psi_r^{(k;j)}(x)$ unterhalb totalstetig ist. Daraus und aus der Stetigkeit von $\psi_r^{(k;j)}(x)$ folgt sofort [6]), dass die in $[a, b]$ zu $\psi_r^{(k;j)}(x)$ gehörende, total-additive Mengenfunktion $\Psi_r^{(k;j)}(e)$ einen nicht-negativen (total-additiven) singulären Bestandteil hat. Dadurch wird $\psi_r^{(k;j)}(x)$ in $[a, b]$ unterhalb totalstetig, somit $\psi_r^{(k)}(x)$ auf P_j unterhalb totalstetig sein.

Ebenso ist jedes $\varphi_l^{(k)}(x)$ auf jeder Menge P_j oberhalb totalstetig.

$F_2(x)$ muss totalstetig sein auf jedem P_j. Sonst existierte für eine dieser Mengen, $P_{j'}$, eine z.B. positive Zahl Δ und eine Folge von Mengen (M_n), welche jede für sich von endlich vielen, nicht übereinander greifenden, abgeschlossenen Intervallen $[a_p^{(n)}, a_{p+1}^{(n)}]$ gebildet würden, deren Endpunkte zu $P_{j'}$ gehörten und so dass gleichzeitig

$$\lim_{n \to \infty} \sum_{(p)} [F_2(a_{p+1}^{(n)}) - F_2(a_p^{(n)})] = \Delta \text{ und } \lim_{n \to \infty} m(M_n) = 0 \quad . \quad . \quad (2)$$

wäre.

$\varphi_l^{(k')}(x)$ sei eine Funktion der Folge $\{\varphi_l^{(k)}(x)\}$ mit

$$F_2(b) - \varphi_l^{(k')}(b) < {}^1/_2 \Delta.$$

Da $F_2(x) - \varphi_l^{(k')}(x)$ eine nicht-abnehmende Funktion ist, wird bei willkürlich, aber fest gewähltem n:

$$\left.\begin{aligned} \sum_{(p)} [F_2(a_{p+1}^{(n)}) - F_2(a_p^{(n)})] - \sum_{(p)} [\varphi_l^{(k')}(a_{p+1}^{(n)}) - \varphi_l^{(k')}(a_p^{(n)})] \\ \leqq F_2(b) - \varphi_l^{(k')}(b) < {}^1/_2 \Delta \end{aligned}\right\} \quad . \quad (3)$$

sein.

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[6]) Siehe C. DE LA VALLÉE POUSSIN, *Intégrales de Lebesgue*, etc., Coll. Borel (2e éd. 1934), Kap. 6, insbes. S. 91.

Aus (2) und (3) folgte

$$\limsup_{n\to\infty} \sum_{(p)} [\varphi_l^{(k')}(a_{p+1}^{(n)}) - \varphi_l^{(k')}(a_p^{(n)})] > {}^1/_2\,\Delta.$$

Da $\varphi_l^{(k')}(x)$ oberhalb totalstetig ist auf $P_{j'}$, gelangen wir zu einem Widerspruch.

Auch ein negativer Wert von Δ ist unmöglich. $F_2(x)$ ist somit auf jedem P_j totalstetig, somit TV in $[a, b]$.

Beweis von 2°: Aus der Eigenschaft einer Oberfunktion $\psi_r(x)$: „$D_+\psi_r(x) \neq -\infty$ in den Punkten von $[a, b]$, abzählbar viele ausgenommen" folgt, dass $\psi_r(x)$ in fast allen Punkten von $[a, b]$ eine endliche approximative Ableitung $D_{\text{appr.}}\psi_r(x) = D_+\psi_r(x) \geqq f(x)$ hat [7]).

Ebenso hat jede Unterfunktion $\varphi_l(x)$ in fast allen Punkten von $[a, b]$ eine endliche approximative Ableitung $D_{\text{appr.}}\varphi_l(x) = D^-\varphi_l(x) \leqq f(x)$.

$\psi_r(x) - F_2(x)$ ist nicht-abnehmend, hat also fast überall in $[a, b]$ eine endliche Ableitung. Somit muss auch $F_2(x)$ fast überall in $[a, b]$ eine endliche approximative Ableitung haben; und in fast allen Punkten von $[a, b]$ ist

$$D_{\text{appr.}}\psi_r \geqq D_{\text{appr.}}F_2 \geqq D_{\text{appr.}}\varphi_l.$$

$F_2(x)$, und die Oberfunktionen und Unterfunktionen der oben eingeführten Folgen $\{\psi_r^{(k)}(x)\}$, $\{\varphi_l^{(k)}(x)\}$ haben a l l e in den Punkten eines massgleichen Kernes K von $[a, b]$ endliche approximative Ableitungen.

Bei willkürlich positivem ε sei $K_k(\varepsilon)$ die Menge derjenigen Punkte von K, in denen die approximativen Ableitungen von $\psi_r^{(k)}$ und $\varphi_l^{(k)}$ eine Differenz $\geqq \varepsilon$ haben. Das Mass dieser Menge wird mit zunehmendem k nach Null konvergieren. Sonst existierte eine positive Zahl δ derart, dass für unendlich viele Werte (k') $m\{K_{k'}(\varepsilon)\} \geqq \delta$ wäre. Anwendung des Vitalischen Ueberdeckungssatzes würde zeigen, dass für die nicht-abnehmenden Funktionen $\{\psi_r^{(k')} - \varphi_l^{(k')}\}$:

$$\psi_r^{(k')}(b) - \varphi_l^{(k')}(b) \geqq \delta \cdot \varepsilon$$

wäre, während doch

$$\lim_{k'\to\infty} \{\psi_r^{(k')}(b) - \varphi_l^{(k')}(b)\} = 0$$

ist. Die approximative Ableitung von $F_2(x)$ weicht somit fast überall in $[a, b]$ um weniger als ε von $f(x)$ ab. Da ε willkürlich positiv ist, muss $D_{\text{appr.}}F_2(x)$ fast überall in $[a, b]$ gleich $f(x)$ sein.

Satz 4. *N i c h t jede über* $[a, b]$ *nach der allgemeinen Denjoyschen Definition integrierbare Funktion hat ein* P_2*-Integral gemäss Definition III. N i c h t jede über* $[a, b]$ *nach der Denjoy-Khintchineschen Definition* [8]) *integrierbare Funktion hat ein* P_2*-Integral gemäss Definition III.*

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[7]) Siehe S. SAKS, *Théorie de l'intégrale*, Warszawa 1933, S. 173 (Th. 7) u. S. 171 (Th. 3).

[8]) In deskriptiver Form lautet diese: $F(x)$ ist das $D.$—$K.$ Integral über $[a, x]$ für jedes x mit $a \leqq x \leqq b$, falls: 1° $F(a) = 0$ ist; 2° $F(x)$ in $[a, b]$ TV ist; 3° die Ableitung von $F(x)$ in fast allen Punkten von $[a, b]$ existiert und gleich $f(x)$ ist.

Beweis. Betrachten wir jede nach der *allgemeinen* Denjoyschen Definition integrierbare Funktion $f(x)$ auch als integrierbar nach Definition III. Dann ist auch $-f(x)$ integrierbar gemäss beiden Definitionen. Die Klassen der zu $f(x)$ in $[a, b]$ adjungierten rechtsseitigen Oberfunktionen und der zu $f(x)$ in $[a, b]$ adjungierten linksseitigen Unterfunktionen, $\{\psi_r(x)\}$ bzw. $\{\varphi_l(x)\}$, sind nicht leer, und es ist:

$$(D)\int_a^b f \cdot dx = (P_\vartheta)\int_a^b f \cdot dx = \text{untere Schranke aller } \psi_r(b)$$
$$= \text{obere Schranke aller } \varphi_l(b).$$

Ebenso sind die Klassen der zu $-f(x)$ in $[a, b]$ adjungierten rechtsseitigen Oberfunktionen und der zu $-f(x)$ in $[a, b]$ adjungierten linksseitigen Unterfunktionen, $\{\bar{\psi}_r(x)\}$ bzw. $\{\bar{\varphi}_l(x)\}$, nicht leer, mit

$$-(D)\int_a^b f \cdot dx = (P_\vartheta)\int_a^b -f \cdot dx = \text{untere Schranke aller } \bar{\psi}_r(b)$$
$$= \text{obere Schranke aller } \bar{\varphi}_l(b).$$

Jede Funktion $-\bar{\psi}_r(x)$ ist in $[a, b]$ zu $f(x)$ adjungierte rechtsseitige Unterfunktion; jede Funktion $-\bar{\varphi}_l(x)$ ist in $[a, b]$ zu $f(x)$ adjungierte linksseitige Oberfunktion. Ausserdem ist

untere Schranke aller $\psi_r(b)$ = untere Schranke aller $-\bar{\varphi}_l(b)$
= obere Schranke aller $\varphi_l(b)$ = obere Schranke aller $-\bar{\psi}_r(b)$.

$f(x)$ wäre dann über $[a, b]$ auch integrierbar nach der Definition I, somit wegen Satz 1 ebenfalls nach der Perron-Bauerschen, und der mit dieser aequivalenten, *speziellen* Denjoyschen Definition. Wir erreichen einen Widerspruch.

Die Behauptung über das Denjoy-Khintchinesche Integral wird in völlig gleichartiger Weise bewiesen.

§ 4. **Satz 5.** *Nicht jede nach Definition III integrierbare Funktion ist es nach der speziellen Denjoyschen Definition; nicht jede nach Definition III integrierbare Funktion ist es nach der Denjoy-Khintchineschen Definition.*

Beweis. Es gibt eine in $[a, b]$ stetige Funktion $F(x)$, welche in den Punkten einer perfekten Teilmenge P von $[a, b]$, mit $m(P) \neq 0$, den Bedingungen genügt:

$$D^+F(x) = +\infty, D_-F(x) = -\infty, D_+F(x) = D^-F(x) = 0,$$

während in den übrigen Punkten von $[a, b]$ eine endliche Ableitung von $F(x)$ existiert [9]).

Die Funktion $f(x)$, welche in den Punkten von P gleich Null, in den

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[9]) Siehe A. DENJOY, Journal de math. (7) **1** (1915), S. 105—240, insbes. S. 196—200; auch E. W. HOBSON, *Theory of functions of a real variable* I (Third Ed. 1927), S. 402 (Ex. 4).

übrigen Punkten von $[a, b]$ gleich der Ableitung von $F(x)$ ist, ist über $[a, b]$ integrierbar nach Definition III; denn $F(x) - F(a)$ ist sowohl eine in $[a, b]$ zu $f(x)$ adjungierte rechtsseitige Oberfunktion wie linksseitige Unterfunktion. Dabei ist

$$(P_2)\int_a^x f\,dx = F(x) - F(a).$$

Da $F(x)$ nicht in fast allen Punkten von $[a, b]$ eine Ableitung hat, gilt dasselbe von $(P_2)\int_a^x f\,dx$.

$f(x)$ kann dadurch nicht integrierbar sein nach der speziellen Denjoyschen Definition [nach der Denjoy-Khintchineschen Definition]. Denn wäre dies wohl der Fall, so wären das P_2-Integral und das spezielle Denjoysche [das D.–K.] Integral über $[a, x]$ beide TV in $[a, b]$, und hätten beide in fast allen Punkten von $[a, b]$ eine approximative oder eine gewöhnliche Ableitung gleich $f(x)$; sie müssten somit zusammenfallen. Da aber $(P_2)\int_a^x f\,dx$ nicht in fast allen Punkten von $[a, b]$ eine gewöhnliche Ableitung haben kann, während dies wohl für das Denjoysche Integral [das D.–K. Integral] von $f(x)$ über $[a, x]$ der Fall sein müsste, gelangten wir zu einem Widerspruch.

§ 4bis. **Definition IIIbis.** Die in $[a, b]$ definierte Funktion $f(x)$ ist *P_2^*-integrierbar über* $[a, b]$, wenn es zu willkürlich positivem ε ein Paar $[\psi_l(x), \varphi_r(x)]$ von linksseitiger Oberfunktion und rechtsseitiger Unterfunktion gibt, beide in $[a, b]$ zu $f(x)$ adjungiert, deren Differenz in jedem Punkte von $[a, b]$, absolut genommen, kleiner als ε ist.

Satz 3bis. *Ist $f(x)$ integrierbar über $[a, b]$ gemäss Definition IIIbis, so ist sie es auch nach der allgemeinen Denjoyschen Integraldefinition; und die Integrale sind einander gleich:*

$$(P_2^*)\int_a^b f(x)\,dx = (D)\int_a^b f(x)\,dx.$$

Satz 4bis. *Nicht jede über $[a, b]$ nach der allgemeinen Denjoyschen Definition integrierbare Funktion hat ein P_2^*-Integral gemäss Definition IIIbis. Nicht jede über $[a, b]$ nach der Denjoy-Khintchineschen Definition* [8]) *integrierbare Funktion hat ein P_2^*-Integral gemäss Definition IIIbis.*

Satz 5bis. *Nicht jede nach Definition IIIbis integrierbare Funktion ist es nach der speziellen Denjoyschen Definition; nicht jede nach Definition IIIbis integrierbare Funktion ist es nach der Denjoy-Khintchineschen Definition* [8]).

Beweis. Dies folgt durch Betrachtung der Funktion $G(y)$, welche aus der im Beweise von Satz 5 benutzte Funktion $F(x)$ hervorgeht durch die Annahme $G(y) = -F(-x)$ mit $y = -x$ [10]).

[10]) In korrespondierenden Punkten y und x ist $D_- G(y) = D_+ F(x)$, $D^+ G(y) = D^- F(x)$.

Satz 6. *Nicht jede über* $[a, b]$ P_2*-integrierbare Funktion ist* P_2^**-integrierbar über* $[a, b]$; *nicht jede über* $[a, b]$ P_2^**-integrierbare Funktion ist* P_2*-integrierbar über* $[a, b]$.

Beweis. Wäre die erste Hälfte des Satzes unrichtig, so wäre für jede über $[a, b]$ P_2-integrierbare Funktion $f(x)$, nach den Sätzen 3 und 3bis,

$$(P_2)\int_a^b f\,dx = (D)\int_a^b f\,dx = (P_2^*)\int_a^b f\,dx.$$

Aus den Definitionen III, IIIbis und I folgte sodann, dass f auch P_1-integrierbar wäre, somit, nach Satz 1, auch integrierbar nach der speziellen Denjoyschen Integraldefinition. Wegen Satz 5 gelangten wir zu einem Widerspruch.

Die zweite Hälfte des Satzes beweist man in analoger Weise.

§ 5. Deuten wir durch P_1, $\mathfrak{P}$, D^*, P_2, P_2^*, D, DK an die Klassen der über $[a, b]$ integrierbaren Funktionen bzw. nach Def. I, Def. II (Perron–Bauer), der Def. des speziellen Denjoyschen Integrals, der Def. III, der Def. IIIbis, der Def. des allgemeinen Denjoyschen Integrals und der Def. des D.–K. Integrals, so liefern die vorangehenden Sätze und Definitionen die folgenden Zusammenhänge zwischen diesen Klassen:

$$P_1 = \mathfrak{P}^* = D^* \subset P_2 \subset D; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4^a)$$

$$P_1 = \mathfrak{P}^* = D^* \subset P_2^* \subset D; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4^b)$$

$$\text{non } DK \subseteqq P_2; \text{ non } P_2 \subseteqq DK; \text{ non } DK \subseteqq P_2^*; \text{ non } P_2^* \subseteqq DK; \quad (4^c)$$

$$\text{non } P_2 \subseteqq P_2^*; \text{ non } P_2^* \subseteqq P_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4^d)$$

Mathematics. — *Non-homogeneous binary quadratic forms.* II. By H. DAVENPORT. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of February 22, 1947.)

1. In a previous paper [1]) with this title (a knowledge of which, however, is not necessary for the understanding of the present paper) I have investigated the possibility of improving in some cases on the result of MINKOWSKI's theorem on the product of two non-homogeneous linear forms. If

$$\xi = \alpha x + \beta y, \ \eta = \gamma x + \delta y$$

are two linear forms, I determined in a few particular cases the least number M such that, for any real a, b one can find integers x, y for which

$$|(\xi - a)(\eta - b)| \leqslant M. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

One example was the case

$$\xi = x + \theta y, \ \eta = \xi' = x + \theta' y, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where

$$\theta = \tfrac{1}{2}(1 + \sqrt{5}), \ \theta' = \tfrac{1}{2}(1 - \sqrt{5}). \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

It was proved that (1) is then always soluble when $M = \frac{1}{4}$, and this is plainly the least such M, as we see from the possibility $a = b = \frac{1}{2}$.

I have since investigated this particular case more deeply, and have discovered a phenomenon which doubtless occurs in many other cases. The results, as far as I have carried them, are as follows:

Theorem 1. *If ξ, ξ' are the linear forms (2), and a, b are any real numbers, then there exist integers x, y such that*

$$|(\xi - a)(\xi' - b)| \leqslant \tfrac{1}{4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The sign of equality is needed here if and only if

$$a = \tfrac{1}{2}\tau + \xi_1, \ b = \tfrac{1}{2}\tau' + \xi_1', \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where τ is any unit, and ξ_1 any integer, of the field $k(\theta)$, and τ', ξ_1' their conjugates.

Theorem 2. *If a, b are not of the form (5), there exist integers x, y such that*

$$|(\xi - a)(\xi' - b)| \leqslant \tfrac{1}{5}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The sign of equality is needed here if and only if

$$a = \frac{\tau}{\sqrt{5}} + \xi_1, \ b = -\frac{\tau'}{\sqrt{5}} + \xi_1', \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where τ, ξ_1 are as before.

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[1]) Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **49**, 815—821 (1946).

Theorem 3. *If a, b are not of the form* (5), *nor of the form* (7), *there exist integers x, y such that*

$$|(\xi-a)(\xi'-b)| < \frac{1}{6.34} \quad \ldots\ldots\ldots \quad (8)$$

The number 6.34 has no unique significance here, and could be improved slightly at the cost of longer calculations. I have not been able to determine whether a "third minimum" exists, but may point out that if it exists it cannot be less than

$$\frac{1}{4\theta} = \frac{1}{6.472\ldots}.$$

This follows from the curious inequality

$$\left|\left(\xi - \frac{1}{2\theta}\right)(\xi' - \tfrac{1}{2})\right| \geqq \frac{1}{4\theta} \quad \ldots\ldots\ldots \quad (9)$$

which will be proved in § 5. It will be seen that this of a different type from the obvious inequalities

$$|(\xi-\tfrac{1}{2})(\xi'-\tfrac{1}{2})| \geqq \tfrac{1}{4}, \quad \left|\left(\xi - \frac{1}{\sqrt{5}}\right)\left(\xi' + \frac{1}{\sqrt{5}}\right)\right| \geqq \tfrac{1}{5}$$

which correspond to the first two minima; for although $\frac{1}{2}$ and $\frac{1}{2\theta}$ are numbers of $k(\theta)$, they are *not* conjugates.

The theorems do not assert anything about the existence of an infinity of solutions of (1). It is, however, possible to modify the method of proof to give the following (less detailed) answer to this question.

Theorem 4. *For any $M > \frac{1}{4}$, the inequality* (1) *has an infinity of solutions with $|\xi - a|$ arbitrarily small, and similarly an infinity of solutions with $|\xi' - b|$ arbitrarily small.*

Theorem 4 can also be expressed in the more concise form:

$$\liminf_{\xi \to a} |(\xi-a)\xi'| \leqq \tfrac{1}{4},$$

where ξ approaches a through integers of $k(\theta)$. Here again $\frac{1}{4}$ is the best possible constant, from the case $a = \frac{1}{2}$. For there are an infinity of arbitrarily small units of the form $2\xi - 1$, and as ξ approaches a through the corresponding values,

$$\lim_{\xi \to a} |(\xi - \tfrac{1}{2})\xi'| = \lim_{\xi \to a} \tfrac{1}{4}|(2\xi-1)(2\xi'-1)| = \tfrac{1}{4}.$$

2. The method of the present paper is a modification and a development of that of another paper [2], on the product of three non-homogeneous linear forms. We first prepare for the proofs of Theorems 1, 2 and 3.

We denote by $M = M(a, b)$ the lower bound of

$$|(\xi-a)(\xi'-b)|$$

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[2] Proc. Cambridge Phil. Soc. (in course of publication).

for all integers ξ of $k(\theta)$, i.e. for all integral x, y in (2). If $M = 0$ there is nothing to prove, so we suppose $M > 0$. For any arbitrarily small positive number ε_0 there exists an integer ξ_0 of $k(\theta)$ such that

$$|(\xi_0 - a)(\xi_0' - b)| = \frac{M}{1-\varepsilon}, \text{ where } 0 \leqq \varepsilon < \varepsilon_0 \quad . \quad . \quad . \quad . \quad (10)$$

We define α, β by

$$\alpha = (a - \xi_0)^{-1}, \ \beta = (b - \xi_0')^{-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Since

$$\alpha\xi - 1 = \frac{\xi + \xi_0 - a}{a - \xi_0},$$

the definition of M tells us that

$$|(\alpha\xi - 1)(\beta\xi' - 1)| \geqq 1 - \varepsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

for all integers ξ of $k(\theta)$. We have also

$$|\alpha\beta| = \frac{1-\varepsilon}{M} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

by (10).

The properties (12) and (13) of α and β are unaltered if we replace α and β by $\alpha\tau$ and $\beta\tau'$, where τ is any unit of $k(\theta)$, since τ can be absorbed in ξ. As θ^n, for any integer n, is a unit of $k(\theta)$, we can modify the numbers α, β in this way to obtain two which satisfy

$$\frac{1}{\theta} < \left|\frac{\beta}{\alpha}\right| \leqq \theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The properties (12) and (13) are also unaltered if we interchange α and β or if we replace α, β by $-\alpha$, $-\beta$. By these operations we can ensure that $|\beta| \leq |\alpha|$ and $\alpha > 0$. Hence we may suppose, without loss of generality, that

$$\alpha > 0, \quad \frac{u}{\theta} \leqq |\beta| \leqq \alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The condition (12) may be expressed in the equivalent form:

$$|(\alpha - \xi^{-1})(\beta - (\xi')^{-1})| \geqq (1-\varepsilon)\,|\xi\xi'|^{-1} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

for all integers $\xi \neq 0$ of $k(\theta)$.

We proceed to investigate the intervals to which α and β are restricted by (15) and (12) [or (16)], and to deduce the possible values of M from (13).

3. We recall that $\theta' = -\theta^{-1} = 1 - \theta$. Some numerical values which are relevant are

$$\theta = 1.618034\ldots, \ \theta' = -0.618034\ldots, \ \theta^2 = 2.618034\ldots, \ \theta^{-2} = 0.381966\ldots.$$

Lemma 1. *It is impossible that* $\alpha < \sqrt{2-\varepsilon}$.

Proof. By (12), with $\xi = 1$, we have $|(\alpha - 1)\ (\beta - 1)| \geq 1 - \varepsilon$. Since $|\beta - 1| \leq \alpha + 1$ by (15), this implies $|\alpha^2 - 1| \geq 1 - \varepsilon$. If

$\alpha < \sqrt{2-\varepsilon}$, then $\alpha \leq \sqrt{\varepsilon}$. Let m be the largest positive integer for which $m\alpha < 1$; then plainly $m\alpha \geq \frac{1}{2}$. But, by (12) with $\xi = m$, and by (15), we have

$$|m^2\alpha^2 - 1| \geqq 1-\varepsilon,$$

and this is a contradiction.

Lemma 2. *If* $\alpha < \sqrt{5} - \varepsilon$ *then*

$$|\alpha - 2| \leqq 2\varepsilon, \quad |\beta - 2| \leqq 2\varepsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Proof. Suppose first $\beta < 0$. By Lemma 1, $\alpha \geq \sqrt{2-\varepsilon}$, whence $|\beta| \geq \sqrt{2-\varepsilon}/\theta > 1/\theta$ by (15). By (16) with $\xi = \theta^{-1}$,

$$|(\alpha - \theta)(\beta - \theta')| \geqq 1-\varepsilon.$$

Since β and θ' are negative, and $|\theta'| < |\beta| \leq \alpha$, this implies

$$|(\alpha - \theta)(\alpha - \theta^{-1})| \geqq 1-\varepsilon,$$

or

$$|\alpha^2 - \sqrt{5}\,\alpha + 1| \geqq 1-\varepsilon.$$

This is impossible, since

$$-\tfrac{1}{4} \leqq (\alpha - \tfrac{1}{2}\sqrt{5})^2 - \tfrac{1}{4} < (\tfrac{1}{2}\sqrt{5} - \varepsilon)^2 - \tfrac{1}{4} \leqq 1-\varepsilon.$$

Hence $\beta > 0$, and $\beta > \theta^{-1}$ as before. We now apply (16) with $\xi = \theta^{-2}$, which gives

$$(\theta^2 - \alpha)(\beta - \theta^{-2}) \geqq 1-\varepsilon,$$

whence

$$(\theta^2 - \alpha)(\alpha - \theta^{-2}) \geqq 1-\varepsilon,$$

or $-\alpha^2 + 3\alpha - 1 \geq 1-\varepsilon$. This implies $(\alpha - \frac{3}{2})^2 \leq \frac{1}{4} + \varepsilon$, whence

$$\alpha \leqq 2 + \varepsilon. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

It follows from $(\alpha - 1)|\beta - 1| \geq 1 - \varepsilon$ that

$$\beta \geqq 1 + (1-\varepsilon)/(1+\varepsilon) \geqq 2 - 2\varepsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Since $\beta \leq \alpha$, (17) follows from (18) and (19).

Lemma 3. *If* $\alpha < \sqrt{5} - \varepsilon$, *then* $\alpha = \beta = 2$.

Proof. We write $\alpha = 2 + \alpha_1$, $\beta = 2 + \beta_1$, so that $|\alpha_1| \leq 2\varepsilon$, $|\beta_1| \leq 2\varepsilon$ by Lemma 2. Substituting in (12),

$$|(2\xi - 1 + \alpha_1\xi)(2\xi' - 1 + \beta_1\xi')| \geqq 1-\varepsilon \quad . \quad . \quad . \quad . \quad (20)$$

for every integer ξ of $k(\theta)$.

Since $\theta^3 = 2\theta + 1 \equiv 1 \pmod 2$, we have $(\pm\theta^3)^n \equiv 1 \pmod 2$ for every integer n (positive or negative). Hence we may take

$$\xi = \frac{(\pm\theta^3)^n + 1}{2}.$$

Then

$$\frac{\xi}{2\xi - 1} = \frac{1 + (\pm\theta^3)^{-n}}{2},$$

and since $(2\xi - 1)(2\xi' - 1) = \pm 1$, (20) becomes

$$\left|\left\{1 + \frac{1 + (\pm\theta^3)^{-n}}{2}\alpha_1\right\}\left\{1 + \frac{1 + (\mp\theta^3)^n}{2}\beta_1\right\}\right| \geqq 1 - \varepsilon, \quad (21)$$

on noting that $\theta' = -\theta^{-1}$.

Suppose first that $\alpha_1 < 0$. We take n to be an even negative integer, say $n = -2m$, in (21), and obtain an inequality of the form

$$|(1 + \lambda_m \alpha_1)(1 + \mu_m \beta_1)| \geqq 1 - \varepsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Here

$$\mu_m = \frac{1 + \theta^{-6m}}{2} < 1,$$

so that

$$|1 + \mu_m \beta_1| \leqq 1 + 2\varepsilon. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Also

$$\lambda_m = \frac{1 + \theta^{6m}}{2},$$

so that

$$\frac{\lambda_{m+1}}{\lambda_m} < \theta^6.$$

Since $\lambda_m \to \infty$ as $m \to \infty$, and $\lambda_1 |\alpha_1| \leq 2\lambda_1 \varepsilon < 1$, we can choose m so that

$$\lambda_m |\alpha_1| \leqq 1 < \lambda_{m+1} |\alpha_1|.$$

Then

$$\lambda_m |\alpha_1| > \lambda_{m+1} |\alpha_1| \theta^{-6} > \theta^{-6},$$

and since we are supposing $\alpha_1 < 0$, we obtain

$$0 \leqq 1 + \lambda_m \alpha_1 < 1 - \theta^{-6} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Now (23) and (24) contradict (22).

Similarly, if $\alpha_1 > 0$ we take n to be an odd negative integer, say $n = -(2m + 1)$, in (21), and taking the lower sign, we obtain an inequality of the form

$$|(1 - \lambda'_m \alpha_1)(1 + \mu'_m \beta_1)| \geqq 1 - \varepsilon,$$

where λ'_m and μ'_m have similar properties to λ_m and μ_m in the preceding. The same proof establishes that $\beta_1 = 0$, by taking n in (21) to be a positive integer, odd if β_1 is positive and even if β_1 is negative.

Lemma 4. *If* $\sqrt{5} - \varepsilon \leq \alpha < 3 - \frac{1}{2}\varepsilon$, *then* $\beta < 0$.

Proof. Suppose $\beta > 0$, so that $\beta \geq (\sqrt{5} - \varepsilon)/\theta > 1$.
The condition (16) with $\xi = \theta^{-2}$ gives

$$|\alpha - \theta^2|(\beta - \theta^{-2}) \geqq 1 - \varepsilon.$$

If $\alpha < \theta^2$, this implies

$$(\theta^2 - \alpha)(\alpha - \theta^{-2}) \geqq 1 - \varepsilon,$$

whence $\alpha \leq 2 + \varepsilon$ as in the proof of Lemma 2. This is impossible.

If $\alpha > \theta^2$, we obtain

$$(\alpha - \theta^2)(\alpha - \theta^{-2}) \geqq 1 - \varepsilon,$$

whence $\alpha^2 - 3\alpha \geq -\varepsilon$, or

$$\alpha - \tfrac{3}{2} \geqq \sqrt{\tfrac{9}{4} - \varepsilon} = \tfrac{3}{2}\sqrt{1 - \tfrac{4}{9}\varepsilon} \geqq \tfrac{3}{2} - \tfrac{1}{2}\varepsilon.$$

This also contradicts the hypothesis.

Lemma 5. *If* $\sqrt{5} - \varepsilon \leq \alpha \leq 2.9208$, then

$$|\alpha - \sqrt{5}| \leqq 11\varepsilon, \quad |\beta + \sqrt{5}| \leqq 11\varepsilon \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Proof. By Lemma 4 we can write $\beta = -\beta'$, where $\beta' > 0$. The inequality (16) with $\xi = \theta^{-2}$ gives

$$|\alpha - \theta^2|(\beta' + \theta^{-2}) \geqq 1 - \varepsilon.$$

If $\alpha > \theta^2$, this implies

$$(\alpha - \theta^2)(\alpha + \theta^{-2}) \geqq 1 - \varepsilon,$$

$$\alpha^2 - \sqrt{5}\,\alpha - (2 - \varepsilon) \geqq 0,$$

$$\alpha \geqq \tfrac{1}{2}(\sqrt{5} + \sqrt{13 - 4\varepsilon}) > 2.9208,$$

which is impossible. Thus we have $\alpha < \theta^2$, and

$$(\theta^2 - \alpha)(\alpha + \theta^{-2}) \geqq 1 - \varepsilon,$$

$$\alpha^2 - \sqrt{5}\,\alpha - \varepsilon \leqq 0,$$

$$\alpha \leqq \tfrac{1}{2}(\sqrt{5} + \sqrt{5 + 4\varepsilon}) \leqq \sqrt{5} + \varepsilon.$$

Also

$$(\theta^2 - \alpha)(\beta' + \theta^{-2}) \geqq 1 - \varepsilon,$$

and since $\theta^2 - \alpha \leq \theta^2 - \sqrt{5} + \varepsilon \leq \theta^{-2}(1 + 3\varepsilon)$, this implies

$$\beta' \geqq -\theta^{-2} + \theta^2(1 - 4\varepsilon) \geqq \sqrt{5} - 11\varepsilon.$$

This suffices to prove the results stated.

Lemma 6. *If* (25) *holds, then* $\alpha = \sqrt{5}$, $\beta = -\sqrt{5}$.

Proof. We write $\alpha = \sqrt{5} + \alpha_1$, $\beta = -\sqrt{5} + \beta_1$, so that $|\alpha_1| \leq 11\varepsilon$, $|\beta_1| \leq 11\varepsilon$, by Lemma 5. Substituting in (12),

$$|(\sqrt{5}\,\xi - 1 + \alpha_1\xi)(-\sqrt{5}\,\xi' - 1 + \beta_1\xi')| \geqq 1 - \varepsilon \quad . \quad . \quad . \quad (26)$$

for every integer ξ of $k(\theta)$.

We have $\theta - 3 \equiv 3(2\theta - 1) \pmod 5$, and since $2\theta - 1 = \sqrt{5}$, it follows that $\theta \equiv 3 \pmod{\sqrt{5}}$. Hence if n is any integer (positive or negative),

$$\theta^{4n+2} \equiv -1 \pmod{\sqrt{5}}, \quad -\theta^{4n} \equiv -1 \pmod{\sqrt{5}}.$$

Thus we may take

$$\xi = (\theta^{4n+2} + 1)/\sqrt{5}$$

in (26), which gives

$$\left|\left\{1+\frac{1+\theta^{-4n-2}}{\sqrt{5}}\alpha_1\right\}\left\{1-\frac{1+\theta^{4n+2}}{\sqrt{5}}\beta_1\right\}\right| \geqq 1-\varepsilon, \quad . \quad . \quad (27)$$

or we may take

$$\xi = (-\theta^{4n}+1)/\sqrt{5},$$

which gives

$$\left|\left\{1+\frac{1-\theta^{-4n}}{\sqrt{5}}\alpha_1\right\}\left\{1-\frac{1-\theta^{4n}}{\sqrt{5}}\beta_1\right\}\right| \geqq 1-\varepsilon. \quad . \quad . \quad . \quad (28)$$

The proof proceeds as in Lemma 3. If $\alpha_1 > 0$ we apply (28) with negative values of n. If $\alpha_1 < 0$ we apply (27) with negative values of n. If $\beta_1 > 0$ we apply (27) with positive values of n. If $\beta_1 < 0$ we apply (28) with positive values of n. In every case we obtain a contradiction, which proves that $\alpha_1 = \beta_1 = 0$.

Lemma 7. *If $\alpha > 2.9208$ and $\beta < 0$, then*

$$|\alpha\beta| > 6.34 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Proof. We write $\beta = -\beta'$, where $\beta' > 0$.

Case 1. *Suppose* $2.9208 < \alpha < 3$. By (16),

$$(\alpha+\theta^{-2})\,|\beta'-\theta^2| \geqq 1-\varepsilon.$$

If $\beta' > \theta^2$, this implies

$$\beta' > \theta^2 + \frac{1-\varepsilon}{3+\theta^{-2}} > 2.618 + \frac{1}{3.382} > 2.913.$$

This is more than sufficient to prove (29), so we can suppose $\beta' < \theta^2$. We now have

$$\beta' < \theta^2 - \frac{1-\varepsilon}{3+\theta^{-2}} < 2.324.$$

We now apply (16) with $\xi = 78 - 48\,\theta$. We have

$$\xi\xi' = 36\,(13-8\,\theta)\,(13-8\,\theta') = 36,$$

and

$$\xi^{-1} = \frac{78-48\,\theta'}{36} = \tfrac{4}{3}\,\theta + \tfrac{5}{6} = 2.99071\ldots,$$

$$(\xi')^{-1} = \frac{78-48\,\theta}{36} = \tfrac{4}{3}\,\theta' + \tfrac{5}{6} = 0.00929\ldots.$$

Hence (16), which takes the form

$$|(\alpha-\xi^{-1})\,(\beta'+(\xi')^{-1})| \geqq \frac{1-\varepsilon}{36},$$

implies

$$|\alpha-\xi^{-1}| \geqq \frac{1-\varepsilon}{36\,(2.334)} > 0.0119.$$

Since $\alpha < 3$, we obtain

$$\alpha < \xi^{-1} - 0.0119 < 2.979.$$

But now

$$(\alpha - \theta^2)(\beta' + \theta^{-2}) < (2.979 - 2.618)(2.324 + 0.382) < 1 - \varepsilon,$$

contrary to (16).

Case 2. *Suppose* $3 \leq \alpha < 3.15$. We apply (16) with $\xi = 7\theta - 11$, so that $\xi\xi' = -5$. We have

$$\xi^{-1} = -\left(\frac{7\theta' - 11}{5}\right) = \frac{7\theta + 4}{5} = 3.06525\ldots,$$

$$(\xi')^{-1} = -\left(\frac{7\theta - 11}{5}\right) = \frac{7\theta' + 4}{5} = -0.06525\ldots.$$

Since $|\alpha - \xi^{-1}| < 3.15 - 3.06525\ldots < 0.085$, we have

$$|\beta' - 0.0625\ldots| > \frac{1-\varepsilon}{5(0.085)} > 2.35.$$

Hence $\beta' > 2.4$, and $\alpha\beta' > 7.2$.

Case 3. *Suppose* $\alpha \geq 3.15$. We have $\beta' \geq \alpha/\theta > \theta$. By (16),

$$(\alpha + \theta')(\beta' - \theta) \geq 1 - \varepsilon,$$

or

$$\beta' \geq \theta + \frac{1-\varepsilon}{\alpha - \theta^{-1}}.$$

Multiplying this by α we obtain a lower bound for $\alpha\beta'$ in terms of α, this lower bound being an increasing function of α, since its derivative with respect to α is

$$\theta - \frac{(1-\varepsilon)\theta^{-1}}{(\alpha - \theta^{-1})^2} > 1.6 - 1 > 0.$$

Thus

$$\alpha\beta' \geq 3.15\,\theta + \frac{(1-\varepsilon)(3.15)}{3.15 - \theta^{-1}}$$

$$> (3.15)(1.618) + \frac{(1-\varepsilon)(3.15)}{3.15 - 0.618} > 6.34.$$

Lemma 8. *If* $\alpha > 2.9208$ *and* $\beta > 0$ *then*

$$\alpha\beta > 6.472. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Proof. We use two inequalities giving lower bounds for β, namely

$$(\alpha - \theta^2)(\beta - \theta^{-2}) \geq 1 - \varepsilon$$

derived from (16), and $\beta \geq \alpha/\theta$ from (15). The former gives

$$\alpha\beta \geq \alpha\theta^{-2} + \frac{(1-\varepsilon)\alpha}{\alpha - \theta^2},$$

This decreases as α increases for $\alpha < \theta^2 + 1$, since its derivative with respect to α is

$$\theta^{-2} - \frac{(1-\varepsilon)\,\theta^2}{(\alpha-\theta^2)^2} < 0.$$

On the other hand, $\alpha\beta \geq \theta^{-1}\,\alpha^2$, which increases with α. The two expressions, if we neglect ε in the former, are equal when $\alpha = 2\,\theta$, since

$$\theta^{-2}(2\,\theta) + \frac{(2\,\theta)}{2\theta-\theta^2} = 2\,\theta^{-1} + \frac{2}{2-\theta} = 2\,(\theta-1) + 2\,(\theta+1) = 4\,\theta.$$

Hence, plainly,

$$\alpha\beta \geqq 4\theta - \varepsilon' = 6.4721\ldots - \varepsilon',$$

where ε' depends only on ε and is arbitrarily small with ε.

4. *Proof of Theorems 1, 2, 3.* In § 2 we derived from the given real numbers a, b two other real numbers α, β, connected with them by (11). We modified α, β by replacing them by $\alpha\tau$, $\beta\tau'$ where τ is a unit of $k(\theta)$, or by interchanging α and β. In Lemmas 3, 5, 6, 7, 8 we have proved that the modified numbers are either 2,2 or $\sqrt{5}$, $-\sqrt{5}$ or else satisfy (29). In the last case, we have

$$M = \frac{1-\varepsilon}{|\alpha\beta|} < \frac{1}{6.34},$$

and the conclusion of Theorem 3 follows. In the two former cases, a, b are either of the form (5) or of the form (7). If they are of the form (5), then (4) is soluble with equality, but not with strict inequality, and if they are of the form (7), the same applies to (6). This proves all that was asserted.

5. We now give a proof of the inequality (9), which may be written

$$|(2\,\theta\xi-1)(2\xi'-1)| \geqq 1.$$

If $\xi = 0$, there is nothing to prove. If $\xi \neq 0$, then $|\xi\xi'| \geq 1$, so that one at least of the following inequalities is valid:

$$|\xi| \geqq \theta^{-2}, \quad |\xi'| \geqq \theta^2.$$

Case 1. *Suppose* $|\xi| \geq \theta^{-2}$. If $\xi \geq \theta^{-2}$, then

$$|2\,\theta\xi-1| = 2\,\theta\xi-1 \geqq \begin{cases} 2\xi-1, \\ 1-2\xi, \end{cases}$$

the latter since $(\theta+1)\,\xi \geq (\theta+1)\,\theta^{-2} = 1$. If $\xi \leq -\theta^{-2}$, then

$$|2\,\theta\xi-1| = 2\,\theta\,|\xi| + 1 > 2\,|\xi| + 1 = |2\xi-1|.$$

Hence, in this case,

$$|(2\,\theta\xi-1)(2\xi'-1)| \geqq |(2\xi-1)(2\xi'-1)| \geqq 1.$$

Case 2. *Suppose* $|\xi'| \geq \theta^2$. If $\xi' \geq \theta^2$, then

$$|2\xi'-1| = 2\xi'-1 \geqq 2\xi'\,\theta^{-1} + 1 = |2\xi'\,\theta'-1|,$$

since $(1-\theta^{-1})\,\xi' \geqslant (1-\theta^{-1})\,\theta^2 = 1$. If $\xi' \leqslant -\theta^2$, then

$$|2\xi'-1| = 2|\xi'| + 1 > 2|\xi'|\,\theta^{-1} - 1 = |2\xi'\theta'-1|.$$

Hence, in this case,

$$|(2\theta\xi-1|)\,(2\xi'-1)| \geqslant |(2\theta\xi-1)\,(2\theta'\xi'-1)| \geqslant 1,$$

since $\theta\xi$, $\theta'\xi'$ are conjugate integers of $k(\theta)$.

This proves the result. It is apparent from the proof that equality occurs in (9) if and only if $\xi = \theta^{-2}$, $\xi' = \theta^2$.

6. Proof of Theorem 4. We define $M(\delta)$, for any $\delta > 0$, as the lower bound of

$$|(\xi-a)\,(\xi'-b)|$$

for all integers ξ of $k(\theta)$ which satisfy

$$|\xi-a| \leqslant \delta.$$

We define M_0 by

$$M_0 = \lim_{\xi \to a} \inf |(\xi-a)\,(\xi'-b)|.$$

Then $M(\delta)$ increases (or does not decrease) as δ decreases, and

$$\lim_{\delta \to 0} M(\delta) = M_0.$$

The theorem asserts that $M_0 \leqslant \frac{1}{4}$, so that in proving it we may suppose that $M_0 > 0$.

For any $\varepsilon_0 > 0$ there exists δ such that

$$M(\delta) > M_0\,(1-\varepsilon_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Having determined δ in this way, there exists an integer ξ_0 of $k(\theta)$ such that

$$|\xi_0-a| \leqslant \delta,$$

$$|(\xi_0-a)\,(\xi_0'-b)| = \frac{M(\delta)}{1-\varepsilon}, \text{ where } 0 \leqslant \varepsilon < \varepsilon_0 \quad . \quad . \quad . \quad . \quad (32)$$

We now define α, β as before by

$$\alpha = (a-\xi_0)^{-1}, \quad \beta = (b-\xi_0')^{-1}.$$

For every integer ξ of $k(\theta)$ satisfying

$$|\alpha\xi-1| \leqslant 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

we have

$$|\xi+\xi_0-a| \leqslant |\xi_0-a| \leqslant \delta,$$

and therefore, by (32) and the definition of $M(\delta)$,

$$|(\alpha\xi-1)\,(\beta\xi'-1)| = \left|\frac{(\xi+\xi_0-a)\,(\xi'+\xi_0'-b)}{(\xi_0-a)\,(\xi_0'-b)}\right| \geqslant \frac{M(\delta)}{M(\delta)/(1-\varepsilon)} = 1-\varepsilon.$$

Hence we have the same situation as in § 2, *except* that the fundamental inequality (12) is now only known to be valid when ξ satisfies (33). In place of (13) we have

$$|\alpha\beta| = \frac{1-\varepsilon}{M(\delta)} < \frac{1-\varepsilon}{(1-\varepsilon_0)M_0} \leqq \frac{1}{(1-\varepsilon_0)M_0}. \quad . \quad . \quad . \quad (34)$$

We can still modify α, β by replacing them by $\alpha\tau$, $\beta\tau'$, where τ is any unit, but we can no longer interchange α, β without disturbing the vital condition (33). We can therefore suppose, without loss of generality, that

$$\alpha > 0, \quad \frac{\alpha}{\theta} < |\beta| \leqq \alpha\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

It will suffice to prove that

$$|\alpha\beta| \geqq 1-\varepsilon_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

where ε_1 depends only on ε and tends to zero with ε (and *is* zero if $\varepsilon = 0$). For then (34) implies $M_0 \leq \frac{1}{4}$, as required.

Case 1. *Suppose* $0 < \alpha \leq 1$. Then $\xi = 1$ satisfies the condition (33), and so

$$(1-\alpha)|\beta-1| \geqq 1-\varepsilon.$$

By (35), $|\beta| \leq \theta\alpha$, hence

$$(1-\alpha)(\theta\alpha+1) \geqq 1-\varepsilon.$$

This gives $\theta\alpha^2 \leq (\theta-1)\alpha + \varepsilon$, whence $\alpha \leq \frac{2}{5}$, since otherwise

$$\theta\alpha^2 > \tfrac{2}{5}\theta\alpha = (\theta-1)\alpha + (1-\tfrac{3}{5}\theta)\alpha > (\theta-1)\alpha + \tfrac{2}{5}(1-\tfrac{3}{5}\theta).$$

Let m be the greatest positive integer for which $m\alpha < 1$, then plainly $m\alpha \geq \frac{1}{2}$. Then $\xi = m$ satisfies the condition (33), and so

$$(1-m\alpha)(m\theta\alpha+1) \geqq 1-\varepsilon.$$

This gives $\theta(m\alpha)^2 \leq (\theta-1)m\alpha + \varepsilon$, and as before we deduce that $m\alpha \leq \frac{2}{5}$. This contradicts $m\alpha \geq \frac{1}{2}$, and so the case is impossible.

Case 2. *Suppose* $1 < \alpha < 2-\varepsilon$ *and* $\beta > 0$. We have again

$$(\alpha-1)|\beta-1| \geqq 1-\varepsilon,$$

which shows that $\beta > 1$. But now, by a well known inequality,

$$\sqrt{\alpha\beta} \geqq 1 + \sqrt{(\alpha-1)(\beta-1)} \geqq 1 + \sqrt{1-\varepsilon},$$

which proves (36).

Case 3. *Suppose* $2-\varepsilon \leq \alpha < 2.55$ *and* $\beta > 0$. Then $\xi = \theta^{-2}$ satisfies the condition (33), whence

$$|(\theta^{-2}\alpha-1)(\theta^2\beta-1)| \geqq 1-\varepsilon.$$

Since $\beta \geq (2-\varepsilon)/\theta > \theta^{-2}$, this implies

$$\beta \geqq \theta^{-2} + \frac{1-\varepsilon}{\theta^2 - \alpha},$$

$$\alpha\beta \geqq \theta^{-2}\alpha - (1-\varepsilon) + \frac{(1-\varepsilon)\theta^2}{\theta^2-\alpha}.$$

This expression plainly increases with α, hence

$$\alpha\beta \geqq \theta^{-2}(2-\varepsilon) - (1-\varepsilon) + \frac{(1-\varepsilon)\theta^2}{\theta^2-2+\varepsilon} \geqq 4-\varepsilon_1,$$

since $2\theta^{-2} - 1 + \theta^2/(\theta^2-2) = 4$.

Case 4. *Suppose* $1 < \alpha < 2.55$ *and* $\beta < 0$. It is impossible that $\alpha < 1.3$, for then $|\beta| \leq \alpha\theta < 2.2$, and

$$|(\alpha-1)(\beta-1)| < (0.3)(2.2+1) < 1-\varepsilon,$$

which is impossible. Hence $\alpha \geq 1.3$.

Since $\xi = \theta^{-1}$ satisfies the condition (33), we have

$$|(\alpha-\theta)(\beta-\theta')| \geqq 1-\varepsilon.$$

Writing $\beta = -\beta'$, where $\beta' > 0$, we have $\beta' \geq \alpha\theta^{-1} > \theta^{-1}$. Hence

$$\beta' \geqq \theta^{-1} + \frac{1-\varepsilon}{|\alpha-\theta|},$$

$$\alpha\beta' \geqq \alpha\theta^{-1} + \frac{(1-\varepsilon)\alpha}{|\alpha-\theta|}.$$

This expression (if we neglect ε) increases as α increases from 1.3 to θ, then decreases as α increases from θ to 2.55. When $\alpha = 1.3$ its value is

$$(1.3)(0.618\ldots) + \frac{(1-\varepsilon)(1.3)}{0.318\ldots} > 4.5,$$

and when $\alpha = 2.55$ its value is

$$(2.55)(0.618)\ldots) + \frac{(1-\varepsilon)(2.55)}{0.932\ldots} > 4.3.$$

Hence (36) holds.

Case 5. *Suppose* $\alpha \geq 2.55$. By (35),

$$\alpha|\beta| \geqq \alpha^2\theta^{-1} \geqq (2.55)^2(0.618) > 4.01.$$

This completes the proof of (36), and so of Theorem 4.

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Mathematics. — *On the theory of simultaneous linear and quadratic representation.* Part V. By F. VAN DER BLIJ. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of February 22, 1947.)

§ 21. *Polygon numbers.*

In this paragraph we consider the representation of rational integers by sums of polygon numbers. We use generalized polygon numbers that is to say numbers $t_m = (\frac{1}{2}m - 1)x^2 - (\frac{1}{2}m - 2)x$, with rational integers x.

We consider the solutions of the equation

$$\mathbf{x}'\mathbf{Q}\mathbf{x} = N, \quad \ldots\ldots\ldots \quad (21.1)$$

which satisfy the congruence

$$\mathbf{x} \equiv \mathbf{a} \pmod{\mu} \quad \ldots\ldots\ldots \quad (21.2)$$

It may be seen readily that the representation by sums of polygon numbers forms a special case of this problem. For we may replace

$$\sum_{i=1}^{r} (\tfrac{1}{2}m-1)x_i^2 - (\tfrac{1}{2}m-2)x_i = N$$

by

$$\sum_{i=1}^{r} \{2(m-2)x_i - (m-4)\}^2 = 8(m-2)N + r(m-4)^2,$$

and this can be written as

$$\sum_{i=1}^{r} y_i^2 = M, \quad y_i \equiv -m + 4 \pmod{2(m-2)}.$$

We are able to determine the number of solutions of this system in some cases, if we replace it by a system of a quadratic equation and a linear one, this system can be treated with the theory developed in the preceding paragraphs. We avoid this roundabout way in the examples given later on.

There is no essential restriction in considering only the system

$$\left.\begin{aligned} \sum_{i=1}^{r} y_i^2 &= N, \\ y_i &\equiv 1 \pmod{\mu}, \end{aligned}\right\} \ldots\ldots\ldots \quad (21.3)$$

if we should study the representation by polygon numbers.

If we consider the system

$$\mathbf{x}'\mathbf{Q}\mathbf{x} = N, \quad \mathbf{x} \equiv \mathbf{a} \pmod{\mu} \quad \ldots\ldots \quad (21.4)$$

we will suppose the congruence

$$\mathbf{a}'\mathbf{Q}\mathbf{a} \equiv N \pmod{\mu} \quad \ldots\ldots\ldots \quad (21.5)$$

to be satisfied, else there are no solutions of the system. We introduce

$$\mathbf{x} = \mu \mathbf{y}_0 + \mathbf{a}\zeta$$

and we obtain

$$\begin{cases} \mu^2\, \mathbf{y}_0' \mathbf{Q}\, \mathbf{y}_0 + 2\,\mu\,\zeta\, \mathbf{y}_0' \mathbf{Q}\, \mathbf{a} + \zeta^2\, \mathbf{a}'\, \mathbf{Q}\, \mathbf{a} = N, \\ \zeta = 1. \end{cases}$$

Let now

$$\begin{pmatrix} \mathbf{y}_0 \\ \zeta \end{pmatrix} = \begin{pmatrix} \mathbf{W} & \mathbf{a} \\ \mathbf{z}' & -\mu \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ \xi \end{pmatrix}, \text{ where } \left| \begin{pmatrix} \mathbf{W} & \mathbf{a} \\ \mathbf{z}' & -\mu \end{pmatrix} \right| = \pm 1;$$

thus we consider

$$\mathbf{y}'\mathbf{Q}_1\mathbf{y} = \mu^2\, \mathbf{y}'\mathbf{W}'\mathbf{Q}\mathbf{W}\mathbf{y} + \mu\mathbf{y}'\mathbf{z}\mathbf{a}'\mathbf{Q}\mathbf{W}\mathbf{y} + \\ + \mu\mathbf{y}'\mathbf{W}'\mathbf{Q}\mathbf{a}\mathbf{z}'\mathbf{y} + \mathbf{y}'\mathbf{z}\mathbf{a}'\mathbf{Q}\mathbf{a}\mathbf{z}'\mathbf{y} = N,$$

$$\mathbf{z}'\mathbf{y} - \mu\xi = 1.$$

Now we search the number of solutions of

$$\mathbf{y}'\mathbf{Q}_1\mathbf{y} = N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21.6)$$

satisfying

$$\mathbf{y}'\mathbf{z} \equiv 1 \pmod{\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21.7)$$

It may be seen readily from the definition of $\mathbf{Q}_1$ that the solutions of (21.6) satisfy

$$(\mathbf{y}'\mathbf{z})^2\, \mathbf{a}'\mathbf{Q}\mathbf{a} \equiv N \pmod{\mu} \quad . \quad . \quad . \quad . \quad . \quad (21.8)$$

And we supposed the congruence

$$\mathbf{a}'\mathbf{Q}\mathbf{a} \equiv N \pmod{\mu}$$

to be satisfied.

If $(\mathbf{a}'\mathbf{Q}\mathbf{a},\ \mu) = 1$ and μ is either a power of a prime of the double of a power of a prime the congruence can be omitted if we halve the number of solutions if $\mu > 2$, if $\mu = 2$ the congruence can be omitted. In some instances we consider systems with $(\mathbf{a}'\mathbf{Q}\mathbf{a},\ \mu) > 1$, too.

First we consider the system (21.3) with $r = 2$.

This may be replaced by

$$\mathbf{y}'\,\mathbf{Q}_1\,\mathbf{y} = \mu^2\, y_1^2 + 2\,\mu\,(1-\mu)\, y_1\, y_2 + (\mu^2 - 2\,\mu + 2)\, y_2^2 = N\ ,\ y_2 \equiv 1 \pmod{\mu}.$$

After an elementary transformation we write this

$$\mu^2\, x^2 + 2\,\mu\, xy + 2\, y^2 = N\ ,\ y \equiv 1 \pmod{\mu}.$$

If μ is even we simplify once more and we get

$$\left(\frac{\mu}{2}\right)^2 x^2 + u^2 = \frac{N}{2}\ ,\ \frac{\mu}{2}\, x + u \equiv 1 \pmod{\mu}.$$

Trigonal numbers being closely connected with odd squares, the representation by sums of three trigonal numbers can be deduced from that by sums of three squares. The formulae being complicated will not be considered here.

According to the researches of STREEFKERK [15] we consider the

representation by a sum of three pentagonal numbers or by a sum of three octagonal numbers. We commence with octagonal numbers

$$\sum_{i=1}^{3}(3x_i^2-2x_i)=N; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21.9)$$

$$\sum_{i=1}^{3}(3x_i-1)^2=3(N+1);$$

$$\sum_{i=1}^{3}z_i^2=3(N+1)\ ,\ z_i\equiv 1 \pmod 3.$$

Now we determine the number of those solutions of

$$\mathbf{y}'\mathbf{Q}\mathbf{y}=(3y_1-2y_3)^2+(3y_2-2y_3)^2+y_3^2=3(N+1), \quad (21.10)$$

which satisfy

$$y_3\equiv 1 \pmod 3.$$

Those solutions of (21.10) which satisfy $y_3\equiv 0 \pmod 3$ are the solutions of

$$(\eta_1-2\eta_3)^2+(\eta_2-2\eta_3)^2+\eta_3^2=\frac{N+1}{3},$$

or of the equation

$$\zeta_1^2+\zeta_2^2+\zeta_3^2=\tfrac{1}{3}(N+1). \quad . \quad . \quad . \quad . \quad . \quad (21.11)$$

Theorem 9a. *The number of solutions of the system* (21.9) *equals* $\frac{1}{2}A(\mathbf{Q})-\frac{1}{2}A_3$, *where*

$A(\mathbf{Q})$ *denotes the number of solutions of* (21.10).

A_3 *denotes the number of solutions of* (21.11).

For the representation by sums of three pentagonal numbers we consider

$$\sum_{i=1}^{3}(\tfrac{3}{2}x_i^2-\tfrac{1}{2}x_i)=N; \quad . \quad . \quad . \quad . \quad . \quad . \quad (21.12)$$

or

$$\sum_{i=1}^{3}z_i^2=3(8N+1), \quad z_i\equiv 1 \pmod 6.$$

We replace this by

$$\mathbf{y}'\mathbf{Q}_2\mathbf{y}=(6y_1+y_3)^2+(6y_2+y_3)^2+y_3^2=24N+3, \quad (21.13)$$

and

$$y_3\equiv 1 \pmod 6.$$

It can be deduced from (21.13) that $y_3\equiv 1 \pmod 2$.

Theorem 9b. *The number of solutions of the system* (21.12) *equals* $\frac{1}{2}A(\mathbf{Q}_2)-\frac{1}{2}A(\mathbf{Q}_3)$, *where*

$A(\mathbf{Q}_2)$ *denotes the number of solutions of* (21.13).

$A(\mathbf{Q}_3)$ *denotes the number of solutions of the equation*

$$(2\eta_1+\eta_3)^2+(2\eta_2+\eta_3)^2+\eta_3^2=\tfrac{1}{3}(8N+1).$$

The numbers $A(\mathbf{Q})$, A_3, $A(\mathbf{Q}_2)$, $A(\mathbf{Q}_3)$ which occur in theorem 9a and 9b can be calculated from the general theory of quadratic forms of SIEGEL [14]. The formulae thus obtained are neither simple nor surveyable.

At last we consider the representation by sums of four trigonal numbers and that by sums of four octogonal numbers. First trigonal numbers.

$$\sum_{l=1}^{4} \left(\tfrac{1}{2} x_l^2 + \tfrac{1}{2} x_l\right) = m$$

We introduce $N = 2m + 1$ and $A_4(\triangle)$ denotes the number of representations of $\triangle$ as a sum of four squares. Considering

$$\sum_{l=1}^{4} (2x_l + 1)^2 = 4N$$

we get the well-known

Theorem 10. *The number of representations of m as a sum of four trigonal numbers equals*

$$A_4(4N) - A_4(N) = 16 \sum_{d \mid 2m+1} d.$$

In the case of octogonal numbers we consider

$$\sum_{l=1}^{4} (3x_l^2 - 2x_l) = m.$$

we replace this by

$$\sum_{l=1}^{4} z_l^2 = 3m + 4 = N, \quad z_l \equiv 1 \pmod 3.$$

Thus we must calculate the number of solutions of the equation

$$(3y_1 + y_4)^2 + (3y_2 + y_4)^2 + (3y_3 + y_4)^2 + y_4^2 = N.$$

We find about this number

$$A_0 = \frac{\pi^2 N}{27} \prod_p \alpha_p,$$

and

$$\alpha_p = \left(1 - \frac{1}{p^2}\right) \sum_{d \mid p^l} d^{-1}, \text{ if } p^l \parallel N \text{ and } (p, 6) = 1.$$

$$\alpha_2 = 1,$$

$$\alpha_3 = 2.$$

Thus

$$A_0 = \tfrac{8}{9} \sum_{d \mid N} d.$$

If we introduce here $N = 7$, the number A_0 becomes a fractional one, so the strict value of the number of solutions is not given by the singular series.

§ 22. *Degenerated cubic forms.*

We study the representation of rational integers by degenerated cubic forms. If we suppose the cubic form to be degenerated into the product of three linear forms the problem is quite elementary. Thus we consider those cubic forms which are degenerated into the product of a linear form and a quadratic one.

First we study the simple equation

$$x^3 + y^3 = N, \quad N > 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (22.1)$$

or

$$(x + y)\,(x^2 - xy + y^2) = N.$$

This may be replaced by the equations

$$\left.\begin{aligned} x^2 - xy + y^2 &= \delta, \\ x + y \quad &= d; \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (22.2)$$

where d runs through all positive divisors of N and $d\,\delta = N$.

There are two rational integral solutions of (22.2) if $12\delta - 3d^2$ a positive square and there is one solution if $12\delta = 3d^2$.

Theorem 11. *The number of solutions of the equation* (22.1) *equals the double of the number of positive divisors* d *of* N *such that* $12\frac{N}{d} - 3d^2$ *is a positive square, this number must be increased by one if* $4N$ *is the cubic of a rational integer.*

This theorem, though it is not very complicated, is not "beautiful", for it is not a multiplicative one.

If the determinant of the quadratic form, which is a factor of the cubic one vanishes the formulae become multiplicative ones. We will give an example.

The equation

$$x^3 + y^3 + z^3 - 3xyz = N, N > 0 \quad . \quad . \quad . \quad (22.3)$$

can be written as

$$(x + y + z)\,(x^2 + y^2 + z^2 - xy - xz - yz) = N.$$

Now we consider the system

$$4(x^2+y^2+z^2-xy-xz-yz) \equiv (2x-y-z)^2 + 3(y-z)^2 = 4\delta, \qquad d\,\delta = N,$$
$$x + y + z = d, \qquad d > 0.$$

First we suppose $N \not\equiv 0 \pmod 3$.

Then half the number of solutions of the equation

$$u^2 + 3v^2 = 4\delta,$$

satisfy the congruence

$$u \equiv d \pmod 3$$

since $d \not\equiv 0$, $\delta \not\equiv 0 \pmod 3$.

The number of solutions of the equation (22.3) equals in this case

$$\sum_{\bar{d}\mid N}\left\{3\sum_{d_0\mid\bar{d}}\left(\frac{d_0}{3}\right)\right\}=3\left(\frac{N}{3}\right)\sum_{d\mid N}\left(\frac{d}{3}\right)\tau(d).$$

where $\tau(d)$ denotes the number of positive divisors of d.

If $N\equiv 0 \pmod 3$ there are no solutions unless $N\equiv 0 \pmod 9$. Thus we suppose

$$N\equiv 0 \pmod 9,\quad N=9N_1.$$

The number of solutions of the quadratic equation $u^2+3v^2=4\bar{d}$ equals

$$6\sum_{d_0\mid\bar{d}}\left(\frac{d_0}{3}\right).$$

Theorem 12. *The number of solutions of the cubic Diophantic equation* (22.3) *equals*

$$\text{if } N\not\equiv 0 \pmod 3 \ldots\ldots 3\left(\frac{N}{3}\right)\sum_{d\mid N}\left(\frac{d}{3}\right)\tau(d).$$

$$\text{if } N\equiv 0 \pmod 9,\ N=9N_1 \ldots\ldots 6\sum_{d\mid N_1}\left(\frac{d}{3}\right)\tau\left(\frac{N_1}{d}\right).$$

Each cubic form which is a symmetric function of it variables and that is degenerated into the product of a linear form and a quadratic one with determinant zero can be written as:

$$\left(\sum_{i=1}^{r}x_i\right)\left\{r\sum_{i=1}^{r}x_i^2-\left(\sum_{i=1}^{r}x_i\right)^2\right\}.$$

The number of the solutions of the equation

$$(r-1)(\Sigma x_i)^3+(r-3)(\Sigma x_i x_j^2)-6\Sigma x_i x_j x_k=N\ ;\ N>0 \quad (22.4)$$

equals

$$\sum_{d\mid N}A(d),$$

where $A(d)$ denotes the number of the solutions of the system

$$\begin{cases} r\sum_{i=1}^{r}x_i^2=\bar{d}+d^2, \\ \qquad\qquad\qquad d\bar{d}=N. \\ \sum_{i=1}^{r}x_i=d, \end{cases}$$

We know that this number $A(d)$ is almost a multiplicative function of r. $r^{-1}(\bar{d}+d^2)-d^2=\bar{d}$.

At last we consider a cubic form, which is not a symmetric function, to know

$$x^3+y^3+z^3+x^2(y+z)+x(y^2-2yz+z^2)-yz(y+z).$$

We consider the equation

$$(x+y+z)\{(x+y-z)^2+(x-y+z)^2\}=2N,\ N>0 \quad . \quad (22.5)$$

and the system

$$\begin{aligned}(x+y-z)^2+(x-y+z)^2 &= 2\,\delta,\\ x+y+z &= d,\end{aligned}\qquad d\,\delta = N.$$

We replace this by the equation

$$u^2+v^2 = 2\,\delta$$

here u and v must satisfy this condition: we must be able to determine rational integers x, y and z from the equations

$$\begin{aligned}x+y-z &= u,\\ x-y+z &= v,\\ x+y+z &= d.\end{aligned}$$

From this we obtain the condition $u \equiv v \equiv d \pmod 2$ so $d \equiv \delta \pmod 2$.

Theorem 13. *The number of solutions of the equation* (22.5) *equals*

$$\text{if } N \equiv 1 \pmod 2 \qquad 4\left(\frac{-1}{N}\right)\sum_{d\mid N}\left(\frac{-1}{d}\right)\tau(d);$$

$$\text{if } N \equiv 2 \pmod 4 \qquad 0$$

$$\text{if } \begin{cases} N \equiv 0 \pmod 4 \\ 2^\alpha \parallel N,\ N = 2^\alpha N_1. \end{cases} \qquad 4(\alpha-1)\left(\frac{-1}{N_1}\right)\sum_{d\mid N}\left(\frac{-1}{d}\right)\tau(d).$$

Applied Mechanics. — *On the restricted applicability of the principle of least work in the plastic domain.* By P. P. BIJLAARD. (Communicated by Prof. J. M. BURGERS.)

(Communicated at the meeting of March 29, 1947.)

In preceding publications [1] [7] we proved, that with large plastic deformations, in relation to which the elastic deformations may be neglected, the stresses in a centrically loaded weakened strip of a steel plate (fig. 1) will adjust themselves in such a manner, that the strain

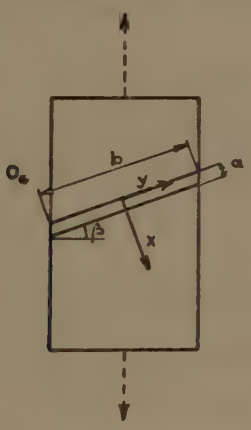

Fig. 1.

energy becomes a minimum. This may be readily understood, as plastic deformations will only occur in the weakened oblique part, where practically a homogeneous stress distribution will prevail. Hence in this region the modulus of plasticity E_p will be a constant, so that the stresses establish themselves in the same way as in the elastic domain, with the only difference that now the coefficient of lateral contraction m is equal to 2, in consequence of which the strain energy is only shearing energy. It follows that the elastic shearing energy V_s, which is the controlling factor for the plastic deformation, is a minimum, so that the plastic deformations will be as small as possible and a maximum resistance will be offered against external forces. These arguments hold equally well in the domain of strain hardening, when the slope φ of the stress-strain graph has a positive value, as at the yield point.

Hence both with elastic and with purely plastic deformations the strain

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1) BIJLAARD. De plastische vervorming van vloei-ijzer en de berekening van ijzerconstructies. De Ingenieur. Nr. 23 (1933).

energy acquires extreme values. In the transitional domain between elastic and mainly plastic deformations the state of stress will have to change from one, corresponding to minimum elastic strain energy, to one, corresponding to minimum strain energy with m equal 2, *i.e.* minimum shearing energy. One might be inclined to think that here the state of stress would correspond to a minimum of the total, *i.e.* elastic plus plastic strain energy, but this is not true.

Let us consider for example a simple case, in which the angle β in fig. 1 equals zero and the weakened strip is bordered by infinitely rigid thin plates. Owing to the junction to the rigid plates, the strain ε_y in the weakened strip will be zero. Furthermore σ_z equals zero. If the weakened strip deforms only elastically the condition $\varepsilon_y = 0$ will give $\sigma_y = \sigma_x/m = 0{,}3\,\sigma_x$, whilst with large plastic deformations (with $m = 2$) the stress σ_y will increase up to $0{,}5\,\sigma_x$. According to the condition of yielding

$$\sigma_x^2 - \sigma_x\,\sigma_y + \sigma_y^2 = \sigma_v^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in the latter case σ_x will increase up to $1{,}154\,\sigma_v$. If for example σ_x equals $1{,}150\,\sigma_v$, the stress σ_y according to eq. (1) will be

$$\sigma_y = \sigma_x/2 - \sqrt{\sigma_v^2 - 3\sigma_x^2/4} = 0{,}485\,\sigma_v;$$

and, according to our former publications [2]),

$$\varepsilon_x = \frac{(m-2)\,(3\sigma_x^2 - 2\sigma_v^2 + \sigma_x\sqrt{4\sigma_v^2 - 3\sigma_x^2})}{2mE\sqrt{4\sigma_v^2 - 3\sigma_x^2}} = 2{,}42\,\sigma_v/E.$$

If, however, σ_y would be a little greater, for example $0{,}5\,\sigma_v$, which would be statically possible just as well, the yield stress would not be reached at all, as may be seen directly from the course of the ellipse (1). In that case the deformations would be purely elastic and ε_x would only amount to

$$\varepsilon_x = \sigma_x/E - 0{,}5\,\sigma_v/mE = (1{,}15 - 0{,}15)\,\sigma_v/E = 1{,}00\,\sigma_v/E.$$

In this imaginary case the total strain energy would be less than half of that which really occurs.

That in this transitional domain the total strain energy V will generally not acquire extreme values, may be proved by means of the calculus of variations. The first variation of the total strain energy V may be written as

$$\delta V = \delta V_e + \delta V_p,$$

V_e and V_p being the strain energies for elastic and plastic deformations respectively. Considering the general case of a three dimensional state of

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[2]) BIJLAARD. A theory of plastic buckling with its application to geophysics. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, Nr. 5 (1938), eq. (8).

stress we have, distinguishing the elastic strains by the subscript e [3]):

$$\delta V_e = \iiint \delta V_{e1}\, dx\, dy\, dz = \iiint (\varepsilon_{xe}\, \delta\sigma_x + \varepsilon_{ye}\, \delta\sigma_y + \varepsilon_{ze}\, \delta\sigma_z + \gamma_{xye}\, \delta\tau_{xy} + \gamma_{yze}\, \delta\tau_{yz} + \gamma_{zxe}\, \delta\tau_{zx})\, dx\, dy\, dz \quad . \quad . \quad . \quad (2)$$

the value V_{e1} being the elastic strain energy per unit of volume.

As already mentioned, in the transitional domain between elastic and mainly plastic deformations the state of stress will change. We assume that under these circumstances the total plastic deformations and the state of stress determine each other reciprocally, in such a way, that the ratio of the total plastic strains is always equal to that of the deviator components of the stresses [4]), so that

$$\varepsilon_{xp} = \bar{\sigma}_x / 2G_p, \text{ etc.}; \quad \gamma_{xyp}/2 = \tau_{xy}/2G_p, \text{ etc.} \quad . \quad . \quad . \quad . \quad (3)$$

the deviator stress $\bar{\sigma}_x$ being for example equal to $\sigma_x - \sigma$, whilst $\sigma = (\sigma_x + \sigma_y + \sigma_z)/3$. In consequence of this assumption the state of stress will be the same as when the plastic strains would have been built up in the ratio they finally acquire. In a former publication [5]) we have compared this mode of deformation with other possibilities and found that with problems of plastic stability we remain at the safe side with it.

We will now compute the first variation of the plastic strain energy V_p in such a way, that the derived formulae at the same time will have a more general value. If in the case of pure extension or compression the relation between the stress and the plastic strain ε_p is an arbitrary one, the work performed by σ_x is $\int \sigma_x d\varepsilon_{xp}$ per unit volume (fig. 2), etc. Hence

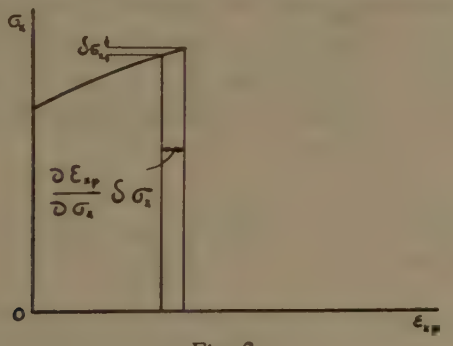

Fig. 2.

the total strain energy for the plastic deformation is

$$V_p = \iiint V_{p1}\, dx\, dy\, dz = \iiint (\sigma_x\, d\varepsilon_{xp} + \sigma_y\, d\varepsilon_{yp} + \sigma_z\, d\varepsilon_{zp} + \tau_{xy}\, d\gamma_{xyp} + \tau_{yz}\, d\gamma_{yzp} + \tau_{zx}\, d\gamma_{zxp})\, dx\, dy\, dz \quad . \quad . \quad . \quad (4)$$

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[3]) FÖPPL. Vorlesungen über Technische Mechanik, Part V (1922), p. 281—286. TIMOSHENKO. Theory of elasticity, p. 145—151.

[4]) PRAGER. Mécanique des solides isotropes au delà du domaine élastique. Mémorial des sciences mathématiques, fasc. 87 (1937).

[5]) BIJLAARD. A theory of plastic stability and its application to thin plates of structural steel. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, Nr. 7 (1938).

and the first variation δV_{p1} per unit volume for arbitrary variations of the stress components can be written:

$$\left.\begin{aligned}\delta V_{p1} = \frac{\partial V_{p1}}{\partial \sigma_x}\delta\sigma_x + \frac{\partial V_{p1}}{\partial \sigma_y}\delta\sigma_y + \frac{\partial V_{p1}}{\partial \sigma_z}\delta\sigma_z + \\ + \frac{\partial V_{p1}}{\partial \tau_{xy}}\delta\tau_{xy} + \frac{\partial V_{p1}}{\partial \tau_{yz}}\delta\tau_{yz} + \frac{\partial V_{p1}}{\partial \tau_{zx}}\delta\tau_{zx}\end{aligned}\right\} \quad \ldots \quad (5)$$

A variation $\delta\sigma_x$ will, according to eq. (4) and fig. 2, produce an excess work $\sigma_x(\partial\varepsilon_{xp}/\partial\sigma_x)\delta\sigma_x$ per unit volume in consequence of the corresponding variation of ε_{xp}, but it will also evoke variations of the other strain components, so that

$$\frac{\partial V_{p1}}{\partial \sigma_x} = \sigma_x \frac{\partial \varepsilon_{xp}}{\partial \sigma_x} + \sigma_y \frac{\partial \varepsilon_{yp}}{\partial \sigma_x} + \sigma_z \frac{\partial \varepsilon_{zp}}{\partial \sigma_x} + \tau_{xy} \frac{\partial \gamma_{xyp}}{\partial \sigma_x} + \tau_{yz} \frac{\partial \gamma_{yzp}}{\partial \sigma_x} + \tau_{zx} \frac{\partial \gamma_{zxp}}{\partial \sigma_x}. \quad (6)$$

In the domain of strain hardening a definite relation $S_1 = \Phi(E_1)$ exists between

$$S_1 = \sqrt{\bar{\sigma}_x^2 + \bar{\sigma}_y^2 + \bar{\sigma}_z^2 + 2(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}$$

and

$$E_1 = \sqrt{\varepsilon_{xp}^2 + \varepsilon_{yp}^2 + \varepsilon_{zp}^2 + \tfrac{1}{2}(\gamma_{xyp}^2 + \gamma_{yzp}^2 + \gamma_{zxp}^2)},$$

S_1^2 being equal to $4GV_s$, in which G is the elastic modulus of rigidity and V_s is the elastic shearing energy. In the case of pure extension for which we may write $\sigma = E_p\varepsilon_p$ (fig. 3), we have $S_1 = \sigma\sqrt{2/3}$, $E_1 = \varepsilon_p\sqrt{3/2}$, so that $S_1 = 2E_pE_1/3 = 2G_pE_1$ (fig. 4), as with plastic deformation $G_p = mE_p/2(m+1) = E_p/3$. Hence for corresponding points of figs. 3 and 4 we have $\tan\varphi_s = \frac{2}{3}\tan\varphi$.

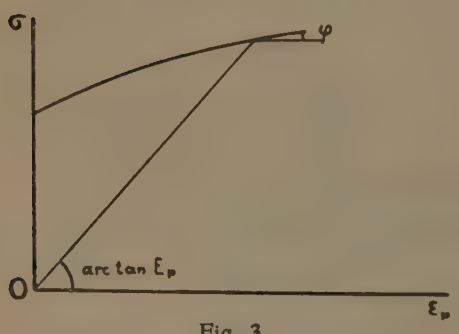

Fig. 3.

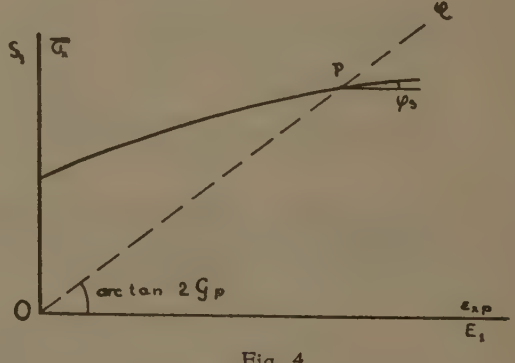

Fig. 4.

From the data given above we compute

$$\frac{\partial\bar{\sigma}_x}{\partial\sigma_x} = \tfrac{2}{3},\quad \frac{\partial\bar{\sigma}_y}{\partial\sigma_x} = \frac{\partial\bar{\sigma}_z}{\partial\sigma_x} = -\tfrac{1}{3},\quad \frac{\partial S_1}{\partial\sigma_x} = \frac{\bar{\sigma}_x}{S_1},$$

and, also according to fig. 4,

$$\left.\begin{aligned}\frac{\partial}{\partial\sigma_x}\left(\frac{1}{G_p}\right)&=\frac{\partial\,(2E_1/S_1)}{\partial\sigma_x}=\frac{d\,(2E_1/S_1)}{dS_1}\frac{\partial S_1}{\partial\sigma_x}=\\&=\frac{2S_1/\tan\varphi_s-2E_1}{S_1^2}\frac{\bar\sigma_x}{S_1}=\frac{\bar\sigma_x}{S_1^2\,G_p}\left(\frac{2G_p}{\tan\varphi_s}-1\right)\end{aligned}\right\}\quad\ldots\quad(7)$$

from which e.g.

$$\left.\begin{aligned}\frac{\partial\varepsilon_{xp}}{\partial\sigma_x}&=\frac{1}{2G_p}\frac{\partial\bar\sigma_x}{\partial\sigma_x}+\frac{\bar\sigma_x}{2}\frac{\partial}{\partial\sigma_x}\left(\frac{1}{G_p}\right)=\frac{1}{E_p}+\frac{\bar\sigma_x^2}{2S_1^2\,G_p}\left(\frac{2G_p}{\tan\varphi_s}-1\right)\\\frac{\partial\varepsilon_{yp}}{\partial\sigma_x}&=\frac{1}{2G_p}\frac{\partial\bar\sigma_y}{\partial\sigma_x}+\frac{\bar\sigma_y}{2}\frac{\partial}{\partial\sigma_x}\left(\frac{1}{G_p}\right)=-\frac{1}{2E_p}+\frac{\bar\sigma_x\,\bar\sigma_y}{2S_1^2\,G_p}\left(\frac{2G_p}{\tan\varphi_s}-1\right)\\\frac{\partial\gamma_{xyp}}{\partial\sigma_x}&=\frac{\bar\sigma_x\,\tau_{xy}}{S_1^2\,G_p}\left(\frac{2G_p}{\tan\varphi_s}-1\right)\end{aligned}\right\}.\quad(8)$$

The other partial derivatives with respect to σ_x will be obvious. Substituting them into eq. (6) we find, after summation,

$$\frac{\partial V_{p1}}{\partial\sigma_x}=\frac{2\sigma_x-\sigma_y-\sigma_z}{2E_p}+$$
$$+\frac{\bar\sigma_x}{2S_1^2\,G_p}\left(\frac{2G_p}{\tan\varphi_s}-1\right)(\sigma_x\,\bar\sigma_x+\sigma_y\,\bar\sigma_y+\sigma_z\,\bar\sigma_z+2\tau_{xy}^2+2\tau_{yz}^2+2\tau_{zx}^2).$$

As

$$\sigma_x\,\bar\sigma_x+\sigma_y\,\bar\sigma_y+\sigma_z\,\bar\sigma_z=\bar\sigma_x^2+\bar\sigma_y^2+\bar\sigma_z^2+\sigma\,(\bar\sigma_x+\bar\sigma_y+\bar\sigma_z)=\bar\sigma_x^2+\bar\sigma_y^2+\bar\sigma_z^2,$$

we have

$$\frac{\partial V_{p1}}{\partial\sigma_x}=\frac{\bar\sigma_x}{2G_p}+\frac{\bar\sigma_x}{2G_p}\left(\frac{2G_p}{\tan\varphi_s}-1\right)=\frac{\bar\sigma_x}{\tan\varphi_s}.\quad\ldots\quad(9)$$

The partial derivatives of V_p with respect to the other stress components are derived in the same way. We find e.g.

$$\left.\begin{aligned}\frac{\partial}{\partial\tau_{xy}}\left(\frac{1}{G_p}\right)&=\frac{2\tau_{xy}}{S_1^2\,G_p}\left(\frac{2G_p}{\tan\varphi_s}-1\right)\\\frac{\partial\varepsilon_{xp}}{\partial\tau_{xy}}&=\frac{\bar\sigma_x\,\tau_{xy}}{S_1^2G_p}\left(\frac{2G_p}{\tan\varphi_s}-1\right)=\frac{\partial\gamma_{xyp}}{\partial\sigma_x}\\\frac{\partial\gamma_{xyp}}{\partial\tau_{xy}}&=\frac{1}{G_p}+\frac{2\tau_{xy}^2}{S_1^2\,G_p}\left(\frac{2G_p}{\tan\varphi_s}-1\right)\\\frac{\partial\gamma_{yzp}}{\partial\tau_{xy}}&=\frac{2\tau_{xy}\,\tau_{yz}}{S_1^2\,G_p}\left(\frac{2G_p}{\tan\varphi_s}-1\right)=\frac{\partial\gamma_{xyp}}{\partial\tau_{yz}}\end{aligned}\right\}.\quad\ldots\quad(10)$$

Thus we obtain e.g.

$$\left.\begin{aligned}\frac{\partial V_{p1}}{\partial \tau_{xy}} = \sigma_x \frac{\partial \varepsilon_{xp}}{\partial \tau_{xy}} + \sigma_y \frac{\partial \varepsilon_{yp}}{\partial \tau_{xy}} + \sigma_z \frac{\partial \varepsilon_{zp}}{\partial \tau_{xy}} + \tau_{xy} \frac{\partial \gamma_{xyp}}{\partial \tau_{xy}} + \tau_{yz} \frac{\partial \gamma_{yzp}}{\partial \tau_{xy}} + \\ + \tau_{zx} \frac{\partial \gamma_{zxp}}{\partial \tau_{xy}} = \frac{\tau_{xy}}{G_p} + \frac{\tau_{xy}}{S_1^2 G_p}\left(\frac{2G_p}{\tan \varphi_s} - 1\right) S_1^2 = \frac{2\tau_{xy}}{\tan \varphi_s}\end{aligned}\right\} \quad (11)$$

so that eqs. (4) and (5) yield

$$\left.\begin{aligned}\delta V_p = \iiint \left(\frac{\bar{\sigma}_x}{\tan \varphi_s} \delta\sigma_x + \frac{\bar{\sigma}_y}{\tan \varphi_s} \delta\sigma_y + \frac{\bar{\sigma}_z}{\tan \varphi_s} \delta\sigma_z + \right. \\ \left. + \frac{2\tau_{xy}}{\tan \varphi_s} \delta\tau_{xy} + \frac{2\tau_{yz}}{\tan \varphi_s} \delta\tau_{yz} + \frac{2\tau_{zx}}{\tan \varphi_s} \delta\tau_{zx}\right) dx\, dy\, dz\end{aligned}\right\}. \quad (12)$$

As $\delta\sigma_x = \delta\bar{\sigma}_x + \delta\sigma$, etc. and $\bar{\sigma}_x + \bar{\sigma}_y + \bar{\sigma}_z = 0$, eq. (12) may also be written as follows

$$\left.\begin{aligned}\delta V_p = \iiint (1/\tan \varphi_s)(\bar{\sigma}_x\, \delta\bar{\sigma}_x + \bar{\sigma}_y\, \delta\bar{\sigma}_y + \bar{\sigma}_z\, \delta\bar{\sigma}_z + 2\tau_{xy}\, \delta\tau_{xy} + \\ + 2\tau_{yz}\, \delta\tau_{yz} + 2\tau_{zx}\, \delta\tau_{zx})\, dx\, dy\, dz = \\ = \iiint \{(1/2\tan \varphi_s)\, \delta(\bar{\sigma}_x^2 + \bar{\sigma}_y^2 + \bar{\sigma}_z^2 + 2\tau_{xy}^2 + 2\tau_{yz}^2 + 2\tau_{zx}^2)\}\, dx\, dy\, dz = \\ = \iiint (\delta S_1^2/2\tan \varphi_s)\, dx\, dy\, dz.\end{aligned}\right\} \quad (13)$$

The variation was arbitrary, so that the ratios $\delta\sigma_x/\sigma_x$, $\delta\sigma_y/\sigma_y$, etc. will not be equal. From eq. (13) we see, however, that we may split up the variation into one, for which the ratio of the stress components and hence also that of the deviator components, does not change, but with varying values of E_1 and S_1; and one, for which E_1 and S_1 remain constant, whilst the ratio in question changes; eq. (13) then says that during the latter variation $\delta V_p = 0$. As with constant E_1 and S_1 the value of G_p is a constant too, so that for example $\delta\varepsilon_{xp} = \delta\bar{\sigma}_x/2G_p$, it follows also more directly, that in this case

$$\delta V_p = \iiint (1/2G_p)(\bar{\sigma}_x\, \delta\bar{\sigma}_x + \bar{\sigma}_y\, \delta\bar{\sigma}_y + \bar{\sigma}_z\, \delta\bar{\sigma}_z + 2\tau_{xy}\, \delta\tau_{xy} + \\ + 2\tau_{yz}\, \delta\tau_{yz} + 2\tau_{zx}\, \delta\tau_{zx})\, dx\, dy\, dz = \iiint (\delta S_1^2/4G_p)\, dx\, dy\, dz = 0.$$

Hence with the original as well as with the varied state of stress V_p will be the same as if the ratio of the stress components had remained constant during the deformation, whilst δV_p presents indeed the difference of the plastic strain energies in the two cases. The fact that no work should be done by the exterior forces in a deformation with constant E_1 is rather contrary to what should be expected; the conclusions obtained above therefore do not plead for a complete behaviour of a material according to the assumed mode of deformation.

According to eqs. (3) we may write eq. (12) as follows

$$\left.\begin{aligned}\delta V_p = \iiint (2G_p/\tan \varphi_s)(\varepsilon_{xp}\, \delta\sigma_x + \varepsilon_{yp}\, \delta\sigma_y + \varepsilon_{zp}\, \delta\sigma_z + \\ + \gamma_{xyp}\, \delta\tau_{xy} + \gamma_{yzp}\, \delta\tau_{yz} + \gamma_{zxp}\, \delta\tau_{zx})\, dx\, dy\, dz.\end{aligned}\right\} \quad (14)$$

The partial strains ε_{xe}, etc. cannot be expressed in the displacements u, v and w; neither can the partial strains ε_{xp}, etc. Only the total strains ε_x, etc. may be expressed by the partial derivatives of the displacements with respect to the coordinates. We can introduce them when eqs. (2) and (14) are combined as follows

$$\left.\begin{aligned}&\iiint\left(\delta V_{e1}+\frac{\tan\varphi_s}{2G_p}\delta V_{p1}\right)dx\,dy\,dz=\\&=\iiint(\varepsilon_x\,\delta\sigma_x+\varepsilon_y\,\delta\sigma_y+\varepsilon_z\,\delta\sigma_z+\gamma_{xy}\,\delta\tau_{xy}+\gamma_{yz}\,\delta\tau_{yz}+\gamma_{zx}\,\delta\tau_{zx})\,dx\,dz\,dz\end{aligned}\right\}\quad(15)$$

To prove the principle of least work in the elastic domain, the strains ε_{xe}, etc. are put equal to $\partial u/\partial x$, etc.; after partial integration, taking into account the conditions of equilibrium and the boundary conditions, it then appears that with unchanged surface forces the integral in eq. (2) is equal to zero [3]). The same reasoning (which we will not repeat here) may be applied to the right hand side of eq. (15), so that it appears that

$$\iiint\left(\delta V_{e1}+\frac{\tan\varphi_s}{2G_p}\delta V_{p1}\right)dx\,dy\,dz=0.\quad\ldots\quad(16)$$

It follows that the relation

$$\delta V=\iiint(\delta V_{e1}+\delta V_{p1})\,dx\,dy\,dz=0,$$

which would mean that the total energy should have an extreme value, can only be obtained if $\tan\varphi_s/2G_p=1$, that is with an S_1-E_1 as well as a $\sigma-\varepsilon_p$ relation given by a straight line through the origin, a case which practically never occurs.

In the same way as we found eq. (13) we may derive that

$$dV_p=\iiint(dS_1^2/2\tan\varphi_s)\,dx\,dy\,dz,$$

or

$$dV_{p1}=dS_1^2/2\tan\varphi_s\quad\text{and}\quad V_{p1}=\int(dS_1^2/2\tan\varphi_s),$$

which equation yields, according to fig. 4,

$$V_{p1}=\int(S_1\,dS_1/\tan\varphi_s)=\int S_1\,dE_1.\quad\ldots\quad(17)$$

As it followed from eq. (13) that a change of the state of stress does not influence V_p when the value of S_1 is kept constant, we may compute V_p just as well upon the assumption that the ratios of the deviator components do not change during the deformation. In that case the ratios between the deviator components and S_1 will be constant. It may be easily understood, that then the relations $\bar{\sigma}_x$—ε_{xp}, etc. will be identical with the relation S_1-E_1, so that $d\varepsilon_{xp}=d\bar{\sigma}_x/\tan\varphi_s$, etc. (fig. 4). Hence it follows from fig. 4 that

$$\begin{aligned}V_{p1}&=\int(\bar{\sigma}_x\,d\varepsilon_{xp}+\bar{\sigma}_y\,d\varepsilon_{yp}+\bar{\sigma}_z\,d\varepsilon_{zp}+\tau_{xy}\,d\gamma_{xyp}+\tau_{yz}\,d\gamma_{yzx}+\tau_{zx}\,d\gamma_{zxp})\\&=\int(1/\tan\varphi_s)(\bar{\sigma}_x\,d\bar{\sigma}_x+\bar{\sigma}_y\,d\bar{\sigma}_y+\bar{\sigma}_z\,d\bar{\sigma}_z+2\tau_{xy}\,d\tau_{xy}+2\tau_{yz}\,d\tau_{yz}+2\tau_{zx}\,d\tau_{zx})\\&=\int(dS_1^2/2\tan\varphi_s)=\int S_1\,dE_1,\end{aligned}$$

in accordance with eq. (17).

We may now derive from fig. 4 that with a $S_1 - E_1$ relation corresponding to the full drawn out curve, a variation of S_1 will cause a variation δV_{p1} which is $2G_p/tan\ \varphi_s$ times the variation δV_{p1s} that would occur, if the $S_1 - E_1$ relation would have been determined by the straight line OPQ. In the latter case we found

$$\delta V = \iiint (\delta V_{e1} + \delta V_{p1s})\, dx\, dy\, dz = 0 \quad . \quad . \quad . \quad . \quad (18)$$

as may be derived likewise from the fact that in this case the stress strain relation is the same as in the elastic domain, with the only difference that now the coefficient of lateral contraction is between m and 2. Substitution of $\delta V_{p1s} = (tan\ \varphi_s/2G_p)\delta V_{p1}$ into eq. (18) will give us inversely the relation expressed by eq. (16).

Furthermore it appears from eq. (16) that with $tan\ \varphi_s = 0$, that is at the yield stress, $\delta V_e = 0$ and hence V_e has an extreme value. This conclusion is in accordance with HENCKY's deductions [6]). However, we have seen above that in the transitional domain the state of stress changes from one, corresponding to minimum elastic strain energy, to one with minimum shearing energy, so that in this domain the elastic strain energy V_e will not obtain a really extreme value. The reason why nevertheless we find δV_e to be zero is, that, although V_e does not become an absolute minimum, it will be (in distinction to what is the case in the elastic domain) a constrained minimum, *i.e.* a minimum for such states of stress only, which besides satisfying the conditions of equilibrium (which have to be satisfied in the elastic domain as well), satisfy the condition of yielding. The latter one here acquires the form $S_1 = \text{const.}$

In the elastic domain $V_p = 0$, so that eq. (16) gives $\delta V_e = 0$, in accordance with the principle of least work valid in that domain.

When the elastic deformations may be neglected relatively to the plastic ones, eq. (16) gives:

$$\iiint \frac{\tan \varphi_s}{2 G_p}\, \delta V_{p1}\, dx\, dy\, dz = 0 \quad . \quad . \quad . \quad . \quad . \quad (19)$$

from which it follows that in this case the principle of least work will hold anyway if $tan\ \varphi_s/2G_p$ is independent of the coordinates x, y and z, in which case eq. (19) reduces to $\delta V_p = 0$. This condition is *i.a.* satisfied in the example we considered in the beginning of this paper, the plastic deformation of a centrically loaded oblique weakened strip, where, also with $tan\ \varphi_s$ surpassing zero, $tan\ \varphi_s$ and G_p everywhere have the same value.

Moreover V_p will become an extreme if δV_{p1} is zero everywhere, also with varying values of $tan\ \varphi_s/2G_p$. This will occur for example, if a weakened strip is deformed arbitrarily in such a way, that the adjacent elastic parts rotate with respect to one another around an arbitrary point O

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[6]) HENCKY. Zur Theorie plastischer Deformationen. Proc. 1st Congress for Applied Mechanics. Delft, (1924).

(fig. 1). In this case the direction of yielding of every point of the strip is given by the perpendicular to a line, connecting that point with O. From our former publications it follows [7]), that in such a case at the yield point the stresses establish themselves such, that the component of the oblique stress $\mu = \sqrt{\sigma_x^2 + \tau_{xy}^2}$ in the direction of yielding (which we have called the "effective oblique stress" $\mu_e = \mu \cos \gamma$) assumes a maximum value. As only these components give a moment with respect to O, the total moment with respect to O will be a maximum too and hence also the resistance of the weakened strip against rotation of the adjacent parts about O. This holds likewise if the weakened section follows a curved course. However, in all these cases the breadth a of the strip has to be small in comparison to its length b (fig. 1), in order to make ε_y everywhere equal to zero. A maximum resistance causes a minimum rotation by a given moment and thus a minimum exterior work, so that the interior work V_p is a minimum as well. At the other hand it follows from eq. (14) that with a plane state of stress, where only σ_x, σ_y and τ_{xy} differ from zero, with $\varepsilon_{yp} = 0$ and given values σ_x and τ_{xy}, hence with $\delta\sigma_x = \delta\tau_{xy} = 0$, we shall have $\delta V_p = 0$. In the latter case the boundary conditions are such that eq. (19) holds also for a length dx of the strip, so that it yields $\delta V_{p1} = 0$ and hence also $\delta V_p = 0$.

If an unweakened plate is loaded in such a way, that the breadth of the dangerous region is small in relation to its length, the same conclusions will hold.

As

$$d\varepsilon_x = \frac{\partial \varepsilon_x}{\partial \sigma_x} d\sigma_x + \frac{\partial \varepsilon_x}{\partial \sigma_y} d\sigma_y + \frac{\partial \varepsilon_x}{\partial \sigma_z} d\sigma_z + \frac{\partial \varepsilon_x}{\partial \tau_{xy}} d\tau_{xy} + \frac{\partial \varepsilon_x}{\partial \tau_{yz}} d\tau_{yz} + \frac{\partial \varepsilon_x}{\partial \tau_{zx}} d\tau_{zx}, \text{ etc.}$$

and

$$\frac{\partial \varepsilon_x}{\partial \sigma_x} = \frac{\partial \varepsilon_{xe}}{\partial \sigma_x} + \frac{\partial \varepsilon_{xp}}{\partial \sigma_x} = \frac{1}{E} + \frac{\partial \varepsilon_{xp}}{\partial \sigma_x}, \text{ etc.,}$$

the differential relations given by eqs. (8) and (10) may be used likewise in order to express the excess stresses $d\sigma_x = \sigma'_x$, etc. by means of the excess strains $d\varepsilon_x = \varepsilon'_x$, etc., if an elasto-plastic body is subjected to an infinitely small excess stress tensor, as we have shown in a communication to the Sixth Congress for Applied Mechanics in Paris, September 1946. It appears, that in such a case, which may occur in three dimensional buckling problems, the behaviour of the elasto-plastic body is anisotropic and is governed by 7 constants. With plane stress the relations reduce to those of our theory of plastic stability of thin plates [8]).

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[7]) BIJLAARD. Weerstand van een verzwakte scheeve doorsnede van een getrokken plaat, berekend volgens de hypothese van HUBER–HENCKY. De Ingenieur. Nr. 37 (1931).
BIJLAARD. Theory of local plastic deformations. Chapter 2. Int. Ass. for Bridge and Structural Engineering. Zürich. Sixth Volume (1940/1941).

[8]) Cf. lit. footnote 2, eqs. (21)—(24).

Applied Mechanics. — *Inequalities for conjugated Maxwell influence numbers.* By O. BOTTEMA. (Communicated by Prof. C. B. BIEZENO.)

(Communicated at the meeting of March 29, 1947.)

1. We consider a two-dimensional framework the joints of which are fixed in space but liable to rotations. The joints are denoted by $A_i (i = 1, 2, \ldots n)$ and the length of the member joining the points A_i and A_j is indicated by $l_{ij} = l_{ji}$; if A_i and A_j are unconnected $l_{ij} = 0$ by definition.

If M_i is the moment acting in A_i and φ_i the rotation that occurs in A_i, the following linear relations hold

$$\left.\begin{aligned} M_1 &= a_{11}\varphi_1 + a_{12}\varphi_2 + \ldots a_{1n}\varphi_n \\ M_2 &= a_{21}\varphi_1 + a_{22}\varphi_2 + \ldots a_{2n}\varphi_n \\ &\ldots\ldots\ldots\ldots \\ M_n &= a_{n1}\varphi_1 + a_{n2}\varphi_2 + \ldots a_{nn}\varphi_n \end{aligned}\right\} (a_{ij} = a_{ji}) \quad . \quad . \quad (1)$$

where the coefficients a_{ij} are characteristic numbers of the framework, depending on the elastic properties of the members. They are called the conjugated Maxwell influence numbers of the system and may be regarded as dualistic to the ordinary Maxwell coefficients [1]). Obviously a_{ij} can be defined as the moment required to act at A_i when A_j is subject to the unit-rotation and if all joints A_k $(k \neq j)$ are locked.

The coefficients a_{ij} fulfill certain inequalities. It is well-known that the determinant

$$\begin{vmatrix} a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \ldots & \ldots \\ a_{n1} & a_{n2} \ldots a_{nn} \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

is positive [2]).

In the course of an investigation concerning the convergence of the so-called relaxation-method for frameworks of the type mentioned above BIEZENO and BOTTEMA [3]) proved in an indirect way

$$\begin{vmatrix} a_{11} & a_{12} & -a_{13} & a_{14} & -a_{15} \ldots \\ a_{21} & a_{22} & a_{23} & -a_{24} & a_{25} \ldots \\ -a_{31} & a_{32} & a_{33} & a_{34} & -a_{35} \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix} > 0 \quad . \quad . \quad (3)$$

[1]) For the conjugated Maxwell numbers see f.i. BIEZENO und GRAMMEL, Technische Dynamik (Berlin 1939), p. 95.

[2]) BIEZENO–GRAMMEL, p. 96.

[3]) BIEZENO and BOTTEMA, The convergence of a specialized iterative process in use in structural analysis. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 49, 489—499 (1946).

thus showing that if in a Maxwell determinant (related to the type of structure mentioned above) the numbers of all diagonals beginning with $a_{13}, a_{15}, \ldots, a_{31}, a_{51}, \ldots$ are replaced by their negative values, the determinant remains positive.

We now add two other inequalities with regard to the coefficients a_{ij} (only for the frameworks under consideration) one of which is a generalization of (3).

2. To this end we give an expression for a_{ij} by means of the values of the characteristic elastic properties of the members of the framework. For the greater part the results have already been given in the above mentioned paper.

We put

$$F_{ij}=\int_{A_i}^{A_j}\frac{dx}{EI},\quad S_{ij}=\int_{A_i}^{A_j}\frac{x\,dx}{EI},\quad I_{ij}=\int_{A_i}^{A_j}\frac{x^2dx}{EI}\quad\ldots\quad(4)$$

where EI represents the local bending rigidity of the member A_iA_j. If all joints A_j $(j\neq i)$ are clamped and the external moment acting on A_i is M_i, if furthermore M_{ij} represents the share of M_i acting on A_iA_j, then the rotation φ_i of A_i is determined by

$$M_{ij}=a_{ij}\,\varphi_i\quad\ldots\quad(5)$$

where

$$a_{ij}=\frac{I_{ij}}{F_{ij}I_{ij}-S_{ij}^2}\quad\ldots\quad(6)$$

Moreover the equilibrium of A_iA_j requires a clamping moment in A_j of the magnitude

$$\beta_{ij}\,M_{ij},$$

where

$$\beta_{ij}=\frac{l_{ij}\,S_{ij}-I_{ij}}{I_{ij}}\quad\ldots\quad(7)$$

From these results it follows that a rotation φ_i in A_i (all other joints being fixed) requires a moment M_j in A_j $(j\neq i)$ of the magnitude

$$M_j=a_{ij}\,\beta_{ij}\,\varphi_i=\frac{l_{ij}\,S_{ij}-I_{ij}}{F_{ij}I_{ij}-S_{ij}^2}\,\varphi_i,$$

the moment M_i in the joint A_i itself being

$$M_i=\sum_k M_{ik}=\sum_k a_{ik}\,\varphi_i$$

Consequently the conjugated Maxwell numbers can be written as:

$$a_{ii}=\sum_k a_{ik}$$

$$a_{ij}=a_{ij}\,\beta_{ij}\quad(i\neq j)$$

If for abbreviation we put

$$\gamma_{ij} = \frac{1}{F_{ij} I_{ij} - S_{ij}^2}, \quad \delta_{ij} = l_{ij} S_{ij} - I_{ij} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

we have

$$\left.\begin{aligned} a_{ii} &= \sum_k \gamma_{ik} I_{ik} \\ a_{ij} &= \gamma_{ij} \delta_{ij} \qquad (i \neq j) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

It can easily be shown that $\gamma_{ij} = \gamma_{ji}$ and $\delta_{ij} = \delta_{ji}$, so that $a_{ij} = a_{ji}$ as required.

3. It follows from (4) and (8) that $\gamma_{ij} > 0$, $\delta_{ij} > 0$ and furthermore that

$$I_{ij} I_{ji} - \delta_{ij}^2 = \frac{l_{ij}^2}{\gamma_{ij}}$$

so that

$$\delta_{ij} < \sqrt{I_{ij} I_{ji}}$$

In view of the theorem on the arithmetical and geometrical mean we have

$$\delta_{ij} < \tfrac{1}{2} (I_{ij} + I_{ji}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Hence

$$\sum_{i \neq j} a_{ij} = \sum_{i,j} \gamma_{ij} \delta_{ij} < \sum_{i,j} \tfrac{1}{2} \gamma_{ij} (I_{ij} + I_{ji}) = \sum_{i,j} \gamma_{ij} I_{ij} = \sum_i \sum_k \gamma_{ik} I_{ik} = \sum_i a_{ii}$$

so that

$$\sum_{i \neq j} a_{ij} < \sum_i a_{ii} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

that means: *the sum of the coefficients in the main diagonal of the Maxwell matrix is greater than the sum of all the other coefficients.*

4. The quadratic form $Q = \sum_{ij} a_{ij} x_i x_j$ can be written as (see (9))

$$Q = \tfrac{1}{2} \sum_{i,j} (\gamma_{ij} I_{ij} x_i^2 + 2 \gamma_{ij} \delta_{ij} x_i x_j + \gamma_{ij} I_{ji} x_j^2) =$$

$$= \tfrac{1}{2} \sum_{i,j} \gamma_{ij} (I_{ij} x_i^2 + 2 \delta_{ij} x_i x_j + I_{ji} x_j^2).$$

The discriminant $\triangle_{ij}$ of

$$q_{ij} = I_{ij} x_i^2 + 2 \delta_{ij} x_i x_j + I_{ji} x_j^2 \quad . \quad . \quad . \quad . \quad . \quad (12)$$

being $\delta^2{}_{ij} - I_{ij} I_{ji}$ we have $\triangle_{ij} < 0$, so that (12) is essentially positive. But then the same holds for

$$q'_{ij} = I_{ij} x_i^2 - 2 \delta_{ij} x_i x_j + I_{ji} x_j^2$$

as well as for Q' if Q' denotes the quadratic form, that is obtained if in $Q = \frac{1}{2} \Sigma \gamma_{ij} q_{ij}$ one or more of the terms q_{ij} are replaced by q'_{ij}. The

coefficients of the quadratic form Q' are thus seen to be the same as those of Q provided that one or more of the coefficients a_{ij} $(i \neq j)$ are replaced by $-a_{ij}$. In view of the well-known properties of positive quadratic forms we have the following theorem: *if in the Maxwell determinant* $|a_{ij}|$ *belonging to the constructions under consideration one or more pairs of symmetrical coefficients* $a_{ij} = a_{ji}$ $(i \neq j)$ *are replaced by their negative values, the determinant remains positive.* The theorem is a generalization of that quoted in § 1.

Comparative Physiology. — *Some Aspects of the Tonus Problem.* (From the Laboratory of comparative Physiology of the University of Utrecht.) By N. POSTMA. (Communicated by Prof. J. M. BURGERS.)

(Communicated at the meeting of March 29, 1947.)

For the foot of the snail (Helix pomatia L.), as a representative of the hollow smooth muscle, we have found that the tonic reactions (lengthening and recovery) are based on reflex-shortening and on reflex-distension of different muscular units. These mutually opposite movements are produced by antagonistic impulses conducted to the muscle along efferent nerve-paths (POSTMA, N. 1945). Innervation and mode of action of the snail foot muscle are similar to those of vertebrate intestinal hollow smooth muscles. The tonic reactions are determined by the ratio between the numbers of shortening and distending muscle fibres. In this respect there is a resemblance to the reflex tonus of striated muscles. With regard to unstriated muscle, however, this does not allow to speak of tetanotonus.

Formerly the theory was accepted that energy had to be supplied by metabolic processes for the shortening of a muscle, the *relaxation* being effected *passively*. Afterwards the opposite conception was held to be true: metabolism restoring the electrical charges in the muscle fibres is now generally held to be the cause of a discontinuation of the shortening. The muscle is stretched ("decontracted") by intrafibrilar electric fields, and *shortening* apparently follows *passively* as soon as these charges are reduced by the stimulus causing contraction (BUCHTHAL, [1]) a.o.). Thus it is possible to maintain tonus, *i.e.* a definite degree of shortening, without demand for extra energy.

Striated muscle decontracts spontaneously (BUCHTHAL); it is probable that this mechanism has been developed to favour the motorial function at the expense of an inexpensive postural one. For decontraction disposes the muscle to immediate new contraction, which is a condition for nimble-fingered action (as required, *e.g.* in playing the piano). As a consequence, tetanotonus, like complete or smooth tetanus, has to make use of the same mechanism. On account of this spontaneous decontraction, the efferent innervation of striated muscle needs only excitatory impulses: stimulation merely causes contraction. The antagonistic stimuli remain afferent; central inhibition can occur in that phase of the phenomenon which is enacted in the spinal chord or in the higher nervous centres.

In addition to BUCHTHAL's investigations on isolated striated muscle fibre those by SZENT–GYÖRGYI on actomyosin threads may be instructive in elucidating the nature of the tonus function, especially of smooth muscle.

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[1]) We do not accept a causal relation between the electric fields and the Z-stripes.

These muscles are characterized: 1. by their slow contraction, which in the model experiment (actomyosin threads) is connected with deficiency of Mg-ions (1941; it is not known whether the fibrils effect a slow shortening caused by Mg-deficiency) [2]; 2. by the need of or at least the supply with impulses to excite distension, whereas in the model experiment relaxation cannot be effected without Mg-ions (1942). Referring to these results and to SZENT–GYÖRGYI's theory of muscle function, according to which contraction is caused by disturbance of the balance of the K-ions, we attributed the distension to a specific tonolytic impulse placing Mg-ions at the disposal of the muscle [3]. So *tonolysis* active in the myogenic region (peripheral) must be distinguished from *inhibition* (active in the neurogenic region) (cf. POSTMA, 1946, p. 477). Relaxation not being spontaneous, maintenance of tonic shortening thus would be assured without a demand for extra energy. SZENT–GYÖRGYI does not pay attention to possible consequences for the tonus problem of the results obtained by his school.

If it were possible to prevent the spontaneous decontraction of skeletal muscle, similarly a higher tonus would not require extra metabolism [3]. The energy-spending tetanotonus, therefore, must be attributed to the luxury of having spontaneous decontraction so as to afford sufficient mobility. Possibly the innervation by the autonomic system sustains tetanotonus by a mechanism similar to the tonus of smooth muscle (cf. POSTMA, 1946, p. 491), but its effect it too weak to fulfil the tonus function apart from tetanus. In our opinion such a mechanism could also play a rôle in the ORBELI-effect. The fact that preceding parasympathetic stimulation augments this effect (BARYSCHNIKOW) has to be attributed to an increased deficit in this supporting component of the tonus function: there is more to supply.

The reflex tonus of hollow smooth muscle, inexpensive as regards energy because of the absence of distension, must be distinguished as *"block" tonus* (German: *Sperrtonus*), the decontraction being blocked (German: gesperrt), in contrast with what occurs in *tetanotonus* of striated muscles.

The above considerations lead to the view that the tonus phenomenon primarily raises a myogenic problem. To solve this problem it is necessary to establish: 1. what determines the delay of relaxation in smooth muscles, while decontraction immediately follows shortening in striated ones; 2. what is the mode of action of the tonolytic stimulus in unstriated muscle; and 3. whether Mg-ions really play a part in the circumstances present in muscle in the sense that they determine the slow tonic contraction (JORDAN) and the delayed distension (POSTMA).

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[2]) In striated as well as in smooth muscles *composed of a number of reacting units* slow or fast contraction shows a relation to the number of units coming into action successively or simultaneously.

[3]) These considerations are derived from an unpublished manuscript of a lecture delivered before the "Committee for the Study of Viscosity" of the Academy of the Sciences in Amsterdam (1944).

LITERATURE.

BARYSCHNIKOW, I. A., (1930), Russian Jl. of Physiol. **V.—13**, 476.
BUCHTHAL, F. et al. cf. JORDAN, H. J., (1945), Arch. néerl. de Zool. **7**, 122.
POSTMA, N., (1945), Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **48**, 394.
————, (1946), Arch. néerl. de Zool. **7**, 471.
SZENT-GYÖRGYI, A., (1941), Stud. Instit Med. Chem. Szeged **I**, 7.
————, (1942), Ibid. **II**, 25.

Paleontology. — *A femur of Manis palaeojavanica Dubois from Western Java.* By D. A. HOOIJER. (Communicated by Prof. H. BOSCHMA.)

(Communicated at the meeting of March 29, 1947.)

STEHN and UMBGROVE (1929) have recorded fossil remains of *Bos, Sus* and *Cervus* from the Tji Taroem valley W. of Batoedjadjar, Preanger, W. Java. The collection was presented to the Geological Museum at Leiden by Prof. Dr. J. H. F. UMBGROVE. The exact geological age of the fossils could not be established; they might have originated from the neighbouring Neogene limestone. VAN ES (1931, p. 15) states that the absence of proboscideans points to a younger age than that of the other occurrences of vertebrates. VON KOENIGSWALD (1933, pp. 44, 47) refers the *Sus* specimen, a right ramus with P_3—M_2, to his species *Sus terhaari*. I cannot see the reason, however; in a later paper (VON KOENIGSWALD, 1935, p. 87) the fossil is left specifically undetermined. The cervine remains, of which two teeth have been figured by STEHN and UMBGROVE (l.c., pl. IV, figs. 1—3), were referred to *Cervus unicolor* Kerr (*C. hippelaphus* Smith) by VON KOENIGSWALD (1933, p. 74). Afterwards, however, they are identified as "*Cervus* (ex aff. *zwaani)*" (the upper molar) and "cf. *Antilope modjokertensis* v. K." (lower molar). The bovine fossils should belong to *Bubalus palaeokerabau* Dubois (VON KOENIGSWALD, 1935, p. 87/88).

To this fauna only *Rhinoceros sondaïcus* Desmarest has been added since by VON KOENIGSWALD (1935, p. 87), who regards the age of the fossils as very probably lower Pleistocene. *Rhinoceros sondaïcus* Desmarest, however, ranges from the lower Pleistocene (Djetis fauna) to the present day (HOOIJER, 1946, pp. 3, 34—81); *Bubalus palaeokerabau* Dubois was not known previously from the lower Pleistocene (cf. VON KOENIGSWALD 1934, p. 191), but indeed the deer remains may belong to species which are said to be typically of lower Pleistocene age.

To my great surprise I found a right femur in the STEHN and UMBGROVE collection (Geol. Mus. Leiden, no. 28069) which does not belong to either of the genera mentioned above. It was catalogued as "*Felis* sp.". The peculiar flatness of the shaft and the comparatively slight prominence of the condyles, however, show this identification to be incorrect. The bone belongs to:

Manis palaeojavanica Dubois.

Manis palaeojavanica Dubois, Tijdschr. Kon. Ned. Aardr. Gen., ser. 2, vol. 24, 1907, p. 455; STREMME, N. Jahrb. f. Min., 1911, p. 55; STREMME, in: SELENKA–BLANCKENHORN, Die Pithecanthropus-Schichten auf Java, Leipzig, 1911, p. 141; MARTIN, Unsere palaeozoologische Kenntnis von Java, Leiden, 1919, p. 107; VAN ES, The Age of Pithecanthropus, The Hague, 1931, p. 31; VON KOENIGSWALD, De Ing. in Ned. Ind., vol. 1, part. 11,

sect. IV, 1934, pp. 190, 191, 196; KORMOS, Folia Zool. et Hydrob., vol. 6, 1934, p. 88; VON KOENIGSWALD, Quartär, vol. 2, 1939, p. 35; BOK, Bijdrage tot de kennis van de raseigenschappen van het Javaansche volk, thesis Utrecht, 1940, p. 4; VON KOENIGSWALD, Wet. Med. Dienst Mijnbouw Ned. Indië, no. 28, 1940, p. 56.

Manis palaejavanica Dubois, Tijdschr. Kon. Ned. Aardr. Gen., ser. 2, vol. 25, 1908, p. 1267, Versl. Verg. Afd. Natuurk. Kon. Akad. v. Wetensch., Amsterdam, Vol. 35, 1926, p. 949, Proc. Kon. Akad. v. Wetensch., Amsterdam, vol. 29, 1926, p. 1233; VAN DER MAAREL, Leidsche Geol. Med., vol. 5, 1931, p. 472; HOOIJER, Zool. Med. Museum Leiden, vol. 26, 1946, p. 107.

M[anis] palaejavanico Weber, Die Säugetiere, vol. 2, Jena, 1928, p. 185.

[*Manis*] *paleojavanica* Raven, Bull. Am. Mus. Nat. Hist., vol. 68, 1935, p. 238.

The present giant pangolin was reported upon first by DUBOIS in 1892 (ANONYMUS, 1892, p. 14) on the evidence of remains of a skeleton from Kedoeng Broeboes, a fossil locality in the Kendeng Beds in Central Java which two years earlier had provided DUBOIS with a mandible fragment which he finally attributed to *Pithecanthropus erectus* (Dubois) but the systematic position of which is still uncertain (WEIDENREICH, 1945, p. 103/104). The total length of the pangolin is stated to have been $2\frac{1}{2}$ m, and it also must have been more heavily built than the recent *Manis javanica* Desmarest. The same statements we find in another paper (DUBOIS, 1907, p. 455) in which the fossil form is designated as *Manis palaeojavanica*. A more elaborate description and comparison with other forms is given in 1908 under the name "*Manis palaejavanica* n. sp." (DUBOIS, 1908, p. 1267). In his final description of the remains DUBOIS (1926) only refers to the latter paper and name.

Manis palaeojavanica Dubois is the best known of the fossil pangolins. We know parts of its skull, atlas, caudal vertebrae, radius, ulna, femur and tibia, as well as metacarpals, phalanges, astragalus, calcaneum, navicular, ectocuneiform and metatarsals. From DUBOIS' paper of 1926 it is clear that the species displays the greatest resemblance to *Manis javanica* Desmarest, though in size it comes nearest to *Manis gigantea* Illiger which is about one-third smaller. From the characters of the endocranial cast and of the foot bones DUBOIS inferred that in all probability the fossil form was covered with scales, that it had large claws on its pes as well as on its manus, and that it walked permanently on the outside of its feet.

Some other fossil pangolins have been based only on isolated phalanges, e.g., *Manis lydekkeri* Trouessart (olim *Manis gigantea* Lydekker) from the Pleistocene Karnul caves near Madras in India and *Manis sindiensis* Lydekker from the Siwaliks of Sind (LYDEKKER, 1886, p. 50), as well as *Manis hungarica* Kormos (1934) from the Villafranchian of Villány in Hungary. HELBING (1938) assembles previous records of Tertiary pangolins in Europe (Quercy, Solnhofen and Wintershof West) and describes remains of *Manis* of Oligocene age from St. André near Marseilles, Montaigu-le-Blin (Allier) and Weisenau near Wiesbaden. None of the species, named or unnamed, is as large as *Manis palaeojavanica* Dubois.

The newly-discovered femur is of the right side. The proximal portion has broken off; the medial surface of the bone is slightly damaged below. The corpus is flattened from before backwards. At the narrowest part, about 4 cm below the broken edge, the width is one-half greater than the antero-posterior diameter. The shaft increases in width distally, its medial border is somewhat more rounded than the lateral and ends in the medial epicondyle which is somewhat less prominent and placed on a slightly higher level than the epicondylus lateralis. Just above the latter the border of the bone is very prominent, rather a crest. The bone belonged to an adult individual, but the suture of the distal epiphysis is still partly shown. Measured across the epicondyles the bone is twice as broad as at the narrowest part of the shaft.

The trochlea is much more prominent medially than laterally; the facies patellaris is rounded off above with its highest point to the medial side of the axis of the corpus.

Of the two condyles the medial is by far the larger; it projects more backwards and is placed slightly higher than the lateral. Its articular surface is more strongly convex from side to side than in the lateral condyle, in which it is narrower and slopes down obliquely to the inside.

The posterior surface, just above the intercondyloid fossa, presents a faint depression, but otherwise the posterior surface is slightly convex from side to side, especially towards the middle of the shaft.

The only parts of the hind limb of *Manis palaeojavanica* Dubois that were known already from DUBOIS' descriptions consist of the medial half of the proximal extremity of a right femur with almost the entire head and the lesser trochanter, and the proximal half of a left tibia. The type specimens are not fully adult; the epiphyses are not yet fused with the shaft. The Preanger femur thus fills up a hiatus in our knowledge of the osteology of this interesting species, and I was anxious to see whether the characters that can be drawn from this part of the skeleton would be in accord with those already found by DUBOIS or not.

DUBOIS compared his fossil specimens with a subadult skeleton of *Manis gigantea* Illiger from Liberia in the Leiden Museum (cat. a) which I have also at my disposal, and with a more than middle-sized skeleton of *Manis javanica* Desmarest in his private collection. It is about one-third larger than the almost adult female skeleton of the same species in the Leiden Museum (reg. no. 1775, from the Rotterdam zoological garden, 13-9-1929) of which the measurements (in mm) are given in the third column of the table below.

From this table it results that the index pilastricus of *palaeojavanica* is intermediate between those of *gigantea* and *javanica*; the figures for the index poplitaeus and the distal width index are greater, and that for the condyle index seems to be smaller than in the recent species. Unfortunately the lack of the proximal part of the fossil femur makes it impossible to compute more and better indices.

Femur	*Manis palaeo-javanica* Dubois	*Manis gigantea* Illiger	*Manis javanica* Desmarest
1. Antero-posterior diameter of caput . .	40	28	11
2. Antero-posterior diameter at trochanter minor	28	14	$7\frac{1}{2}$
3. Distance from top of trochanter minor to top of caput	52	40	16
4. Width at narrowest part of shaft . .	$32\frac{1}{2}$	22	9
5. Antero-posterior diameter at same level	$21\frac{1}{2}$	13	7
6. Index pilastricus $\left(\frac{\text{meas. no. } 5 \times 100}{\text{meas. no. } 4}\right)$	66	59	78
7. Width of distal end of corpus	60	49	19
8. Antero-posterior diameter of the latter .	41	29	12
9. Index poplitaeus $\left(\frac{\text{meas. no. } 8 \times 100}{\text{meas. no. } 7}\right)$	68	59	63
10. Greatest distal width	66	53	20
11. Distal width index $\left(\frac{\text{meas. no. } 4 \times 100}{\text{meas. no. } 10}\right)$	49	42	45
12. Width of condyles	57	46	18
13. Greatest width of facies patellaris . .	37	31	10
14. Distal antero-posterior diameter from medial ridge of trochlea to line connecting posterior surfaces of condyles	54	39	16
15. Height of condylus medialis	ca. 27	24	9
16. Condyle index $\left(\frac{\text{meas. no. } 15 \times 100}{\text{meas. no. } 12}\right)$	ca. 46	52	50

In search for other recent species of *Manis* in which the distal extremity of the shaft of the femur is less compressed antero-posteriorly, and in which the distal epiphysis is more broad than *gigantea* and *javanica*, I found *Manis temminckii* Smuts (Leiden Museum, cat. a) to have a distal width index of 48; unfortunately the index poplitaeus could not be computed because the sutures are invisible. In this adult specimen and in an adult of *Manis longicaudata* (Brisson) (Leiden Museum, cat. a) the index pilastricus is 65 to 66. Both femora also possess a distinct supracondyloid crest which has not so well developed in the other, younger, specimens I had for comparison. In *Manis aurita* Hodgson (Leiden Museum, cat. a) the popliteal index of the femur is 56, the same figure I found in *Manis tricuspis* Rafinesque (Leiden Museum, cat. a).

Thus a distinctive feature of the femur of *Manis palaeojavanica* Dubois, as compared to the other species of its genus, seems to be the thickness of the lower end of the shaft. But certainly there is an amount of variation in this character; so, e.g., in two specimens of *Tamandua tetradactyla* (L.) in the Leiden Museum (reg. nos. 1163 and 1773) the index poplitaeus is 63 and 56 respectively. Both specimens are females.

The difference between the distal antero-posterior diameter (meas.

Manis palaeojavanica Dubois, distal portion of femur dext., Tji Taroem valley W. of Batoedjadjar, Preanger, Western Java. STEHN and UMBGROVE coll., Geological Museum Leiden no. 28069. Anterior, posterior, lateral and distal views, 4/5 natural size.

no. 14) and the condyle width (no. 12 in the table) is less in the fossil form than that in the recent. This ratio is 0.95, against 0.85 and 0.89 in *gigantea* and *javanica*. This is remarkably well in accord with the shape of the approximal surface of the tibia of *palaeojavanica* in the Dubois collection. The proximal extremity of the tibia, just like the distal extremity of the femur here described, is thicker in proportion to its width than in *gigantea* and *javanica*. The ratio's of the antero-posterior to the transverse diameter of the upper epiphysis of the tibia (see DUBOIS, 1926, p. 1238) for the three species in the above given order are 0.69, 0.60 and 0.63 respectively. The difference between the figures found for the ratio's of *palaeojavanica* and *gigantea* is more than twice as great as that between the figures in the former and *javanica*, i.e., exactly the same result as I arrived at with the Preanger femur.

DUBOIS' type specimens are from Kedoeng Broeboes, a locality with both lower Pleistocene (Djetis-) fauna and middle Pleistocene (Trinil-) fauna (VON KOENIGSWALD, 1934, p. 188). VON KOENIGSWALD (l.c., p. 190) records a phalanx of *Manis palaeojavanica* Dubois from the Djetis deposits of Goenoeng Boetak, which is very near the type locality. He states that, the species not being found in any of the rich fossil localities with Trinil or Ngandong (upper Pleistocene) fauna, it is not to be doubted at that DUBOIS' specimens originated from the Djetis deposits. If *Manis palaeojavanica* Dubois indeed is typical of the Djetis fauna, the femur in the STEHN and UMBGROVE collection supports VON KOENIGSWALD's view that the Tji Taroem valley fauna belongs to the lower Pleistocene.

LITERATURE.

ANONYMUS, 1892. Palaeontologische Onderzoekingen op Java. Verslag Mijnwezen 2e kwartaal 1892, pp. 14—17.

DUBOIS, E., 1907. Eenige van Nederlandschen kant verkregen uitkomsten met betrekking tot de kennis der Kĕndeng-Fauna (Fauna van Trinil). Tijdschr. Kon. Ned. Aardr. Gen., ser. 2, vol. **24**, pp. 449—458.

———, 1908. Das Geologische Alter der Kendeng- oder Trinil-Fauna. Ibid., vol. **25**, pp. 1235—1270, pl. 39.

———, 1926. Manis Palaejavanica, the Giant Pangolin of the Kendeng Fauna. Proc. Kon. Akad. v. Wetensch., Amsterdam, vol. **29**, pp. 1233—1243, 1 plate, 2 figs.

ES, L. J. C. VAN, 1931, The Age of Pithecanthropus. The Hague, 142 pp., 11 maps and sections.

HELBING, H., 1938. Nachweis manisartiger Säugetiere im stratifizierten europäischen Oligocaen. Eclogae Geol. Helv., vol. **31**, pp. 296—303, 4 figs.

HOOIJER, D. A., 1946. Prehistoric and Fossil Rhinoceroses from the Malay Archipelago and India. Zool. Med. Museum Leiden, vol. **26**, pp. 1—138, pls. I—X, 1 fig., 8 tables.

KOENIGSWALD, G. H. R. VON, 1933. Beitrag zur Kenntnis der fossilen Wirbeltiere Javas I. Wet. Med. Dienst Mijnb. Ned. Ind., no. **23**, 127 pp. 28 pls., 9 figs.

———, 1934. Zur Stratigraphie des javanischen Pleistocän. De Ing. in Ned. Ind., vol. **11**, sect. IV, pp. 185—201, pls. I—IV.

———, 1935. Bemerkungen zur fossilen Saugetierfauna Javas. II. Ibid., vol. **2**, no. 10, sect. IV, pp. 85—88, 3 + 10 figs.

KORMOS, TH., 1934. Manis hungarica n. sp., das erste Schuppentier aus dem europäischen Oberpliozän. Folia Zool. et Hydrob., vol. 6, pp. 87—94, 14 figs.

LYDEKKER, R., 1886. The Fauna of the Karnul caves. Mem. Geol. Surv. Ind., ser. 10, vol. 4, part 2, pp. 23—58, pls. VII—XI, 1 fig.

STEHN, CH. E. and J. H. F. UMBGROVE, 1929. Bijdrage tot de geologie der vlakte van Bandoeng. Tijdschr. Kon. Ned. Aardr. Gen., ser. 2, vol. 46, pp. 301—314, 2 maps, 5 pls., 3 figs.

WEIDENREICH, F., 1945. Giant Early Man from Java and South China. Anthr. Papers Am. Mus. Nat. Hist., vol. 40, part 1, pp. 1—134, pls. 1—12, 28 figs., 19 tables.

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Zoology. — *On the Identity of Maticora intermedia Westermann and Dipsadoides decipiens Annandale.* By L. D. BRONGERSMA. (Communicated by Prof. H. BOSCHMA.)

(Communicated at the meeting of March 29, 1947.)

In his notes on the snakes of the islands of Banka and Billiton, WESTERMANN (1942) mentions two species, the identity of which will be discussed in the present paper.

Maticora intermedia Westermann

This new species is described by WESTERMANN (1942, p. 617) from two specimens, one from Billiton, the other from Borneo. With its ventral count the species comes within the range of variation of *Maticora intestinalis* (Laur.), while it agrees with *Maticora bivirgata* (Boie) in the number of subcaudals. However, it differs from both *M. intestinalis* and *M. bivirgata* in having seven upper labials, and in having the subcaudals partly single, partly paired. A further difference from these two *Maticora* species is found in the enlarged vertebral scales. This character *M. intermedia* has in common with the species of the genus *Bungarus* Daudin. Thus the species would be intermediate between *Maticora intestinalis* and *M. bivirgata* in some characters, while in others it would be intermediate between the genera *Maticora* Gray and *Bungarus* Daudin. The new species is referred to *Maticora,* however, "as the heart of the examined specimens is shifted far backwards on account of the poison glands".

The author mentions that the coloration of *Maticora intermedia* is similar to that of *M. bivirgata* in several respects. When recently handling a number of specimens of *Bungarus flaviceps* Reinh., I was struck by the superficial resemblance, which this species bears to *Maticora bivirgata.* It became clear to me that *Maticora intermedia* probably had been based on specimens of *Bungarus flaviceps.* Owing to conditions due to the war the types of *M. intermedia* were not at hand, when in 1946 I paid a short visit to the Zoological Museum at Buitenzorg (Java). Therefore, the following discussion must be based on WESTERMANN's description.

A comparison of the description of *Maticora intermedia* to that given for *Bungarus flaviceps* by DE ROOIJ (1917, p. 245) shows that in all characters of lepidosis the two species are identical. DE ROOIJ mentions 193—240 ventrals (*intermedia* 206—214) and 42—54 subcaudals (*intermedia* 48—54), the latter partly single and partly paired. Like *M. intermedia, Bungarus flaviceps* has seven upper labials, of which the third and fourth enter the orbit. In coloration they are exactly alike, the red colour of the tail (white in alcohol) extending on to the posterior part of the back.

To have a further check, I examined the following specimens of *Bungarus flaviceps* in the collections of the Leiden Museum:

2 ex., Sumatra, leg. G. F. WIENECKE, 1865, Herp. reg. no. 5772.
1 ex., Sumatra, leg. S. MÜLLER, Herp. reg. no. 1437.
2 ex., Nias, leg. J. D. PASTEUR, Herp. reg. no. 4333.
1 ex., Banka, leg. J. F. R. S. VAN DEN BOSSCHE, Herp. reg. no. 559.
1 ex., Banka, leg. P. BUITENDIJK, Herp. reg. no. 5628.
1 ex., Smitau on the Kapuas river, Borneo, December 11, 1893, leg. J. BÜTTIKOFER, Herp. reg. no. 8392.

The counts of ventrals and subcaudals of these specimens, together with those of the types of *Maticora intermedia* are given in table I. All specimens of *Bungarus flaviceps* have an undivided anal, one preocular, two postoculars, temporals 1 + 2, and seven upper labials of which the third and fourth enter the orbit. In one of the Nias specimens the second upper labial of the right side is separated from the border of the lip by the first and third labial forming a short suture. In the specimen from Smitau the sixth upper labial of the right side just reaches the border of the lip, while on the left side it is separated from this by the fifth and seventh labial forming a suture.

TABLE I.

Reg. No.	Ventrals	Subcaudals
5772	224	$23 + {}^{28}/_{28} + 1$
	221	${}^{1}/_{1} + 17 + {}^{31}/_{31} + 1$
1437	225	?
4333	219	$26 + {}^{20}/_{20} + 1 + {}^{2}/_{2} + 1$
	219	$10 + {}^{2}/_{2} + 2 + {}^{36}/_{36} + 1$
559	217	$19 + {}^{2}/_{2} + 1 + {}^{23}/_{22} + 5 + {}^{1}/_{1} + 1$
5628	211	$20 + {}^{32}/_{32} + 1$
8392	228	$24 + {}^{1}/_{1} + 2 + {}^{24}/_{24} + 1$
M. intermedia	206	$33 + {}^{1}/_{1} + 2 + {}^{12}/_{12} (+ 1\,?)$
	214	$21 + {}^{1}/_{1} + 2 + {}^{6}/_{6} + 9 + {}^{3}/_{3} + 6 + {}^{6}/_{6} (+ 1\,?)$

The chief argument for referring *intermedia* to the genus *Maticora* was the position of the heart (WESTERMANN, 1942, p. 618). The types of *M. intermedia* have a head and body length of 413 and 369 mm respectively. The front of the heart is situated at 114 and 105 mm from the tip of the snout. For these specimens and for specimens of some other species I calculated the ratio: $\frac{\text{distance from tip of snout}}{\text{length of head and body}}$ with the following results.

Maticora intermedia Westerm.	0.28—0.29	(2 specimens)
Maticora bivirgata (Boie)	0.26—0.35	(22 specimens)
Maticora intestinalis (Laur.)	0.37—0.44	(10 specimens)

Bungarus flaviceps Reinh.	0.24—0.29 (7 specimens)
Bungarus fasciatus (Schn.)	0.22—0.26 (6 specimens)

In the value of this ratio *Maticora intermedia* comes within the range of both *Maticora bivirgata* and *Bungarus flaviceps*. Therefore this ratio cannot be used to assign *intermedia* either to the genus *Maticora* or to the genus *Bungarus*. If for the types of *M. intermedia* it was known at the level of which ventral the front of the heart was situated, this might prove more helpful. For the other species mentioned I find the following values:

Maticora bivirgata (Boie)	79th — 94th ventral
Maticora intestinalis (Laur.)	83rd—106th ventral
Bungarus flaviceps Reinh.	54th — 58th ventral
Bungarus fasciatus (Schn.)	52nd— 53rd ventral

WESTERMANN accounts the position of the heart to the length of the poison glands, but the author does not state that he actually checked whether the poison glands extended into the body cavity. I feel convinced that from the position of the heart, WESTERMANN took the length of the poison gland for granted. The largest specimen of *Bungarus flaviceps* (Smitau, head and body 1720 mm, tail 235 mm) was examined in this respect. The poison gland is of the same type as in other *Bungarus* species and does not extend into the body cavity.

If the similarity in outward appearance had not misled this author, so that he firmly believed that he was dealing with a *Maticora* species, but if instead he had used the keys by DE ROOIJ (1917, pp. 241, 243) this would have led his identification straight to *Bungarus flaviceps* Reinh. In my opinion there can remain no doubt that *Maticora intermedia* Westermann is a strict synonym of *Bungarus flaviceps* Reinh.

Dipsadoides decipiens Annandale

This species was described by ANNANDALE (1905, p. 213) from a single specimen from the Malay Archipelago. The type was re-examined by DE ROOIJ (1917, p. 210, fig. 81), who mentions that the scales are placed in 21 rows, and not in 19 rows as stated by ANNANDALE. Further DE ROOIJ mentions that the chinshields of the second pair are separated from each other. Since the original description was published, no new specimens became known, until VAN HEURN (1932, p. 14) mentions a specimen as presumably belonging to this species [1]). Later Jhr. W. C. VAN HEURN sent me the head of a snake, taken by Mr. J. H. WESTERMANN on Mt. Menumbing, Banka, in June 1933 (Herp. reg. no. 8428); Jhr. VAN HEURN was not sure whether it represented *Dipsadoides decipiens* or a *Boiga* species. WESTERMANN (1942, p. 611, fig. 1) describes and figures a snake from Banka, which he refers to *Dipsadoides decipiens*; on p. 612 this author mentions a second specimen from Pendopo, South Sumatra.

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1) This specimen afterwards proved to be *Boiga drapiezii* (Boie).

As in the case of *Maticora intermedia*, the specimen was not at hand when I visited Buitenzorg, and we have to depend on the description by WESTERMANN. However, in a collection of snakes from Banka, presented to the Leiden Museum in 1874 by Mr. I. H. G. VOSMAER, I found two snakes (Herp. reg. no. 8427), which agree well with that described by WESTERMANN. They have the same distinctive colour pattern of the upper surface of the head, and the same arrangement of the posterior chinshields. These two snakes, as well as the head sent by Jhr. VAN HEURN, I identify as *Boiga jaspidea* (Dum., Bibr. & Dum.). For comparison I used the following additional specimens of *Boiga jaspidea*:

1 ex., Java, leg. S. MÜLLER, Herp. reg. no. 982.

2 ex., Mt. Simpai, W. Java, 2100 ft., leg. C. P. J. DE HAAS, Herp. reg. no. 6834.

1 ex., Muara Siberut, Siberut, Mentawei Ids., leg. W. RUINEN, Herp. reg. no. 5847.

The scale counts of all specimens examined by me, together with those mentioned by WESTERMANN are given in table II.

If these specimens are compared to the descriptions given by DE ROOIJ (1917, pp. 199, 210) for *Boiga jaspidea* (*Dipsadomorphus jaspideus*) and *Dipsadoides decipiens*, it is clear that they agree about equally well with both. The point may be raised, whether *Dipsadoides decipiens* really represents a distinct genus and species.

DE ROOIJ (1917, p. 210) mentions the following characters in which the genus *Dipsadoides* differs from *Boiga*: "pupil round; nasal undivided. Maxillary teeth 6, ...; mandibular teeth subequal". These characters may be discussed first.

WESTERMANN (1942, p. 612) states that in the living specimen the pupil was vertically elliptic. In the preserved specimens of *Boiga jaspidea* examined by me, some have a distinctly vertically elliptic pupil, while in others the pupil has expanded to nearly round. In fact a slight angle in the upper and lower border of the pupil is the only indication of its elliptic origin. The round pupil of the type of *Dipsadoides decipiens* may be well due to post mortal changes as assumed by WESTERMANN.

In the specimens of *Boiga jaspidea* examined by me the nostril is very large and it is directed backwards. Above and below the nostril, the nasal is very narrow, but it is not completely divided into two shields. In some specimens an incisure is indicated, and the nasal may be termed semidivided. Posterior to the nostril the nasal is concave. In the structure of the nasal and nostril *Boiga jaspidea* and *Dipsadoides decipiens* apparently agree completely.

Maxillary teeth. It is a well known fact that in snakes often part of the teeth are ankylosed to the jaw, while others are only loosely attached to the bone. Generally the loosely attached teeth alternate with the ankylosed

teeth. In counting the number of maxillary teeth, both the ankylosed and the loosely attached teeth must be considered. If the maxillary has been extracted from the mouth, and if it has been cleaned, all or most of the loose teeth will have got lost; their place between the ankylosed teeth is indicated by shallow depressions in the jaw. To ascertain the exact number of maxillary teeth it is necessary to count those ankylosed to the jaw, as well as the depressions in the bone, which mark the lost teeth. In this connection I may cite BOULENGER (1896, pp. 614—615), who wrote: "It even often happens that every alternate tooth having dropped out, the jaw appears, on a superficial examination, to possess only half the real number."

In *Boiga jaspidea* the maxillary teeth number 12 solid ones, followed after an interspace by two or three larger grooved teeth. Of the solid teeth 6 are ankylosed to the jaw, and 6 are only loosely attached to it. The figure of a maxillary of *Dipsadoides decipiens* given by ANNANDALE (1905, fig. 3) shows that the teeth stand wide apart. The wide interspaces between these teeth are a distinct indication that only the ankylosed teeth are represented, and at the same time they indicate that 6 loosely attached teeth got lost. In some places the contour of the maxillary shows slight inward curves, probably indicating the depressions. Therefore, we may safely conclude, that 12 solid teeth followed by two grooved ones is the number of maxillary teeth originally present in the type of *Dipsadoides decipiens*. In this respect there is also complete agreement between *Dipsadoides decipiens* and *Boiga jaspidea*.

Mandibular teeth. DE ROOIJ (1917, p. 195) mentions that in *Boiga* (*Dipsadomorphus*) the anterior mandibular teeth are longest, while (l.c., p. 210) it is stated that in *Dipsadoides* the mandibular teeth are subequal. In *Boiga jaspidea* the difference in size between the anterior and posterior mandibular teeth is only slight, especially if this species is compared to other *Boiga* species, e.g., *Boiga dendrophila* (Boie), in which there is a well marked difference. It is well understandable that some authors would describe the mandibular teeth of *Boiga jaspidea* as subequal. The supposed difference between *Boiga* and *Dipsadoides* may be explained in this way.

Of the specific characters mentioned by DE ROOIJ (1917, pp. 199, 210) for *Boiga jaspidea* (*Dipsadomorphus jaspideus*) and *Dipsadoides*, the relative size of the internasals and the number of postoculars might provide differences. For *B. jaspidea* DE ROOIJ mentions: "internasals as long as prefrontals", and for *D. decipiens* "internasals larger than prefrontals". In the specimens of *B. jaspidea* examined by me, the greatest length of the internasals is equal to, or slightly exceeds that of the prefrontals; however, the medium suture between the internasals is much longer than that between the prefrontals; indeed, the former may be even up to three times as long as the latter.

The type of *Dipsadoides decipiens* has a very large supraocular and one small postocular. The specimen described by WESTERMANN has two post-

oculars, and so have all but one of the specimens of *Boiga jaspidea* examined by me; one specimen has three postoculars. The figure of *Dipsadoides decipiens* given by DE ROOIJ (1917, fig. 81) makes it likely that in the type the upper postocular has fused with the supraocular, a variation known from other snakes too.

The length of the eye in *Boiga jaspidea* varies from very slightly shorter to slightly longer than the length of the snout. In this respect the type of *Dipsadoides decipiens* comes within the range of variation of *Boiga jaspidea* too.

The colour pattern is also the same in *Dipsadoides decipiens* and *Boiga jaspidea.*

Thus I arrive at the conclusion that *Dipsadoides decipiens* agrees in all essential characters with *Boiga jaspidea,* and that there is no reason to separate them specifically. *Dipsadoides decipiens* Annandale is, therefore, referred to the synonymy of *Boiga jaspidea* (Dum., Bibr. & Dum.). I compared my specimens of *Boiga jaspidea* to the original description of *Triglyphodon jaspideum* by DUMÉRIL, BIBRON and DUMÉRIL (1854, p. 1093), and there can be no doubt that they belong to the species described by these authors.

It may be remarked that ANNANDALE compared his specimen to *Boiga cynodon* (Boie). According to DUMÉRIL, BIBRON and DUMÉRIL (1854, p. 1096), SCHLEGEL even confused this species with *Boiga cynodon* (Boie). The specific name *decipiens* was well chosen indeed.

TABLE II.

Reg. No.	Scales	Ventrals	Subcaudals	Upper Labials
8427	21/21/13	257	150/150 + 1	r: 6 (3, 4) l: 8 (3, 4, 5)
	23/23/15	255	151/151 + 1	8 (3, 4, 5)
982	21/21/15	246	142/142 + 1	8 (3. 4, 5)
6834	23/21/15	249	147/147 + 1	8 (3, 4, 5)
	21/21/15	253	149/149 + 1	r: 8 (3, 4, 5) l: 7 (3, 4)
5847	23/21/15	253	147/147 + 1	8 (3, 4, 5)
8428	23/—/—	—	—	7 (3, 4, 5)
Banka, WESTERMANN	—/21/13	262	159/159 (+ 1 ?)	8 (3, 4, 5)
Pendopo, S. Sumatra	21/21/15	250	147/147 (+ 1 ?)	—
D. decipiens, type	—/21/—	258	152/152 (+ 1 ?)	8 (3, 4, 5)

With regard to the specimens examined by me, I may make the following notes on variations observed. The scales may be placed in 23 rows behind the head, this number generally diminishes towards the middle of the body, where I count 21 rows. Only in one specimen from Banka (Herp. reg. no. 8427) I count 23 rows at mid-body. Further backwards the number of scale rows is reduced to 15 or 13 in front of the vent. One of the Banka specimens has two superposed loreals, instead of only one. One of the

specimens from Mt. Simpai, W. Java, has three postoculars. The number of upper labials may vary from 6 to 8 (table II).

LITERATURE

ANNANDALE, N., 1905. Additions to the Collection of Oriental Snakes in the Indian Museum, Part 3. Journal and Proc. As. Soc. Bengal, n.s., vol. **1**, pt. 8, Journal pp. 208—214, figs. 1—3.

BOULENGER, G. A., 1896. Remarks on the Dentition of Snakes and on the Evolution of the Poison-fangs. Proc. Zool. Soc. London, pp. 614—616.

DUMÉRIL, A. M. C., G. BIBRON & A. DUMÉRIL, 1854. Erpétologie Générale ou Histoire Naturelle complète des Reptiles, vol. **7**, pt. 2, pp. I—XII + 781—1536.

HEURN, W. C. VAN, 1932. Dipsadoides decipiens (Voorloopig bericht). De Tropische Natuur, vol. **21**, pt. 1, p. 14.

ROOIJ, N. DE, 1917. The Reptiles of the Indo-Australian Archipelago, vol. 2, Ophidia, XIV + 331 pp., 117 figs.

WESTERMANN, J. H., 1942. Snakes from Bangka and Billiton. Treubia, vol. **18**, pt. 3, pp. 611—619, 2 figs.

Mathematics. — *On sets of integers.* (Third communication.) By J. G. VAN DER CORPUT.

(Communicated at the meeting of April 26, 1947.)

§ 4. *Theorems on more than two sets.*

The "sumset" $A_1 + \ldots + A_n$ of n sets $A_1, \ldots, A_n$, each consisting of integers $\geqq 0$, is defined as the set of all integers of the form $a_1 + \ldots + a_n$, where, for each value of h, the integer a_h is a term of A_h.

By $A(m)$ I denote again the number of positive elements $\leqq m$ of A.

In order to deduce a general result, I consider l systems $s_1, \ldots, s_l$ $(l \geqq 1)$, where s_h, for each value of h, is formed by n_h sets A_{hk} $(k = 1, \ldots, n_h)$, each consisting of integers $\geqq 0$, each containing zero and satisfying the inequality

$$A_{h,1}(m) + \ldots + A_{h,n_h}(m) \geqq \gamma_h m \qquad (m = 1, \ldots, g), \quad . \quad . \quad (46)$$

where $\gamma_1 + \ldots + \gamma_l \leqq 1$. Further I consider a sum of the form $\Sigma_1 T(m)$, extended over a finite number of sets T with the property, that any of these sets T is the sumset of a number of systems A_{hk}. I suppose that in any T each system A_{hk} is counted at most once, so that for instance T may have the form A_{11} or $A_{12} + A_{13}$ or $A_{12} + A_{41} + A_{52}$, but not $A_{11} + A_{11}$. In connection with the fact that the given inequalities (46) are symmetrical in the sets A_{1h} belonging to s_1, I will assume that $\Sigma_1 T(m)$ is symmetrical in these sets, also symmetrical in the sets belonging to $s_2, \ldots$, finally symmetrical in the sets, belonging to s_l. Then it is possible to deduce for $m = 1, \ldots, g$ a convenient lower bound for the sum $\Sigma_1 T(m)$, namely

Theorem 8. *Under these conditions we have for* $m = 1, \ldots, g$

$$\Sigma_1 T(m) \geqq m \, \Sigma_1 \tau(T),$$

if

$$\tau(T) = \frac{\lambda_1 \gamma_1}{n_1} + \ldots + \frac{\lambda_l \gamma_l}{n_l},$$

where λ_h *denotes, for each value of* h, *the number of terms* A_{hk}, *occurring in* T.

As corollaries of the special case $l = 1$ we obtain the theorems 9 and 10, found by F. J. DYSON:

Theorem 9. *If each system* $A_1, \ldots, A_n$ *contains zero and*

$$A_1(m) + \ldots + A_n(m) \geqq \gamma m \qquad (m = 1, \ldots, g),$$

where $\gamma \leqq 1$, *then*

$$(A_1 + \ldots + A_n)(m) \geqq \gamma m \qquad (m = 1, \ldots, g).$$

In fact, the left hand side is symmetrical in $A_1, \ldots, A_n$ and is the sum of n terms $T(m)$, where $\tau(T) = \frac{\gamma}{n}$

Theorem 10. *If the conditions of the preceding proposition are satisfied, we have for every natural number $r \leqq n$ and for $m = 1, \ldots, g$*

$$\Sigma_2 (A_{t_1} + \ldots + A_{t_r})(m) \geqq \binom{n-1}{r-1} \gamma m,$$

where Σ_2 is extended over the $\binom{n}{r}$ systems of natural numbers $t_1, \ldots, t_r$ with $t_1 < t_2 < \ldots < t_r \leqq n$.

In fact, the left hand side is symmetrical in $A_1, \ldots, A_n$ and is the sum Σ_1 of $\binom{n}{r}$ terms $T(m)$, where $\tau(T) = \frac{r\gamma}{n}$, so that

$$\Sigma_1 \tau(T) = \binom{n}{r} \cdot \frac{r\gamma}{n} = \binom{n-1}{r-1} \gamma.$$

Another example of theorem 8: If each of the systems A, B, C and D contains zero,

$$A(m) + B(m) \geqq \gamma m \text{ and } C(m) + D(m) \geqq \delta m \qquad (m = 1, \ldots, g),$$

where $\gamma + \delta \leqq 1$, then we obtain for $m = 1, \ldots, g$

$$(A + C)(m) + (A + D)(m) + (B + C)(m) + (B + D)(m) \geqq 2(\gamma + \delta) m.$$

In fact, the left hand side is symmetrical in A and B, also in C and D and is the sum of four terms $T(m)$, where $\tau(T) = \frac{1}{2}(\gamma + \delta)$.

We have treated the last problem already in § 2.

I propose not only to prove theorem 8, but simultaneously the following proposition, involving positive weights $f(m)$. These weights satisfy the inequalities

$$\left.\begin{array}{l} f(m+1) \geqq f(m) \quad (m = 1, \ldots, g-1) \text{ and } f^2(m+1) \geqq f(m) f(m+2) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (m = 1, \ldots, g-2), \end{array}\right\} \quad (29)$$

whereas $A(m)$ denotes the sum $\sum\limits_{a \leqq m} f(a)$, extended over all positive elements $a \leqq m$ of A.

Theorem 11. *If the positive numbers $f(m)$ $(m = 1, \ldots, g)$ satisfy the inequalities (29), if the sets $A_1, \ldots, A_n$ consist of integers $\geqq 0$, if each of these sets contains zero and if*

$$A_1(m) + \ldots + A_n(m) \geqq \gamma \sum_{h=1}^{m} f(h) \qquad (m = 1, \ldots, g),$$

where $\gamma \leqq 1$, then for these values of m

$$(A_1 + \ldots + A_n)(m) \geqq \gamma \sum_{h=1}^{m} f(h) \qquad (m = 1, \ldots, g).$$

To prove these theorems I make use of two lemmas. In these lemmas $A(m)$ denotes the sum of the weigths of the positive elements $\leqq m$ of A, where for $m > g$ the weight $f(m)$ is defined such that

$$f(m+1) \geqq f(m) \text{ and } f^2(m+1) \geqq f(m) f(m+2) \qquad (m = 1, \ldots, g).$$

Let us consider $n (n \geqq 2)$ systems $A_1, \ldots, A_n$ each of which consists of integers $\geqq 0$ and $\leqq g$ and contains the number zero, whereas A_n contains moreover at least one positive integer. Be e the smallest integer $\geqq 0$, such that a natural number $q \leqq n-1$ can be found with the property that e is an element of A_q and that A_n contains at least one positive element b, such that $e + b$ is not contained in A_q. Such an element e exists, since the greatest element a of A_1 and any positive element b of A have the property that $a + b$ does not belong to A_1. After having fixed e and q, I cancel in A_n a positive element b, such that $e + b$ does not belong to A_q, and I add to A_q this integer $e + b$, if it is $\leqq g$. If A_q and A_n are transformed in this manner into A'_q and A'_n and if further $A'_h = A_h$ for $h \neq q$ and $\neq n$, I say that the system $A_1, \ldots, A_n$ is transformed by an elementary transformation into $A'_1, \ldots, A'_n$. Thus A'_n is the set of elements $\neq b$ of A_n; if $e + b > g$, then $A'_q = A_q$, and if $e + b \leqq g$, then A'_q is the set formed by $e + b$ and the elements of A_q.

Lemma 8. *If each of the systems $A_1, \ldots, A_n$ $(n \geqq 2)$ contains zero, if A_n contains moreover at least one positive integer $\leqq g$, and if*

$$A_1(m) + \ldots + A_n(m) \geqq \gamma \sum_{h=1}^{m} f(h) \qquad (m = 1, \ldots, g), \quad (47)$$

where $\gamma \leqq 1$, then each elementary transformation transforms $A_1, \ldots, A_n$ into a system $A'_1, \ldots, A'_n$, with

$$A'_1(m) + \ldots + A'_n(m) \geqq \gamma \sum_{h=1}^{m} f(h) \qquad (m = 1, \ldots, g).$$

We obtain even for any integer $r \geqq 0$

$$\sum_{a'_1 \leqq m} f(a'_1 + r) + \ldots + \sum_{a'_n \leqq m} f(a'_n + r) \geqq \gamma \sum_{h=1}^{m} f(h + r) \quad (m = 1, \ldots, g), \quad (48)$$

if $\sum_{a'_h \leqq m}$ is extended, for each value of h, over the positive elements $a'_h \leqq m$ of A'_h.

Let us suppose that this assertion is not true. Be k the smallest positive integer $\leqq g$, for which the lemma is not valid, more precisely: the positive integer $k \leqq g$ possesses the following properties:

1. It is possible to find a system $A_1, \ldots, A_n$, such that each set A_h contains zero, that A_n contains at least one positive element $\leqq g$, that the inequalities (47) are true for $m = 1, \ldots, k$ and that a suitably chosen

elementary transformation transforms the system $A_1, \ldots, A_n$ into a system $A_1', \ldots, A_n'$, for which at least one of the inequalities

$$\sum_{a_1' \leqq m} f(a_1' + r) + \ldots + \sum_{a_n' \leqq m} f(a_n' + r) \geqq \gamma \sum_{h=1}^{m} f(h + r) \quad (m = 1, \ldots, k), \tag{49}$$

where r denotes a convenient integer $\geqq 0$, is not true.

2. Be l an arbitrary positive integer $< k$. If each of the systems $C_1, \ldots, C_n$ contains zero, if moreover C_n contains at least one positive integer $\leqq g$ and if finally

$$C_1(m) + \ldots + C_n(m) \geqq \gamma \sum_{h=1}^{m} f(h) \quad (m = 1, \ldots, l),$$

then each elementary transformation transforms $C_1, \ldots, C_n$ into $C_1', \ldots, C_n'$, such that

$$\sum_{c_1' \leqq m} f(c_1' + t) + \ldots + \sum_{c_n' \leqq m} f(c_n' + t) \geqq \gamma \sum_{h=1}^{m} f(h + t) \quad (m = 1, \ldots, l) \tag{50}$$

for each integer $t \geqq 0$.

The special case $l = k - 1$, $t = r$, $C_h = A_h$ gives, that the inequalities (49) are true for $m = 1, \ldots, k-1$. Consequently (49) is not valid for $m = k$, that is

$$\sum_{a_1' \leqq m} f(a_1' + r) + \ldots + \sum_{a_n' \leqq m} f(a_n' + r) < \gamma \sum_{h=1}^{m} f(h + r). \quad . \quad . \tag{51}$$

From (47) and lemma 5 we conclude for $m = 1, \ldots, k$

$$\sum_{a_1 \leqq m} f(a_1 + r) + \ldots + \sum_{a_n \leqq m} f(a_n + r) \geqq \gamma \sum_{h=1}^{m} f(h + r) \quad . \quad . \tag{52}$$

Just as in the proof of lemma 6 (§ 3) we show, that the number b, cancelled in A_n, is $\leqq k$ and that the number $e + b$, possibly added to A_q, is $> k$.

From the definition of e it follows that each element $a < e$ of each set A_h $(h = 1, \ldots, n-1)$ has the property, that $a + b$ belongs to A_h; in particular b belongs to each of the sets $A_1, \ldots, A_{n-1}$. The system A_h $(h < n)$ contains $1 + A_h\,(k - b)$ elements $a \leqq k - b < e$ and each of these elements furnishes an element $a + b$ of A_h, which is $\geqq b$ and $\leqq k$, so that we find for $h = 1, \ldots, n-1$

$$\sum_{a_h \leqq k} f(a_h + r) - \sum_{a_h \leqq b-1} f(a_h + r) \geqq f(b + r) + \sum_{a_h \leqq k-b} f(a_h + r + b). \tag{53}$$

The inequality

$$\sum_{a_1 \leqq b-1} f(a_1 + r) + \ldots + \sum_{a_n \leqq b-1} f(a_n + r) \geqq \gamma \sum_{h=1}^{b-1} f(h + r). \quad . \tag{54}$$

is obvious for $b = 1$ and follows for $b > 1$ from (52). For $h = 1, \ldots, n-1$ the sets A_h and A_h' contain the same integers $\leqq k$; the two

sets A_n and A'_n contain the same integers $\leqq b-1$, so that we obtain for $h = 1, \ldots, n-1$ from (53)

$$\sum_{a'_h \leqq k} f(a'_h + r) \geqq f(b+r) + \sum_{a'_h \leqq b-1} f(a'_h + r) + \sum_{a_h \leqq k-b} f(a_h + r + b) \quad (55)$$

and further from (54)

$$\sum_{a'_1 \leqq b-1} f(a'_1 + r) + \ldots + \sum_{a'_n \leqq b-1} f(a'_n + r) \geqq \gamma \sum_{h=1}^{b-1} f(h+r). \quad . \quad (56)$$

The proof is established if we show

$$\sum_{a_1 \leqq k-b} f(a_1+r+b) + \ldots + \sum_{a_{n-1} \leqq k-b} f(a_{n-1}+r+b) \geqq \gamma \sum_{h=1}^{k-b} f(h+r+b), \quad (57)$$

for then we may replace in this inequality $a_1, \ldots, a_{n-1}$ by $a'_1, \ldots, a'_{n-1}$, hence by (55) and (56)

$$\begin{aligned}
&\sum_{a'_1 \leqq k} f(a'_1 + r) + \ldots + \sum_{a'_n \leqq k} f(a'_n + r) \\
&\geqq \sum_{a'_1 \leqq k} f(a'_1 + r) + \ldots + \sum_{a'_{n-1} \leqq k} f(a'_{n-1} + r) + \sum_{a'_n \leqq b-1} f(a'_n + r) \\
&\geqq (n-1) f(b+r) + \gamma \sum_{h=1}^{b-1} f(h+r) + \gamma \sum_{h=1}^{k-b} f(h+r+b) \\
&\geqq \gamma \sum_{h=1}^{k} f(h+r)
\end{aligned}$$

(in virtue of $n-1 \geqq 1 \geqq \gamma$), contrary to (51).

The proof of (57) runs precisely as that of (6) in lemma 1 (§ 1). Thus we find the assertion of our lemma.

Let us again consider n $(n \geqq 2)$ systems $A_1, \ldots, A_n$, each of which consists of integers $\geqq 0$ and $\leqq g$ and contains the number zero, whereas A_n contains moreover at least one positive integer. Be e again the smallest integer $\geqq 0$ such that a natural number $q \leqq n-1$ can be found with the property that e is an element of A_q and that A_n contains at least one positive element b, such that $e + b$ is not contained in A_q. I cancel in A_n all these elements b and I add $e + b$ to A_q, as far as they are $\leqq g$. In this manner A_n and A_q are transformed into the sets A^*_n and A^*_q; I put $A^*_h = A_h$ for $h \neq q$ and $\neq n$. I say that the system $A_1, \ldots, A_n$ is transformed into the system $A^*_1, \ldots, A^*_n$ by an e-transformation. Since this transformation can be decomposed into a finite number of elementary transformations, the preceding lemma gives:

Lemma 9. *Suppose that each of the n $(n \geqq 2)$ systems $A_1, \ldots, A_n$ consists of integers $\geqq 0$ and $\leqq g$, and contains the number zero, whereas moreover A_n contains at least one positive integer. If the inequalities (47) are true, where $\gamma \leqq 1$, we have also*

$$A^*_1(m) + \ldots + A^*_n(m) \geqq \gamma \sum_{h=1}^{m} f(h) \qquad (m = 1, \ldots, g). \quad . \quad (58)$$

The proof of theorem 11 can now be given in a few lines. The assertion of this theorem is obvious if $n = 1$, so that I may suppose $n > 1$ and I may assume that the theorem is true for every value of n less than the actual value. Consequently the theorem holds, if A_n consists only of the number zero. I may therefore suppose, that A_n contains at least one positive element $\leqq g$ and that the theorem is true with the actual value of n, whenever the value of $A_n(g)$ is less than its actual value. I transform the system $A_1, \ldots, A_n$ by an e-transformation into a system $A_1^*, \ldots, A_n^*$. Then $A_n^*(g) < A_n(g)$ and the preceding lemma gives the inequalities (58), so that it follows from the inductive hypothesis

$$(A_1^* + \ldots + A_n^*)(m) \geqq \gamma \sum_{h=1}^{m} f(h) \qquad (m = 1, \ldots, g).$$

We have $A_h^* = A_h$ for each value $h \leqq n-1$, with the exception of one value $h = q$ and by lemma 3 (§ 1) $A_q + A_n$ contains each element $\leqq g$ of $A_q^* + A_n^*$, hence

$$(A_1 + \ldots + A_n)(m) \geqq (A_1^* + \ldots + A_n^*)(m).$$

This proves theorem 11.

Now the proof of theorem 8. I put $n_1 + \ldots + n_l = p$. If $p = 1$, then $l = 1$ and $n_1 = 1$, so that each set T, occurring in the sum Σ_1, denotes the set A_{11} with $\tau(T) = \gamma_1$, hence for $m = 1, \ldots, g$

$$T(m) = A_{11}(m) \geqq \gamma_1 m = m\tau(T).$$

Consequently I may suppose $p > 1$ and assume that the theorem is true for any value of p less than the actual value. If one of the p sets A_{hk} consists only of the number zero, each set T, occurring in the sum Σ_1, is the sumset of $p-1$ systems A_{hk}, so that the assertion of theorem 8 follows from the inductive hypothesis. Consequently I may suppose, that each set A_{hk} contains at least one positive integer and I may assume that the theorem holds with the actual value of p, whenever the value of $A_{l,n_l}(g)$ is less than its actual value. I distinguish two cases:

1. If $n_1 = \ldots = n_l = 1$, then by (46)

$$A_{h,1}(m) \geqq \gamma_h m$$

and T is the sumset $\Sigma_3 A_{h,1}$ of a number of sets $A_{h,1}$. In virtue of $\gamma_1 + \ldots + \gamma_l \leqq 1$ and $n_h = 1$ it follows from the theorem of MANN

$$T(m) \geqq m \Sigma_3 \gamma_h = m(\lambda_1 \gamma_1 + \ldots + \lambda_l \gamma_l) = m\tau(T),$$

consequently

$$\Sigma_1 T(m) \geqq m \Sigma_1 \tau(T).$$

2. In the remaining case at least one of the integers $n_1, \ldots, n_l$ is greater than 1. Without loss of generality I may assume $n_l > 1$. I transform the system $A_{l,1}, \ldots, A_{l,n_l}$ by an e-transformation into a system $A_{l,1}^*, \ldots, A_{l,n_l}^*$. Then $A_{l,n_l}^*(g) < A_{l,n_l}(g)$. By the preceding lemma, applied with $f(m) = 1$,

the inequalities (46) remain true, if $A_{l,h}$ $(1 \leqq h \leqq n_l)$ is replaced by $A^*_{l,h}$. Consequently we deduce from the inductive hypothesis for $m = 1, \ldots, g$

$$\Sigma_1 T^*(m) \geqq m\, \Sigma_1 \tau(T^*) = m\, \Sigma_1 \tau(T);$$

here T^* denotes the set into which T is transformed, if $A_{l,h}$ $(1 \leqq h \leqq n_l)$ is replaced by $A^*_{l,h}$. It is therefore sufficient to deduce the inequalities

$$\Sigma_1 T(m) \geqq \Sigma_1 T^*(m) \qquad (m = 1, \ldots, g).$$

It follows from the definition of the e-transformation, that $A^*_{l,h} = A_{l,h}$ for each $h \leqq n_l - 1$, only one value $h = q$ excepted. I write

$$\Sigma_1 T = \Sigma_4 T + \Sigma_5 T + \Sigma_6 T;$$

The summation Σ_4 is extended over the sets T, involving neither $A_{l,q}$ nor A_{l,n_l}; the summation Σ_5 over the sets T, involving one and only one of the two sets $A_{l,q}$ and A_{l,n_l}; finally Σ_6 is extended over the sets T involving both $A_{l,q}$ and A_{l,n_l}.

For the sets T, occurring in Σ_4, we have $T = T^*$, hence $\Sigma_4 T(m) = \Sigma_4 T^*(m)$. Each set T, occurring in Σ_6, is a sumset containing both sets $A_{l,q}$ and A_{l,n_l}; by convention, made at the beginning of this section, each of these two sets is counted only once, hence

$$T = A_{l,q} + A_{l,n_l} + U \text{ and } T^* = A^*_{l,q} + A^*_{l,n_l} + U.$$

By lemma 3 (§ 1) $A_{l,q} + A_{l,n_l}$ contains each element $\leqq g$ of $A^*_{l,q} + A^*_{l,n_l}$, so that

$$T(m) \geqq T^*(m), \text{ hence } \Sigma_6 T(m) \geqq \Sigma_6 T^*(m).$$

The sum $\Sigma_5 T(m)$, which is symmetrical in $A_{l,q}$ and A_{l,n_l} can be written as a sum of terms of the form $(U + A_{l,q})(m) + (U + A_{l,n_l})(m)$, and lemma 4 (§ 2) gives for $m = 1, \ldots, g$

$$(U + A_{l,q})(m) + (U + A_{l,n_l})(m) \geqq (U + A^*_{l,q})(m) + (U + A^*_{l,n_l})(m),$$

hence

$$\Sigma_5 T(m) \geqq \Sigma_5 T^*(m).$$

This completes the proof.

Biochemistry. — *Tissues of prismatic celloidin cells containing biocolloids.* VIII. *Gelation of the parietal gelatine-gum arabic complex-coacervate and behaviour of the objects obtained with regard to neutral red at various pH's.* By H. G. BUNGENBERG DE JONG and R. C. BAKHUIZEN VAN DEN BRINK.

(Communicated at the meeting of February 22, 1947.)

1. *Introduction.*

In previous communications [1]) we described the complex-coacervation of a mixture of gelatine and gum arabic sols, present in the cells of an artificial tissue, and studied the influence of pH, neutral salts and non-electrolytes. During these investigations the temperature (33—40°) was always kept above the temperature of gelation of the coacervate (± 28°). In this communication we describe the gelation of the coacervate present in the cells and some properties of the objects obtained, especially shifting of colloid from the gelated coacervate-layer to the central vacuole.

2. *Preparation of the membranes.*

We made use of new experience [2]) with regard to a method of spreading that could be better reproduced. As an emulsionizing fluid we used a solution of 20 gr celloidin-gel in a mixture of 100 cc ether + 50 cc amyl-alcohol + 50 cc benzene. In a graduated stoppered measure of 10 cc, fastened to a revolving disk with 22 rotations a minute, 1 cc of a solution of 3 gr gum arabic + 3 gr gelatine + 100 cc distilled water is emulsionized for 5 minutes in 4 cc of the emulsionizing fluid indicated above. Of the emulsion obtained 0,33 cc is spread on a basin of a circular shape, 30 cm in diameter, filled with fresh tap-water (distance of the lower end of the pipette to the surface of the water 3 cm). Under these circumstances a membrane formed, with an average diameter of about 14 cm, in which the separate round cell-groups are still connected by bundles of little folds, but in which as a rule only two such bundles of little folds spring from each cell-group.

A short time after the preparation, part of the membrane is scooped up with an object-glass, the superfluous appending membrane is removed and the object-glass covered with membrane put as a cover on a cuvette as described in previous communications, which is filled with hot (35°) 2 m aeq. acetate-buffer of pH 3.7 (2 m aeq. Na-acetate + 20 millimol acetic acid p. L.)

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[1]) H. G. BUNGENBERG DE JONG and cooperators, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **43**, 512 (1940), **45**, 51, 67 (1942).

[2]) In a later communication in these Proceedings we hope to give further details about this experience.

After this buffer-mixture has passed through the cuvette for 2 minutes, we switch over to a 20 m aeq. acetate-buffer of the same pH (20 m aeq. Na acetate + 200 millimol acetic acid p. L.). After 5 minutes the morphologically ideal final stage, the parietal coacervate being free from vacuoles (fig. 1) has been reached. The picture remains unchanged even if this 20 m aeq. buffer (35°) has passed for some hours.

3. *Undesired lesions of the tissue-cells.*

It was found that the procedure indicated: 2 min with 2 m aeq. buffer followed by 5 min 20 m aeq. buffer, led much more quickly to a parietal coacervate free from vacuoles than if one works exclusively with 20 m aeq. buffer.

It may occur that in spite of this a few cells in a cell-group do have a vacuolised parietal coacervate, while all the cells surrounding them look normal. Such cells in which vacuolisation continues a long time or even increases are damaged and often microscopically one may observe a perforation of the cell-wall between the cell-cavity and the cuvette-liquid (fig. 1a). It was also observed that such lesions occur more frequently if

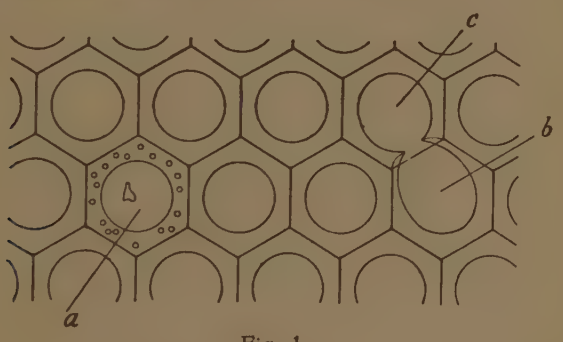

Fig. 1.

an emulsion kept one or more days at room-temperature (± 17°), is used for the preparation of a membrane. Apparently this is correlated with the gelation of the emulsionized drops. Such an un suitable emulsion is much improved by warming to 35° and cooling to room-temperature. In general we recommend using freshly prepared emulsion, as we got the impression that a different kind of lesion may also occur much more frequently when spreading such emulsions that are 1 or more days old. We mean lesions of the side-walls between the cells themselves. Fig. 1 gives the morphological picture of such a lesion of the side-wall between the cells *b* and *c*. The two central vacuoles communicate here, because of a perforation-hole in the side-wall that bulges toward *c*.

Apparently the lesion has been caused by an over-pressure in cell *b* relatively with regard to the pressure in *c*. Such side-wall lesions may sometimes become very frequent and whole series of cells in the cell-group communicate with one another in the same way as *b* and *c*.

4. *Gelation of the parietal coacervate.*

With reduction to room-temperature, while the 20 m aeq. buffer continues to pass along the membrane, vacuolisation occurs in the parietal coacervate. It depends further on the rate of reduction in temperature whether many small (quick reduction in temperature) resp. relatively few larger vacuoles (slow reduction in temperature) are ultimately present in the gelated parietal coacervate. Vacuolisation is a process that also occurs in a coacervate layer in a sedimentation tube, beginning to appear at about 28° [3]).

While the original central vacuole was round, fig. 2*a*, it has now become more or less angular in shape, fig. 2*b*, and the gelated coacervate-layer is less thick than the fluid one.

With warming up the original situation ($b \rightarrow a$) is re-established.

5. *Morphological changes of the gelated complex-coacervate-layer, when the complex-relations are removed, shifting of colloid to the vacuole.*

When now, starting from the condition *b* of fig. 2, KCl is added to the buffer (pH 3.7) at room-temperature, to an amount of 200 m aeq. p. L.,

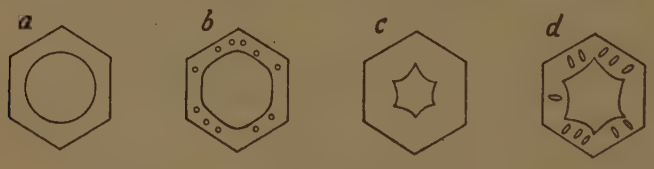

Fig. 2.

the parietal coacervate-layer swells, the central vacuole becomes irregular in shape, while the optic contrast between vacuole and gelated coacervate-layer diminishes strongly. The boundary of the central vacuole is not distinctly visible and the vacuoles in the pariotal coacervate layer become also invisible (fig. $b \rightarrow c$).

When the KCl concentration is sufficiently reduced, the vacuoles become visible again, however. The KCl action is based on a removal of the electrostatic interaction of gelatine (positive) and gum arabic (negative). We know from other experiments that under these circumstances gum arabic goes out of the gelated coacervate and so in our case the central vacuole must begin to contain gum arabic.

This cannot be shown, however, with a basic dye, as the accumulation is prevented by the high KCl concentration.

The complex-relations can also be removed by increasing the pH above the I.E.P. of the gelatine used (I.E.P. $\pm$ 5). If, starting from the condition

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[3]) H. G. BUNGENBERG DE JONG, E. G. HOSKAM and L. H. v. D. BRANDHOF-SCHAEGEN, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **44**, 1104 (1941).

b at room-temperature a very much diluted borax-solution ($\frac{1}{4000}$ mol. p. L.) is led along the membrane, the gelated coacervate-layer also swells, the central vacuole being strongly curved in and the vacuoles of gelation stretched out in the shape of a pear or elliptically (fig. 2*d*).

After this, when rinsing with a diluted solution of a basic dye, it can be proved that the gum arabic has gone over into the central vacuole, because this vacuole colours more strongly than the gelated coacervate-layer.

The results depend on the dye used and on the pH. With methylene blue the accumulation can be realised even at pH 9, but the coacervate-layer also still colours weakly here. A better contrast can be obtained with neutral red. For this reason we have examined this more in detail.

6. *Accumulation of neutral red in the central vacuole.*

In the case of neutral red the choice of the pH is limited on account of the easy flocculation at higher pH's. Borax exclusively (pH 9.24) is therefore completely unsuitable and even in borax-KH_2PO_4 buffers of pH 7.5 flocculation still occurs. pH 7.0 can just be used but pH 6.0 is more favourable in every respect.

In 5) colouring of the vacuoles was attained in two stages: a treatment beforehand with a buffer, followed by a treatment with the diluted dye-solution. It appears that it is also possible to obtain the same result by leading only a buffered neutral red solution along the membrane. It takes a rather long time then, however, as apparently at the lower pH's (7 resp. 6) the expulsion of the gum arabic to the central vacuole occurs more slowly than at pH 9.24.

Use was made of a liquid consisting of 10 cc (borax + KH_2PO_4) buffer [4]) + 5 cc neutral red 0.1 % + 85 cc distilled water.

Starting from condition *b* (fig. 2), within half an hour strong accumulation in the vacuoles is obtained at pH 7.0 and strong accumulation followed by coacervation at pH 6.0. From now on we shall indicate the first type by *diffuse accumulation*, the second by *grain-accumulation*. We found further that with these $\frac{1}{200}$ % neutral red solutions at pH 6 diffuse accumulation occurs instead of grain-accumulation, when the passing liquid also contains 50 m aeq. p. L. KCl. When switching over to the same liquid without KCl, the diffuse accumulation changes into grain-accumulation. When the liquid containing KCl is carried over, grain-accumulation is again replaced by diffuse accumulation.

The reversible change of grain-accumulation into diffuse accumulation is not only possible by increasing the salt-concentration, there being constant dye-concentration, but can also be realised by exclusively changing the dye-concentration.

If at pH 6 with $\frac{1}{200}$ % neutral red grain-accumulation (fig. 3*b*) has been obtained, it changes into diffuse accumulation when $\frac{1}{4000}$ % neutral red

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[4]) By mixing 0.05 molar borax with 0.1 molar KH_2PO_4.

(pH 6) passes. Increasing the neutral red concentration to $\frac{1}{200}$ % then again gives grain-accumulation (fig. 3$a \rightarrow b$).

7. *Colouring of the gelated coacervate-layer with neutral red.*

If one starts from the grain-accumulation with $\frac{1}{800}$ % neutral red at pH 6 and if the liquid that passes through is replaced by a $\frac{1}{200}$ % neutral red solution of pH 5, the darkly coloured grains disappear in the vacuole and the vacuole liquid loses its colour, while on the other hand the gelated coacervate layer itself is coloured (fig. 3$b \rightarrow c$).

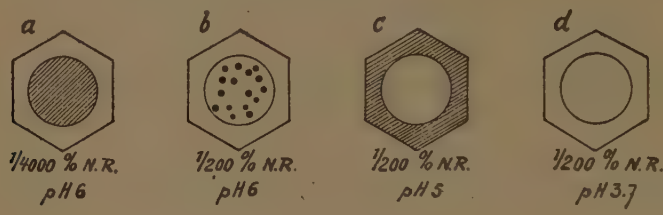

Fig. 3.

So at this pH, which about agrees with the I.E.P. of the gelatine used, practically all the gum arabic already goes again from the vacuole into the parietal gelated coacervate layer, but as it is here only very imperfectly compensated by the positive partial charges of the gelatine, the possibility of colouring with neutral red remains.

At pH 3.7, where complete compensation does take place, neutral red does not colour the gelated coacervate layer at all (fig. 3d).

That this non-colouring is indeed due to the fact that gum arabic combines with gelatine and not to a possibly unfavourable pH is clear, because with membranes which only include gum arabic, at pH 3.7 strong accumulation and coacervation do take place. It is only at a lower pH value (2.3), where the ionisation of the COOH groups of the gum arabic is practically suppressed, that no accumulation occurs any longer.

Essentially the same phenomena as with neutral red were also observed with methylene blue.

Summary.

1. Undesired lesion-symptoms of the cells filled with gelatine and gum arabic, and their prevention, are described.

2. The morphological changes which accompany the gelation of the parietal complex coacervate and the influence on this gelated coacervate of a neutral salt addition or a pH increase are described.

3. At pH's higher than the I.E.P. of the gelatine, gum arabic passes into the central vacuole, as a consequence of which accumulation, sometimes followed by coacervation, with a basic dye (neutral red $\frac{1}{4000} - \frac{1}{800}$ %) may occur.

4. The reversible change (neutral red, pH 6) between diffuse and

grain-accumulation as a consequence of a change of the salt resp. of the dye-concentration is described.

5. The change of the colouring of the vacuole (neutral red pH 6) into the colouring of the gelated coacervate layer (neutral red pH 5) points to an absorption of gum arabic by the gelatine, the gum arabic being compensated very insufficiently, so that it can still combine with neutral red.

6. At pH 3.7 the gelated coacervate layer is no longer coloured by neutral red, though gum arabic, if only present in the cell compartiments, may still accumulate neutral red at this pH. So because of the combination with the positive gelatine the power to combine with neutral red is suppressed.

7. Gum arabic, if only present in the cell compartiments, no longer accumulates neutral red at pH 2.3. This can be explained by suppression of the ionisation of the COOH groups of the gum arabic.

Aerodynamics. — *Aerodynamical problems connected with the motion of a cloud of gas emitted by Nova Persei.* III *). By J. M. BURGERS. (Mededeling no. 49 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft.)

(Communicated at the meeting of March 29, 1947.)

6. *Summary of results of Part I.* — In the first part of this paper we have investigated the phenomena of compression and subsequent expansion produced in a mass of interstellar gas, upon which a thin sheet of a gas of far greater density was imagined to have struck. The results for the density, pressure and temperature of the gas in the compression region are given by the formulae:

$$\varrho = 4\varrho_0/\Phi \quad ; \quad p = 12\,\varrho_0 S'^2/\Phi^{5/3} \quad ; \quad RT = 3\,S'^2/\Phi^{2/3} \; . \quad . \quad (41)$$

where ϱ_0 = original density of the uncompressed interstellar gas. In these expressions Φ is a function of t and s given by the first line of (21); S'^2 is a function of s given by (24*a*), s being a parameter characterizing the paths of the various layers of the compressed gas, with $s = 0$ at the front of the advancing cloud. For $s = 0$ the function S'^2 reduces to $\frac{1}{9} V_0^2$, so that at the front $p = \frac{4}{3}\varrho_0 V_0^2/\Phi_0^{5/3}$; $RT = \frac{1}{3} V_0^2/\Phi_0^{2/3}$ (Φ_0 is the value of Φ for $s = 0$; it is a function of the time). In the formulae referred to the quantities $\tau = \frac{1}{3} aV_0 t$ and $\sigma = a\,s$ are non-dimensional variables; V_0 is the velocity of the thin sheet or cloud at the moment it struck upon the mass of interstellar gas (assumed originally to have constant density with a sufficiently well defined plane front), while a is a constant, proportional to $\varrho_0/\bar{\varrho}$ ($\bar{\varrho}$ being the mass of the advancing cloud per cm² of frontal area), as indicated in (14*b*) and (22). Distances expressed in σ-measure can be reduced to distances in x-measure (*i.e.* real distances) by means of the relation: $\partial x/\partial\sigma = \Phi/a$. — The velocity V of the front of the advancing cloud as a function of the time is given by (23).

The calculations leading to these expressions had been based upon the assumption that the processes taking place in the gas could be discussed with the aid of the classical equations of aerodynamics, treating this gas as an ideal monatomic gas and neglecting radiation, heat conduction and effects of viscous forces. Investigations in the meantime carried out by OORT have thrown doubt upon this assumption; its seems probable that in consequence of the high velocities of the atoms the ordinary calculations concerning the mean free path do not apply and that phenomena of quite a different nature make their appearance. The process is complicated by

*) Parts I and II appeared in these Proceedings **50**, pp. 262, 332, 1947.

the dissociation of the hydrogen atoms into protons and electrons. These circumstances require totally different methods of treatment, which for a large part still have to be developed. As such a development falls outside the original scope of the paper, and on the other hand the methods applied here may have some interest in themselves, the author has permitted himself to finish the paper in the way originally intended, by considering the effect of heat conduction from the compressed gas to the advancing cloud and by working out an energy balance for the latter. To retain continuity with the preceding part of the paper, the numerical values found there have been used also here ($V_0 = 1{,}8 \cdot 10^8$ cm/sec; $\tau = 0{,}55$; $\Phi_0 = 1{,}53$; $a = 2{,}6 \cdot 10^{-17}$; $T = 97{,}5 \cdot 10^6$). For ϱ_0 instead of OORT's original value $12{,}1 \cdot 10^{-24}$ gr/cm^3 (7,3 H-atoms per cm^3) we take a much lower estimate $2{,}85 \cdot 10^{-24}$ gr/cm^3 (1,7 H-atom per cm^3), which is deduced below in section 8 where the temperature of the cloud is supposed to be 25.000° (instead of 50.000°, which may be too high). We then find: $\varrho = 4\varrho_0/\Phi_0 = 7{,}43 \cdot 10^{-24}$ gr/cm^3. But it must be kept in mind that probably the equations can only be applied to cases of a less extreme nature.

7. Heat conduction in the compressed gas. — From the kinetic theory of gases we take the following expressions, referring to a monatomic gas with smooth spherical molecules:

coefficient of viscosity: $\mu = 0{,}33\, l\, \varrho\, c_m$;

coefficient of heat conduction: $\lambda = 2{,}5\, \mu\, c_v$,

where $c_m = \sqrt{3RT}$ = (quadratic) mean molecular velocity; $c_v = 1{,}5\, R$ = specific heat at constant volume; and l = mean free path, defined by $l = 1/(\pi n \sigma^2) = 4{,}38 \cdot 10^{-9}/\varrho$, with σ = diameter of an atom = $1{,}1 \cdot 10^{-8}$ cm and $n = 6{,}03 \cdot 10^{23}\, \varrho$ = number of atoms per cm^3. Taking $\varrho = 7{,}43 \cdot 10^{-24}$ gr/cm^3 and $c_m = 1{,}56 \cdot 10^8$ cm/sec, corresponding to $RT = 0{,}81 \cdot 10^{16}$, we roughly find: $l = 5{,}9 \cdot 10^{14}$ cm; $\nu = \mu/\varrho = 3{,}04 \cdot 10^{22}$ cm^2/sec; $\lambda/\varrho c_v = 7{,}6 \cdot 10^{22}$ cm^2/sec. The mean free path is small in comparison with the distances over which there is an appreciable change of temperature according to the results of Part I, as those distances are of the order $\Phi/a = 6.10^{16}$ cm. — In the neighbourhood of the front of the advancing cloud a much lower temperature will appear in consequence of loss of heat through conduction; this will entail a decrease of λ proportional to $\sqrt{T}$, while at the same time the density will change proportionally to $1/T$, if we assume that the pressure near the front remains equal to the value $\frac{4}{3}\varrho_0 V_0^2/\Phi_0^{5/3}$. In that case we shall write:

$$\left.\begin{aligned} \lambda/c_v &= c_1\, T^{1/2} \\ \varrho = p/RT &= \tfrac{4}{3}\varrho_0\, V_0^2/R\, T\, \Phi_0^{5/3} = c_2\, T^{-1}\, \Phi_0^{-5/3} \end{aligned}\right\} \quad . \quad . \quad . \quad (42)$$

where $c_1 = 2{,}5 \cdot 0{,}33 \cdot 4{,}38 \cdot 10^{-9} \sqrt{3R} = 5{,}7 \cdot 10^{-5}$, and $c_2 = \frac{4}{3}\varrho_0\, V_0^2/R = 14{,}8 \cdot 10^{-16}$.

The equation for the change of temperature, through expansion of the

gas and heat conduction combined, when written in EULERian coordinates, has the form:

$$\varrho\, c_v \left(\frac{\partial T}{\partial t} + u\,\frac{\partial T}{\partial x}\right) + p\,\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\lambda\,\frac{\partial T}{\partial x}\right) \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Writing $p = \varrho RT$ and transforming to the LAGRANGian coordinates t, s as used previously we find:

$$\varrho\, c_v \left(\frac{\partial T}{\partial t} + \tfrac{2}{3}\,\frac{T}{\Phi}\,\frac{\partial u}{\partial s}\right) = \frac{1}{\Phi}\,\frac{\partial}{\partial s}\left(\frac{\lambda}{\Phi}\,\frac{\partial T}{\partial s}\right) \quad . \quad . \quad . \quad . \quad . \quad (43a)$$

Noticing that $\partial u/\partial s = \partial \Phi/\partial t$ and introducing the non-dimensional variables τ and σ, we can also write this equation as follows:

$$\frac{\partial T}{\partial \tau} + \tfrac{2}{3}\,\frac{T}{\Phi}\,\frac{\partial \Phi}{\partial \tau} = \frac{3a}{\varrho\, c_v\, V_0}\,\frac{1}{\Phi}\,\frac{\partial}{\partial \sigma}\left(\frac{\lambda}{\Phi}\,\frac{\partial T}{\partial \sigma}\right) \quad . \quad . \quad . \quad . \quad (43b)$$

From the value of $\lambda/\varrho c_v$ given above, combined with the data for V_0 and a, we deduce that in the interior of the compression region the value of $3a\lambda/\varrho c_v V_0$ is approximately 0,033. Hence in this region, where the derivatives of T, λ and Φ with respect to τ and σ are of the same order of magnitude as the quantities themselves (τ and σ having values of order unity), the right hand side of (43b) can be neglected, so that we arrive at the result:

$$T\,\Phi^{2/3} = \text{constant along a line of constant } \sigma,$$

which is the adiabatic law applied in Part I.

The situation becomes different in the neighbourhood of the frontier with the cloud. As the density of the latter, at least in the layers most near to the front, is about a thousand times greater than the density of the gas before the front, the temperature of the cloud will be about a thousand times less, as otherwise there could not be equality of pressure on both sides of the boundary surface. (This high density of the cloud is the cause of the observable radiation by the material of the cloud, as the radiation per unit volume is proportional to the square of the density). We assume that the temperature of the cloud, although it may be changing, nevertheless will remain considerably lower than that of the interstellar gas at some distance from the front; consequently we can consider the presence of the cloud simply as a boundary condition for a problem of heat conduction confined to the interstellar gas only. In view of the expected ratio between the temperatures it is allowed to take this boundary condition in the simplified form that the temperature of the interstellar gas decreases nearly to zero at the surface where this gas meets the cloud.

Combining this argument with what had been deduced in connection with eq. (43b) we conclude that in the interstellar gas, in the immediate neighbourhood of the frontier with the cloud, the temperature of this gas

will change considerably over a region, the width of which in σ-measure can be expected to be of the order 0,18 [7]). As this corresponds to a distance of about 10^{16} cm in x-measure, there is no danger of coming too near to the length of the mean free path, the more so as in consequence of the greater density in the region considered the length of this path will decrease far below the value $5{,}9 \cdot 10^{14}$ cm mentioned above. On the other hand the small width in σ-measure allows us to replace Φ by Φ_0, which is independent of σ. Multiplying eq. (43*b*) on both sides by $\frac{3}{2} T^{1/2} \Phi_0$ and introducing the expressions for λ and ϱ given by (42), we can now bring this equation into the form:

$$\frac{\partial}{\partial \tau}(T^{3/2}\,\Phi_0) = \frac{3\,c_1\,a\,T^{3/2}}{c_2\,V_0\,\Phi_0^{1/3}}\,\frac{\partial^2}{\partial \sigma^2}(T^{3/2}\,\Phi_0). \quad . \quad . \quad . \quad . \quad . \quad (43c)$$

We put:

$$T^{3/2}\,\Phi_0 = Y\,; \qquad \int_0^{\tau} d\tau\,\Phi_0^{-4/3} = z. \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

The equation can then be written:

$$\frac{\partial Y}{\partial z} = \frac{1}{2\,\varkappa^2}\,Y\,\frac{\partial^2 Y}{\partial \sigma^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

where $\varkappa = \sqrt{c_2 V_0 / 6 c_1 a} = 5{,}48 \cdot 10^6$.

In analogy with the Error-function solution of the ordinary equation of heat conduction, we look for a solution of (45) in which Y is a function f of the variable $\zeta = \varkappa\sigma/\sqrt{z}$ [the factor $\varkappa$ has been inserted to get rid of the coefficient in (45)]. It is found that f must satisfy:

$$f'' f = -\zeta f'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

As boundary conditions we assume: $f = 0$ for $\zeta = 0$; $f =$ *constant* for $\zeta = \infty$. Solutions of (46) satisfying these conditions are of the form:

$$f = a^2\,F(\eta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

where $\eta = \zeta/a$, a being an arbitrary numerical constant, while F is defined in such a way that $F'(0) = 1$. For small values of η the following series can be constructed:

$$F = \eta - \tfrac{1}{2}\,\eta^2 + \tfrac{1}{12}\,\eta^3 + \tfrac{1}{144}\,\eta^4 - \tfrac{1}{960}\,\eta^5 - \tfrac{1}{1800}\,\eta^6. \quad . \quad . \quad (48)$$

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[7]) It might be useful to check whether there is sufficient space for this region between the front of the cloud and the shock wave at the head of the compression region. It can be that with small values of τ this should not be the case; we might then expect a somewhat smaller value for q_{front} than is obtained from eq. (50). This point has not been investigated.

The solution can be continued by a simple numerical method of integration, giving the following approximate values:

$\eta = 0{,}5$	$F = 0{,}386$ (from the series)
1,0	0,59
1,5	0,68
2,0	0,71
∞	0,72 (estimated).

To apply this solution to the problem of heat conduction before us, the following boundary conditions are assumed for the latter: at the initial instant $\tau = 0$ the quantity $T\,\Phi^{2/3}$ has the constant value $T_1 = \frac{1}{5} V_0^2/R = 1{,}3 \cdot 10^8$ for the relevant values of σ; for $\tau > 0$ the value of $T\,\Phi^{2/3}$ is zero at the frontier $\sigma = 0$; while for values of σ large compared with 0,18 it has the value $3S'^2/R$, which remains practically equal to T_1. With these boundary conditions we find $Y_\infty = a^2 F(\infty) = T_1^{3/2}$; so that, with $F(\infty) = 0{,}72$, we obtain $a = 1{,}18\, T_1^{3/4} = 1{,}43 \cdot 10^6$. The solution becomes:

$$T^{3/2}\,\Phi_0 = Y = a^2\, F(\varkappa\sigma/a\sqrt{z}). \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Hence the heat current at the front (taken positive when directed towards negative values of x) amounts to:

$$\left.\begin{aligned} q_{\text{front}} = \lambda \left(\frac{\partial T}{\partial x}\right)_{s=0} &= \frac{a\lambda}{\Phi_0}\left(\frac{\partial T}{\partial \sigma}\right)_{\sigma=0} = \\ = \tfrac{2}{5}\,\frac{a\lambda}{T^{1/2}\,\Phi_0^2}\left(\frac{\partial Y}{\partial \sigma}\right)_{\sigma=0} &= \tfrac{2}{5}\,\frac{c_1\, c_v\, a}{\Phi_0^2}\,\frac{a\varkappa}{\sqrt{z}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (50)$$

With $\tau = 0{,}55$ we find roughly $z = 0{,}41$. Inserting numerical values we obtain:

$$q_{\text{front}} = 0{,}65\ \text{erg/cm}^2\,\text{sec}.$$

This amount is about 3 times higher than the amount of the radiation estimated from the astronomical data, *viz.* 0,2 erg/cm² sec. The decrease of heat conductivity in consequence of the decrease of the temperature near the frontier of the domain appears to be compensated by the increased steepness of the temperature gradient, which according to (49) becomes infinite for $\sigma = 0$ [8]). If a lower value is taken for the temperature of the

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[8]) Formula (50) for the heat current can be brought into the form:

$$q_{\text{front}} = \frac{2 \cdot 1{,}18}{3\sqrt{6}}\, T\, \frac{\sqrt{a\lambda\varrho c_v V_0}}{\Phi_0^{2/3}\,\sqrt{z}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (I)$$

where T represents the temperature of the gas at sufficient distance from the frontier ($T = T_1\,\Phi^{-2/3}$), while the values of λ and ϱ correspond to this same temperature.

When λ and ϱ should have been treated as constants, and at the same time the temperature T at a sufficient distance from the frontier and the linear scale of the field

interstellar gas, q_{front} changes proportionally to $T^{3/4}$ through the influence of a. Such a lower temperature might be obtained if we could assume a smaller value for the initial velocity V_0. It should further be remarked that the value of q_{front} as given by (50) decreases with time. If we use the approximate expression for Φ mentioned in section 4, so that $\Phi_0 = (1 + \frac{4}{3}\tau)^{3/4}$ we find:

$$z = \tfrac{3}{4} ln\,(1 + \tfrac{4}{3}\tau); \qquad \Phi_0^2 \sqrt{z} = 0{,}87\,(1 + \tfrac{4}{3}\tau)^{3/2} \sqrt{ln\,(1 + \tfrac{4}{3}\tau)},$$

which for $\tau = 1$ becomes 2,85.

8. *Energy balance for the cloud.* — When the energy received by conduction from the interstellar gas is not wholly radiated away, a gradual rise of temperature of the cloud must be expected, accompanied by expansion. In order to obtain a picture of these phenomena, we construct a semi-stationary solution, according to the method indicated in a previous paper [9]). It is assumed that the expansion proceeds sufficiently slowly, so that in first approximation the whole mass of the cloud suffers the same deceleration $-dV/dt$. As OORT had pointed out this deceleration has the same effect as a gravity field directed towards the front of the cloud, and gives rise to the appearance of a "barometric" gradient of the pressure. We assume that the differences of temperature between the various parts of the cloud are not great, and will regard the temperature as approximately constant throughout its interior.

The equations of motion, when written in EULERIAN coordinates, have the form [10]):

$$\varrho \frac{Du}{Dt} = \varrho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) = - \frac{\partial p}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

$$\frac{D\varrho}{Dt} = \frac{\partial \varrho}{\partial t} + u \frac{\partial \varrho}{\partial x} = - \varrho \frac{\partial u}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

$$\varrho\, c_v \frac{DT}{Dt} = \varrho\, c_v \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} \right) = - p \frac{\partial u}{\partial x} + \frac{\partial q}{\partial x} - \psi \, . \quad . \quad . \quad (53)$$

Here q is the current of heat, reckoned positive when directed towards

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$(\partial x/\partial \sigma)$ could be considered as independent of the time, the familiar solution of the ordinary equation of heat conduction would have given:

$$q_{front} = T \sqrt{\frac{\lambda \varrho c_v}{\pi t}} = \frac{1}{\sqrt{3\pi}} T \frac{\sqrt{a \lambda \varrho c_v V_0}}{\sqrt{\tau}} \, . \quad . \quad . \quad . \quad . \quad (II)$$

The numerical factors in formulae (I) and (II) differ very little (resp. 0,321 and 0,326); the only difference of some importance is that between $\Phi_0^{2/3} \sqrt{z} = 0{,}84$ and $\sqrt{\tau} = 0{,}74$, which is connected with the change of the field with time.

9) Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 49, 600 (1946).

10) Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 49, 591 (1946).

$-x$, while ψ represents the radiation per unit mass in unit time. It is assumed that the cloud stretches out from $x = \varphi_0(t)$ (= front of the cloud) to some negative value of x.

The solution will be constructed so as to satisfy the equation of continuity (52) and eq. (51). To find an exact solution of (53) would require precise knowledge of the phenomena of heat conduction and radiation as functions of x throughout the cloud; as this is not available we replace (53) by the integral equation of energy for the whole cloud, which is obtained by adding to (53) the product of (51) with u, and integrating the resulting expression with respect to x. When we suppose that at the back of the cloud (practically for $x = -\infty$) the pressure and the value of q can be neglected, we find:

$$q_{\text{front}} = \int_{-\infty}^{\varphi_0} dx \left(\varrho\, c_v \frac{DT}{Dt} + \varrho\, u \frac{Du}{Dt} \right) + p_{\text{front}}\, V + \int_{-\infty}^{\varphi_0} \psi\, dx.$$

We can eliminate p_{front} by making use of the integral of (51):

$$p_{\text{front}} = -\int_{-\infty}^{\varphi_0} dx\, \varrho \frac{Du}{Dt}.$$

In this way the resulting equation takes the form:

$$q_{\text{front}} = \int_{-\infty}^{\varphi_0} dx \left\{ \varrho\, c_v \frac{DT}{Dt} - \varrho\,(V-u) \frac{Du}{Dt} \right\} + \int_{-\infty}^{\varphi_0} \psi\, dx \quad . \quad . \quad (54)$$

expressing that the heat received by conduction from the interstellar gas is divided into a part serving for increase of temperature, a second part for expansion, and a third part for radiation. — The lack of information which we suffer by the rejection of the differential equation (53) will be compensated by an assumption concerning the distribution of the density, to be introduced subsequently.

We assume the expression:

$$x = \varphi_0(t) - s\, \varphi_1(t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

to represent the paths described by the various elements of the cloud, s being a parameter distinguishing between the paths of different elements, in such a way that $s = 0$ at the front of the cloud, while positive values of s refer to elements at some distance behind the front. We take $t = t_0$ at the epoch of observation (1934), and determine $\varphi_1(t)$ in such a way that $\varphi_1(t_0) = 1$, so that for t_0 the value of s is equal to the distance behind the front. The velocity of an element of the cloud is given by:

$$u = (dx/dt)_{s=\text{const.}} = \dot{\varphi}_0 - s\, \dot{\varphi}_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

Hence at the front: $V = \dot{\varphi}_0$. We further have: $\partial u/\partial x = -(1/\varphi_1)\cdot(\partial u/\partial s) = \dot{\varphi}_1/\varphi_1$, from which it will be seen that (55) implies a homogeneous expansion of the whole cloud [a non homogeneous expansion would have been obtained, if to (55) we should have added terms with higher powers of s]. In connection with what has been said concerning the rate of expansion, it will be evident that we must suppose $s\dot{\varphi}_1 \ll \dot{\varphi}_0$ for the relevant values of s.

In order to satisfy the equation of continuity the expression for the density must be of the form: $\varrho = \varrho^*(s)/\varphi_1$; making use of the freedom we have gained by discarding (53), we take an exponential function for $\varrho^*(s)$ and write:

$$\varrho = \varrho_m e^{-\beta s}/\varphi_1(t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

where ϱ_m is a constant. When this expression is substituted into (51), integration gives:

$$p = -\varrho_m e^{-\beta s}\left(\frac{\ddot{\varphi}_0}{\beta} - \frac{1+\beta s}{\beta^2}\ddot{\varphi}_1\right) \quad . \quad . \quad . \quad . \quad . \quad (58)$$

from which the temperature is found by means of:

$$RT = \frac{p}{\varrho} = -\varphi_1\left(\frac{\ddot{\varphi}_0}{\beta} - \frac{1+\beta s}{\beta^2}\ddot{\varphi}_1\right) \quad . \quad . \quad . \quad . \quad . \quad (59)$$

For many purposes it is sufficient to reduce these expressions to their most important parts as follows:

$$p_m \cong -\varrho_m e^{-\beta s}\ddot{\varphi}_0/\beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60a)$$

$$RT \cong -\ddot{\varphi}_0\varphi_1/\beta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60b)$$

We might write: $-\ddot{\varphi}_0 = g$, representing the deceleration of the cloud; from (60b) we then find: $\beta = g\varphi_1/RT$, so that (60a) can be written:

$$p \cong (\varrho_m RT/\varphi_1)\cdot e^{-gs\varphi_1/RT} \quad . \quad . \quad . \quad . \quad . \quad . \quad (60c)$$

showing the analogy with the formula for the barometric pressure gradient under the action of gravity.

We now calculate the terms of the right hand side of (54). The first one becomes, to the same approximation as used in (60):

$$\int_{-\infty}^{\varphi_0} dx\,\varrho\,c_v\frac{DT}{Dt} = \int_0^{\infty} ds\,\tfrac{2}{3}R\varrho\frac{DT}{Dt}\varphi_1 = -\tfrac{2}{3}\varrho_m\frac{\ddot{\varphi}_0\dot{\varphi}_1 + \dddot{\varphi}_0\varphi_1}{\beta^2}.$$

In order to calculate the second one, which depends upon the difference between V and u, we use the full expression for u, but reduce Du/Dt to its principal term. This gives:

$$-\int_{-\infty}^{\varphi_0} dx\,\varrho\,(V-u)\frac{Du}{Dt} = -\int_0^{\infty} ds\,\varrho\,(V-u)\frac{Du}{Dt}\varphi_1 = -\varrho_m\frac{\ddot{\varphi}_0\dot{\varphi}_1}{\beta^2}.$$

In the third term the radiation per unit volume will be represented by a formula of the type $\varrho^2\psi_0(T)$. Neglecting differences of temperature in the interior of the cloud we find:

$$\int_{-\infty}^{\varphi_0} dx\,\psi = \int_0^{\infty} ds\,\varrho^2\,\psi_0\,\varphi_1 = \frac{\varrho_m^2\,\psi_0(T)}{2\beta\,\varphi_1}.$$

Adding together the various terms we arrive at:

$$q_{\text{front}} = -\tfrac{5}{2}\,\varrho_m\,\frac{\ddot{\varphi}_0\,\dot{\varphi}_1}{\beta^2} - \tfrac{3}{2}\,\varrho_m\,\frac{\ddot{\varphi}_0\,\varphi_1}{\beta^2} + \frac{\varrho_m^2\,\psi_0(T)}{2\beta\,\varphi_1}. \quad . \quad . \quad (61)$$

Along with this equation we use the condition that the pressure at the front of the cloud, to be obtained from (60*a*) for $s = 0$, must be equal to the pressure in the interstellar gas at this front, which has been found in Part I. This leads to the equation:

$$-\varrho_m\,\ddot{\varphi}_0/\beta = \tfrac{4}{3}\,\varrho_0\,V_0^2\,\Phi_0^{-5/3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

This equation, which in fact is the same as equation (6) of section 2 [the reader will observe that the mass $\bar{\varrho}$ of the cloud per unit area of frontal surface, in the present notation is equal to ϱ_m/β] gives the value of $\ddot{\varphi}_0$, which thus can be considered as a known function of the time. — The value of q_{front} is given by (50). — As the character of the function $\psi_0(T)$ is known from physical considerations, this quantity can be considered as a function of Φ_0 and φ_1 in consequence of (60*b*) and (62). Hence (61) constitutes a differential equation for φ_1; when φ_1 has been found, the course of temperature and density as functions of the time can be deduced as well.

In order to have numerical values, we take the temperature of the cloud to be 25.000° at the epoch of observation. At this same epoch $-\ddot{\varphi}_0 = 0{,}11$ cm/sec², while φ_1 is assumed to have the value 1; hence eq. (60*b*) gives: $\beta = 5{,}29 \cdot 10^{-14}$ cm^{-1} [11]). Calculations made by OORT give for this temperature: $\psi_0 = 6{,}5 \cdot 10^{-23}$ erg/cm³ for a density of 1 H-atom per cm³. Reducing the density to gr/cm³, we find: $\psi_0 \cong 2{,}4 \cdot 10^{25}$. The expression for the integrated radiation per cm² of frontal area, $\varrho_m^2\,\psi_0/2\beta\varphi_1$, must be equal to 0,2 erg/cm² sec. With these values we obtain:

$$\varrho_m^2\,\psi_0 = 2{,}12 \cdot 10^{-14}; \qquad \varrho_m \cong 3{,}0 \cdot 10^{-20}; \qquad \bar{\varrho} = \varrho_m/\beta \cong 5{,}6 \cdot 10^{-7};$$

and finally: $\varrho_0 = \frac{7}{36}\,a\,\bar{\varrho} = 2{,}85 \cdot 10^{-24}$, as was already mentioned before.

For a rough discussion of eq. (61) we shall assume ψ_0 to be independent of the temperature, which can be done in the range from 15.000° to 50.000°, a suitable mean value being $1{,}8 \cdot 10^{25}$.

When the expressions for $\ddot{\varphi}_0$ [from (62)] and for q_{front} [given by

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[11]) It should be checked afterwards whether the condition $\dot{\varphi}_1/\beta \ll \dot{\varphi}_0$ is satisfied with this value of β. As $\dot{\varphi}_1$ will be of the order aV_0, this probably will be the case.

(50)] are inserted into (61), this equation after some reductions can be brought into the form:

$$\frac{d\xi}{dt} = \frac{c_1 c_v a \alpha \varkappa \beta}{5 \varrho_0 V_0^2 \Phi_0^{4/3} \sqrt{z}} - \frac{3 \varrho_m^2 \psi_0}{20 \varrho_0 V_0^2 \Phi_0^{1/3}} \frac{1}{\xi} \quad . \quad . \quad . \quad . \quad (63)$$

ξ having been written for φ_1/Φ_0. Introducing the numerical data, and changing from the time t in seconds to the dimensionless variable $\tau = \frac{1}{3} a V_0 t$, we obtain:

$$\frac{d\xi}{d\tau} = \frac{107}{\Phi_0^{4/3}\sqrt{z}} - \frac{22{,}2}{\Phi_0^{1/3}\xi} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63a)$$

In this equation Φ_0 and z are known functions of the time. It is sufficient here to use the approximate expressions mentioned at the end of section 7; then:

$$\frac{d\xi}{d\tau} = \frac{124}{(1+\frac{4}{3}\tau)\sqrt{ln\,(1+\frac{4}{3}\tau)}} - \frac{22{,}2}{(1+\frac{4}{3}\tau)^{1/4}\xi} . \quad . \quad . \quad . \quad (63b)$$

It will be seen that with the present numerical data, which did not give a balance of heat conduction and radiation, the first member on the right hand side of (63*b*) is preponderant, so that in course of time ξ will increase; this in its turn will strengthen the preponderance of the first term. An approximate solution is obtained by neglecting the second term; adjusting the constant of integration in such a way that at $\tau = 0{,}55$ (epoch of observation) we shall have $\xi = 1/\Phi_0 = 0{,}65$, we find:

$$\xi < 185\,[\sqrt{ln\,(1+\tfrac{4}{3}\tau)} - \sqrt{ln\,1{,}73}] + 0{,}65. \quad . \quad . \quad . \quad . \quad (64)$$

From this expression φ_1 is obtained by multiplication with $(1+\frac{4}{3}\tau)^{3/4}$. Taking as a rough approximation to the value of $\ddot{\varphi}_0$ the formula:

$$\ddot{\varphi}_0 = -\tfrac{7}{27}\, a\, V_0^2\, (1+\tfrac{4}{3}\tau)^{-5/4},$$

equation (60*b*) finally gives:

$$RT < 48\,\frac{a V_0^2}{\beta}\,(1+\tfrac{4}{3}\tau)^{-1/2}\,[\sqrt{ln\,(1+\tfrac{4}{3}\tau)} - 0{,}740].$$

The temperature would be rapidly rising during the present period.

The approximations applied here are very rough, but they show that the behaviour of the temperature of the cloud is very sensitive to changes in the energy balance.

Physics. — *Recovery and recrystallization viewed as processes of dissolution and movement of dislocations.* I. By W. G. BURGERS. (Laboratorium voor Physische Scheikunde der Technische Hoogeschool, Delft.) (Communicated by Prof. J. M. BURGERS.)

(Communicated at the meeting of March 29, 1947.)

Summary.

In this paper recovery and recrystallization are treated as processes of dissolution and movement of dislocations present in the cold-worked state.

In I some general features of both processes are discussed, especially with a view to ascertain whether they are fundamentally different or whether recovery has to be viewed as recrystallization on an "invisible" scale. Some experimental facts are brought forward, relating to the dependence of both phenomena on time and temperature of annealing and degree of deformation, which point to fundamental differences between both processes.

In order to get a closer insight into these differences, in II the block-structure of the crystalline state is discussed. Attention is drawn to the fact that single crystal, polycrystal and cold-worked state differ only by degrees: they all are built up of lattice blocks, eventually in a stressed condition, connected and separated by transition layers of dislocated atoms. Some considerations regarding the stability of such structures, as far as available from the literature, are given.

Finally in III the atomic movements, induced in the cold-worked state by heating, are considered. A difference is made between (*a*) displacements which, by "dissolution" of dislocations, change the stresses in the lattice-blocks, and (*b*) displacements of boundary layers as a whole, causing growth of one domain at the cost of an adjoining one. The former are essential for recovery, the latter for recrystallization. With the aid of both types of displacements several experimental facts, observed with recovery and recrystallization, are discussed. As to recrystallization it is mentioned that the mean rate of growth of crystals, grown in fine grained quasi-isotropic polycrystalline testpieces, although being constant in all directions, is different for different crystals. The possibility is to be considered that such differences are due to differences in imperfection ("mosaic character") of these crystals. This supposition would imply the assumption that a crystal, when starting its growth from an "imperfect" nucleus, retains this imperfection throughout its development.

Finally the occurrence of "stimulated" crystals is brought forward. It is shown that their presence explains the increase in rate of nucleation in the course of isothermal recrystallization, observed recently in an investigation of the recrystallization of aluminium by ANDERSON and MEHL.

Introduction.

When a metal is cold-worked, its physical and mechanical properties generally suffer considerable changes. By a subsequent annealing treatment these changes are again annihilated, either gradually or, at a sufficiently high temperature, in a more or less abrupt manner. We can thus speak of a process of recovery, taking this word in its general sense. In practice, however, there is reason to discriminate between two apparently different processes, one most easily observed at lower temperatures, in which a return of the properties to their normal values takes place without "visible" change in structure of the metal, and a second process, in which we observe the formation and the growth of new crystallites. The expression *recovery* is generally confined to the first mentioned process, whereas the second is called *recrystallization*. This expression embraces in its "primary" stage the formation and growth of new crystallites in the deformed matrix. It is often followed by a "secundary" growth of the new crystallites at the cost of their equally new neighbours, a process indicated as grain-growth.

Although both recovery and recrystallization have been extensively studied, for example in their dependence on duration and temperature of annealing (for a summarizing review we may refer to (1)), it must be stated that our knowledge and insight in the mechanism of the underlying atomic processes are still very defective, notwithstanding valuable contributions to this question by various authors, of whom we mention in particular KORNFELD (2) and DEHLINGER (3).

In what follows we intend to discuss recovery and recrystallization, starting from a picture of the cold-worked state as consisting of a system of slightly deformed lattice blocks, separated and connected by more severely dislocated transition layers. One of our objects will be to ascertain in how far such a picture can be of help to understand some of the differences observed between these two annealing processes.

I. *Recovery versus recrystallization in their dependence on time and temperature of heating.*

I, 1. *Analogy.*

The question can be raised what, physically speaking, the difference between recovery and recrystallization amounts to, as undoubtedly both processes are due to the shifting of a certain number of atoms in the deformed matrix to positions of more stable equilibrium. In fact it is often very difficult to decide whether during a given heat-treatment recrystallization does not occur at all. The dissolving power of the experimental method applied to ascertain the presence or absence of structural changes (microscopic observation, X-ray diffraction, etc.) plays an important part here. Often what is called recovery according to one method, is beginning recrystallization according to the other.

In practice a discrimination between recovery and recrystallization is

certainly justified. Especially at lower temperatures of heating a considerable recovery of physical and mechanical properties can be brought about without "visible" structural changes. This is particularly true with deformed *single* crystals, where, according to measurements of HAASE and SCHMID (4) with bismuth and zinc, of KORNFELD (5) with aluminium and of KOREF (6) with tungsten, an approximately complete recovery of the shear- or tensile-stress could be brought about. With polycrystalline testpieces such a "pure" recovery cannot be realized so completely.

On the other hand it was brought forward in particular by VAN LIEMPT (7), that in many annealing experiments the relation between time (t) and absolute temperature (T) of heating, required to cause a definite percentage of recovery of a physical property (for example the electric resistance) or in other cases a definite state of recrystallization (for example half of the matrix recrystallized) obeys to the same formula, of type

$$T \ln c \cdot t = \frac{Q}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which Q has the meaning of an activation energy for atomic interchange and c is proportional to the atomic frequency of the metal considered.

This "law of corresponding states of recovery (or recrystallization)" may said to be based on the very simplest picture of the processes involved, in which the atomic movements from a metastable "deformed" into a stable "undeformed" position are governed by a single activation energy Q. In such a picture the recovered respectively recrystallized fraction (x) [1]) at any moment obeys a relation

$$\frac{dx}{dt} = k(1-x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

with the velocity constant k determined by

$$k = c \cdot e^{-\frac{Q}{RT}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or integrated

$$x = 1 - e^{-ct \cdot e^{-Q/RT}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

[see KRUPKOWSKI and BALICKI (8); also (1) (§§ 75, 77, 151)]. For the quantity Q values are found of the order of magnitude of those met with in diffusion experiments in the solid state, that is, values to be expected for processes, consisting of interchange of individual atoms.

Similar activation energies occur in the relations, which in "pure" recrystallization experiments are found to govern the rate of nucleation and the rate of growth of the new crystals, for example with aluminium by

[1]) It is further assumed that the relative recovery of the property considered is at every moment proportional to this fraction.

KORNFELD and PAWLOW (9) and by ANDERSON and MEHL (10), for rock-salt by MÜLLER (11) [2].

This apparently closely analogous behaviour of both recovery and recrystallization in their dependence on time and temperature of heating might be considered to support the conception, that what is called recovery is in reality recrystallization on such a small scale, that it is still "invisible", so that there would be no reason to discriminate between both phenomena in a more fundamental way.

I, 2. *Differences.*

This conclusion breaks down, however, if we consider what happens after *prolonged times of heating.* If, under such circumstances, "visible" recrystallization sets in, we observe a return of the properties to their normal values for the completely annealed state, this state being attained the sooner the higher the chosen temperature, as might be expected after [1], which is confirmed in the more precise treatment given in (10). In "pure" recovery-anneals, on the contrary, after an initial return of the properties, the rate of the process decreases so much below the magnitude

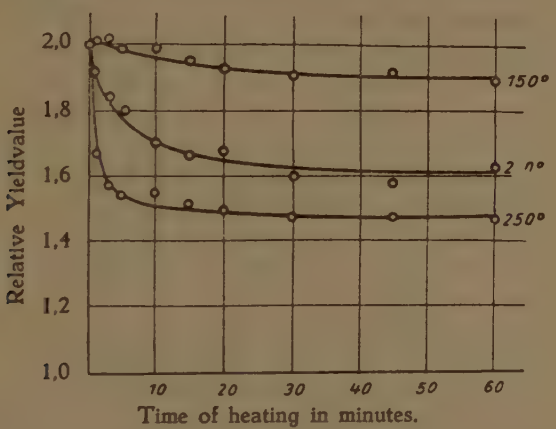

Fig. 1. Isothermal recovery curves of aluminium single crystals. The original deformation was in all cases the same (100 % increase of yield value = relative yield value 2). The ordinates give the decrease of this latter quantity as a function of time of heating for three different temperatures: *at each temperature a nearly constant "rest value" is approached,* which lies the lower the higher the temperature of annealing (after KORNFELD (2) (5)).

[2] In ANDERSON and MEHL's paper, which appeared as recently as 1945, the course of recrystallization in aluminium sheet was analysed in terms of rate of nucleation (N = number of new crystals formed per unit of time per unit of area of the unrecrystallized matrix) and rate of growth (G). It follows from their investigation that formula [1] given above is valid only under simplifying assumptions (for that case the paper gives a relation between Q and the activation energies Q_N and Q_G for nucleation and growth separately). ANDERSON and MEHL show that the whole course of isothermal recrystallization is more complicated than follows from [2] and [3], in particular in consequence of the circumstance, that N increases with time, a fact to which we shall return at the end of section III of this paper.

expected according to [4], that a value is attained which remains approximately constant for much longer durations of heating. Although the difference between this "restvalue" and the value for the completely annealed state diminishes with increasing temperature of annealing, nevertheless at each temperature, if not too low, a definite residual effect of the cold-working remains. Fig. 1, which is taken from KORNFELD (2) (5), may serve to illustrate this for annealed aluminium crystals [for other examples we refer to (1) (§ 69)].

This different behaviour of "recovering" and "recrystallizing" anneals is still more accentuated by the fact that, according to KORNFELD (2), the rate at which the "restvalues" are attained, is greater for single crystals than for polycrystals: this is contrary to what is generally found in recrystallization experiments: here for a given deformation polycrystalline material recrystallizes "easier" (for example at lower temperatures) than unicrystalline material [KARNOP and SACHS (11a); (1) (§ 150)].

A difference is also observed when we consider annealing experiments carried out with test-pieces, subjected to *different degrees of cold work*. On general grounds one would expect the activation energy Q to decrease with increasing deformation and thus, according to [4], the more severely deformed test-piece to approach the completely annealed state for a given temperature in a shorter time, or for a given time at a lower temperature. This is actually observed in pure recrystallization experiments

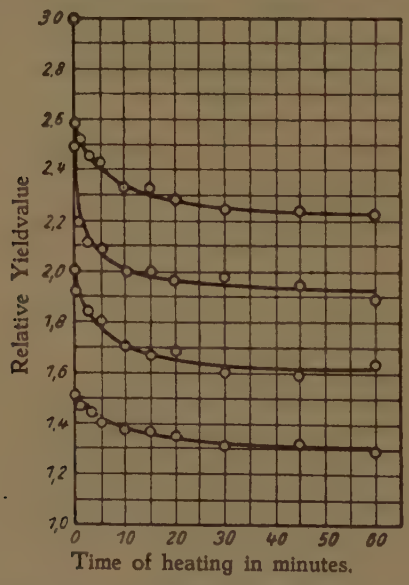

Fig. 2. As fig. 1. Here, however, all curves relate to the same temperature (200° C.) but to four different states of deformation, the original relative yield values before annealing being 3, 2.5, 2 and 1.5. *The "rest values" lie higher, the larger the original degree of cold-work* (the relative recovery [1]) is approximately the same in all four cases, namely 0.4) (after KORNFELD (5)).

[1]) Calculated as: $\frac{\text{yieldvalue before anneal} - \text{yieldvalue after anneal}}{\text{yieldvalue before anneal} - \text{original yieldvalue}}$

[see again ANDERSON and MEHL (10)] and also in such "recovery" experiments, in which it is highly probable that the recovery was actually accompanied by recrystallization: for example by SAUERWALD (12), BRINDLEY (12a). In KORNFELD's (5) recovery experiments with aluminium single crystals, however, a different result is obtained, as shown in fig. 2. Here the four "recovery isotherms" relate to four crystals, extended so far that the yield-value was raised to respectively 1.5, 2, 2.5 and 3 times its original value. The curves show the decrease of these "relative" elastic limits as a function of the time of heating at 200° C., where no "visible" recrystallization occurred. It will be seen that the quantities considered approach to residual values which lie *higher the more the crystal had been deformed* (the *relative* recovery, however, was approximately the same for all degrees of deformation). This result again is quite different from that generally found in the case of recrystallizing test-pieces [3]). MASING (12b) has already drawn attention to this curious behaviour and considered it as an indication that recovery and recrystallization are essentially different phenomena [4]).

Leaving this question as it stands, it seems in any case to follow from what has been said that "pure" recovery at constant temperature of annealing leads to states, which are intermediate between the cold-worked and the completely annealed state and which possess a considerable stability.

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[3]) A to some extent similar result was, however, obtained by BRINDLEY (12a) in annealing experiments with rolled copper with different percentage reduction, where the thermal E.M.F. between the cold-worked and the undeformed state was measured, although in these experiments the occurrence of at least partial recrystallization seems possible. According to BRINDLEY it is likely, that the effect is due to different grain sizes and (or) grain orientations in the partly annealed states as compared to those present in the original undeformed metal.

[4]) It must be remarked here that also DEHLINGER (3) has advanced the view that recovery and recrystallization are different processes. To his conceptions we shall return in section III. DEHLINGER's arguments were partly based on the assumption of an essentially different dependence of both phenomena on temperature, in the sense that the temperature coefficient of the velocity for recrystallization was much larger than that for recovery. As far back as 1929 DEHLINGER in a fundamental paper on recrystallization put forward the conception of an "instability-temperature" (a kind of CURIE-temperature), above which the rate of recrystallization should increase jumpingly. As we exposed in (1), (§ 152), we do not know experiments which support this assumption in an unambiguous way; also its theoretical basis is not clear.

Mathematics. — *On the zeros of a polynomial and of its derivative II.* By N. G. DE BRUIJN and T. A. SPRINGER. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of April 26, 1947.)

1. In a previous paper [1]) (referred to as I), the following theorem was proved for some special classes of polynomials:

Theorem 1. *Let the polynomial $f(z)$ of degree $n > 1$, have the zeros $\xi_1, \ldots, \xi_n$, and let $\eta_1, \ldots, \eta_{n-1}$ be those of $f'(z)$. Then we have*

$$\frac{1}{n-1} \sum_{\nu=1}^{n-1} |Im\, \eta_\nu| \leqslant \frac{1}{n} \sum_{\nu=1}^{n} |Im\, \xi_\nu|, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

the sign of equality holding if and only if no two zeros of $f(z)$ are separated by the real axis.

Here we shall prove the theorem in the general case, namely for polynomials with arbitrary real or complex coefficients. In our proof we introduce an auxiliary function $f^*(z)$ obtained from $f(z)$ by replacing the zeros of $f(z)$ in the lower half-plane by their complex conjugates.

Theorem 1 can be generalized in several ways. In the first place we may ask for the class C of real continuous functions $\psi(z)$ of the complex variable z, such that

$$\frac{1}{n-1} \sum_{\nu=1}^{n-1} \psi(\eta_\nu) \leqslant \frac{1}{n} \sum_{\nu=1}^{n} \psi(\xi_\nu). \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

holds for any polynomial $f(z)$. We have not been able to characterize this class C; it is, however, likely, that C consists of all convex functions $\psi(z)$ [2]). Anyhow, all functions of the class C are convex.

It is possible to derive from theorem 1, by superposition, a large subclass C^* of functions $\psi(z)$ belonging to C. Important items are $\psi(z) = |z|^p$ and $\psi(z) = |Im\, z|^p$ $(p \geqslant 1)$. This will be shown in section 3.

A second generalisation of theorem 1 is to rational functions with positive residues (section 4).

Other generalisations, concerning the zeros of "composition-polynomials", will be given in a next paper.

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[1]) N. G. DE BRUIJN, On the zeros of a polynomial and of its derivative, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **49**, 1037—1044 (1946). In that paper, our theorem 1 was proved in the following two cases:

a) if all coefficients of $f(z)$ are real, and

b) if all zeros of $f(z)$ are purely imaginary.

[2]) $\psi(z)$ is called convex, if $\psi(\lambda_1 z_1 + \lambda_2 z_2) \leqslant \lambda_1 \psi(z_1) + \lambda_2 \psi(z_2)$ for all values of $z_1, z_2, \lambda_1, \lambda_2$, satisfying $\lambda_1 \geqslant 0, \lambda_2 \geqslant 0, \lambda_1 + \lambda_2 = 1$.

2. Proof of theorem 1.

We remark, in the first place, that the theorem is trivial when all zeros of $f(z)$ lie in $Im\, z \geqq 0$. For then, by the well-known Gausz–Lucas theorem, the same holds for the zeros of $f'(z)$, so that the imaginary parts of $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_{n-1}$ all have the same sign. The theorem then follows from the relation

$$\frac{1}{n-1} \sum_{\nu=1}^{n-1} \eta_\nu = \frac{1}{n} \sum_{\nu=1}^{n} \xi_\nu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This shows that in (1) the sign of equality holds in this case.

The general case is reduced to this one by means of the following

Lemma. *Let* $f(z) = a \prod_{\nu=1}^{k} (z-\xi_\nu) \prod_{\nu=k+1}^{n} (z-\xi_\nu)$ $(0 \leq k \leq n)$, *where* $Im\, \xi_\nu \geq 0$ $(\nu = 1, 2, \ldots, k)$, $Im\, \xi_\nu < 0$ $(\nu = k+1, \ldots, n)$. *Putting*

$$f^*(z) = a \prod_{\nu=1}^{k} (z-\xi_\nu) \prod_{\nu=k+1}^{n} (z-\bar{\xi}_\nu), \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

we have

$$|f'(x)| \leqq |f^{*\prime}(x)| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

for all real values of x.

There is equality for all real x *if and only if no two zeros of* $f(z)$ *are separated by the real axis.*

Proof. Writing

$$\sum_{\nu=1}^{k} \frac{1}{x-\xi_\nu} = P + Qi, \quad \sum_{\nu=k+1}^{n} \frac{1}{x-\xi_\nu} = R + Si \quad (x, P, Q, R, S \text{ real})$$

we have

$$\frac{f'(x)}{f(x)} = (P+R) + i(Q+S), \quad \frac{f^{*\prime}(x)}{f^*(x)} = (P+R) + i(Q-S).$$

Now it follows from

$$Im\, \xi_\nu \geqq 0\ (\nu = 1, \ldots, k), \quad Im\, \xi_\nu < 0 \quad (\nu = k+1, \ldots, n)$$

that $Q \geq 0$, $S \leq 0$. We obtain $|Q+S| \leq |Q-S|$, which gives

$$\left| \frac{f'(x)}{f(x)} \right| \leqq \left| \frac{f^{*\prime}(x)}{f^*(x)} \right|.$$

For real x we have

$$|f(x)| = |f^*(x)|,$$

and hence

$$|f'(x)| \leqq |f^{*\prime}(x)|.$$

There is equality (for all real x) only if either $Q = 0$ or $S = 0$, that is to say, if all zeros of $f(z)$ lie either in $Im\, z \leq 0$ or in $Im\, z \geq 0$.

With this lemma the proof of theorem 1 is quite simple. Since all zeros of $f^*(z)$ lie in $Im\, z \geq 0$, it follows from our remark above that

$$\frac{1}{n-1}\sum_{\nu=1}^{n-1} |Im\, \eta_\nu^*| = \frac{1}{n}\sum_{\nu=1}^{n} |Im\, \xi_\nu| \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where $\eta_1^*, \ldots, \eta_{n-1}^*$ denote the zeros of $f^{*\prime}(z)$. Further, by the lemma,

$$\int_{-A}^{A} \log |f'(x)|\, dx \leqq \int_{-A}^{A} \log |f^{*\prime}(x)|\, dx \qquad (A > 0) \quad . \quad . \quad . \quad (7)$$

or

$$\sum_{\nu=1}^{n-1} \int_{-A}^{A} \log |x-\eta_\nu|\, dx \leqq \sum_{\nu=1}^{n-1} \int_{-A}^{A} \log |x-\eta_\nu^*|\, dx \quad . \quad . \quad . \quad . \quad (8)$$

It is easily seen that

$$\int_{-A}^{A} \log |x-a|\, dx = 2(A \log A - A) + \pi |Im\, a| + O\left(\frac{1}{A}\right) . \quad . \quad (9)$$

Substituting this into (8), and making $A \to \infty$ we find

$$\sum_{\nu=1}^{n-1} |Im\, \eta_\nu| \leqq \sum_{\nu=1}^{n-1} |Im\, \eta_\nu^*| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Combining this inequality with (6) we obtain (1).

There is equality in (10) if and only if there is equality in (7), that is, if $|f'(x)| = |f^{*\prime}(x)|$ for all real values of x, and then, by the lemma, all zeros of $f(z)$ lie either in $Im\, z \geq 0$ or in $Im\, z \leq 0$.

Thus theorem 1 is completely proved.

3. Theorem 1 means, geometrically, that the zeros of $f'(z)$ lie, in the mean, closer to the real axis, than the zeros of $f(z)$. The same can be said about any line, that is to say, theorem 1 remains true, when we replace $|Im\, z|$ by $|Im(\alpha z + \beta)|$, α and β being complex numbers. (This is easily proved by applying theorem 1 to $f\left(\frac{z-\beta}{\alpha}\right)$). Hence the functions $|Im(\alpha z + \beta)|$ belong to the class C, defined in section 1. Furthermore, it follows from (3) that the functions $Im\,(\alpha z + \beta)$ also belong to C. We can obtain new functions of C by superposition of these special ones. We thus obtain a sub-class C* of C, which consists of all real continuous functions $\psi(z)$ of the complex variable z, which are sums of functions of the types $|Im(\alpha z + \beta)|$, $Im\,(\alpha z + \beta)$ with positive weights. For instance, C* contains all convex functions of $Im\, z$. We have, namely

Theorem 2. *Let $\xi_1, \ldots, \xi_n$ be the zeros of $f(z)$, $\eta_1, \ldots, \eta_{n-1}$ those of*

$f'(z)$, and let $Im\,\xi_1 \leq Im\,\xi_2 \leq \ldots \leq Im\,\xi_n$. If $\psi(x)$ is a convex real function of x in the interval $Im\,\xi_1 \leq x \leq Im\,\xi_n$, and if

$$D(\psi, f) = \frac{1}{n}\sum_{\nu=1}^{n} \psi(Im\,\xi_\nu) - \frac{1}{n-1}\sum_{\nu=1}^{n-1} \psi(Im\,\eta_\nu)$$

then

$$D(\psi, f) \geqq 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$D(\psi, f) = 0$ holds only if $\psi(x)$ *is linear for* $Im\,\xi_1 \leq x \leq Im\,\xi_n$ (which implies the case $Im\,\xi_1 = \ldots = Im\,\xi_n$).

Theorem 2 can be proved in the same way as theorem 7 in I. A special case is $\psi(x) = |x|^p$ $(p \geq 1)$, giving

Theorem 3. *With the notations of theorem 1, we have, if* $p \geq 1$

$$\frac{1}{n-1}\sum_{\nu=1}^{n-1} |Im\,\eta_\nu|^p \leqq \frac{1}{n}\sum_{\nu=1}^{n} |Im\,\xi_\nu|^p \quad . \quad . \quad . \quad . \quad . \quad (12)$$

There is equality in the following two cases only: a) if $p = 1$ *and all zeros of* $f(z)$ *lie in the same half-plane* $Im\,z \geq 0$ *or* $Im\,z \leq 0$, *and b) if* $p \geq 1$ *and* $Im\,\xi_1 = \ldots = Im\,\xi_n$.

Obviously this theorem remains true when $|Im(az + \beta)|$ is substituted for $|Im\,z|$. This remark is used for the proof of

Theorem 4. *With the assumptions of theorem 1, we have, if* $p \geq 1$

$$\frac{1}{n-1}\sum_{\nu=1}^{n-1} |\eta_\nu|^p \leqq \frac{1}{n}\sum_{\nu=1}^{n} |\xi_\nu|^p \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

There is equality in the following two cases only: a) if $p = 1$ *and all zeros of* $f(z)$ *lie on the same half-line with endpoint 0, and b) if* $\xi_1 = \ldots = \xi_n$.

Proof. The distance of the point z in the complex plane to the line through the point 0 making an angle φ with the positive real axis, is

$$|\cos\varphi \cdot Im\,z - \sin\varphi \cdot Re\,z|.$$

By theorem 3 we have

$$\frac{1}{n-1}\sum_{\nu=1}^{n-1} |\cos\varphi \cdot Im\,\eta_\nu - \sin\varphi \cdot Re\,\eta_\nu|^p \leqq \frac{1}{n}\sum_{\nu=1}^{n} |\cos\varphi \cdot Im\,\xi_\nu - \sin\varphi \cdot Re\,\xi_\nu|^p.$$

By integrating this inequality we obtain

$$\frac{1}{n-1}\sum_{\nu=1}^{n-1} \int_0^{2\pi} |\cos\varphi \cdot Im\,\eta_\nu - \sin\varphi \cdot Re\,\eta_\nu|^p\, d\varphi \leqq$$

$$\leqq \frac{1}{n}\sum_{\nu=1}^{n} \int_0^{2\pi} |\cos\varphi \cdot Im\,\xi_\nu - \sin\varphi \cdot Re\,\xi_\nu|^p\, d\varphi$$

or

$$\frac{1}{n-1}\sum_{\nu=1}^{n-1} |\eta_\nu|^p \leqq \frac{1}{n}\sum_{\nu=1}^{n} |\xi_\nu|^p.$$

The cases of equality are easily deduced from those of theorem 3.

A direct consequence of theorem 4 is

Theorem 5. *With the assumptions of theorem 1 and $p>0$, we have*

$$\sum_{\nu=1}^{n-1}|\eta_\nu|^p \leqq \frac{n-\frac{1}{k}}{n}\sum_{\nu=1}^{n}|\xi_\nu|^p, \text{ if } k \text{ is an integer} > \frac{1}{p} \quad . \quad . \quad (14)$$

This may be proved by application of theorem 4 to $f(z^k)$. (Cf. I, theorem 3).

It is probable, that, more generally, the inequality

$$\sum_{\nu=1}^{n-1}|\eta_\nu|^p \leqq \frac{n-p}{n}\sum_{\nu=1}^{n}|\xi_\nu|^p \qquad (0\leqq p\leqq 1). \quad . \quad . \quad . \quad (15)$$

holds. We have, however, not been able to prove this. Anyhow, if ϱ is a fixed number ($\varrho>p$), an inequality

$$\sum_{\nu=1}^{n-1}|\eta_\nu|^p \leqq \frac{n-\varrho}{n}\sum_{\nu=1}^{n}|\xi_\nu|^p \qquad (0\leqq p\leqq 1) \quad . \quad . \quad . \quad . \quad (16)$$

cannot be true for arbitrary $f(z)$. This is easily seen by considering $f(z)=z^{n-1}(z-1)$ for large integers n.

4. Rational functions of the type

$$\varphi(z) = -az+b+\sum_{\nu=1}^{n}\frac{t_\nu}{z-a_\nu} \qquad (a\geqq 0,\ t_\nu>0,\ n\geqq 0) \quad . \quad (17)$$

have properties analogous to that expressed in theorem 1 (Cf. I, theorem 2).

In the first place, we obtain

Theorem 5. *If $n>1$,*

$$\varphi(z) = \sum_{\nu=1}^{n}\frac{t_\nu}{z-a_\nu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

and if $\beta_1, \ldots, \beta_{n-1}$ are the zeros of $\varphi(z)$, then we have [3])

$$\sum_{\nu=1}^{n-1}|Im\,\beta_\nu| \leqq \sum_{\nu=1}^{n}|Im\,a_\nu| - \frac{\sum_{\nu=1}^{n} t_\nu\,|Im\,a_\nu|}{\sum_{\nu=1}^{n} t_\nu} \quad . \quad . \quad . \quad . \quad (19)$$

Proof. This may be proved by the same method as theorem 1, but it is also possible to deduce (19) directly from theorem 1. For, by applying (1)

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[3]) The case $t_1=\ldots=t_n=1$ is embodied in theorem 1.

to the polynomial $\prod_{\nu=1}^{n} (z-a_\nu)^{k}$ (where the k_ν are natural numbers) we obtain

$$\sum_{\nu=1}^{n-1} |Im\, \beta_\nu| + \sum_{\nu=1}^{n} (k_\nu - 1)\, |Im\, a_\nu| \leqq \frac{-1 + \sum_{\nu=1}^{n} k_\nu}{\sum_{\nu=1}^{n} k_\nu} \cdot \sum_{\nu=1}^{n} k_\nu\, |Im\, a_\nu|,$$

where $\beta_1, \ldots, \beta_{n-1}$ are the zeros of $\sum_{\nu=1}^{n} \frac{k_\nu}{x-a_\nu}$. Hence (19) follows for rational t_ν, and the general case follows by an argument of continuity.

Since (1) holds for all functions $\psi(z)$ of the class C, the same argument shows

Theorem 6. *Under the assumptions of theorem 5, we have*

$$\sum_{\nu=1}^{n-1} \psi(\beta_\nu) \leqq \sum_{\nu=1}^{n} \psi(a_\nu) - \frac{\sum_{\nu=1}^{n} t_\nu \psi(a_\nu)}{\sum_{\nu=1}^{n} t_\nu}, \quad \ldots \quad (20)$$

for any function $\psi(z)$ of the class C.

To obtain a corresponding inequality for the function

$$\varphi(z) = b + \sum_{\nu=1}^{n} \frac{t_\nu}{z-a_\nu} \quad (b \neq 0,\ t_\nu > 0,\ n > 0) \quad \ldots \quad (21)$$

we apply theorem 6 to

$$\varphi_T(z) = \frac{T}{z + \frac{T}{b}} + \sum_{\nu=1}^{n} \frac{t_\nu}{z-a_\nu}.$$

From $\lim_{T \to +\infty} \varphi_T(z) = \varphi(z)$ it follows, that

$$\sum_{\nu=1}^{n} \psi(\beta_\nu) \leqq \sum_{\nu=1}^{n} \psi(a_\nu) + \left(\sum_{\nu=1}^{n} t_\nu \right) \cdot \lim_{T \to +\infty} \frac{1}{T} \psi\left(-\frac{T}{b}\right)$$

($\beta_1, \ldots, \beta_n$ denoting the zeros of $\varphi(z)$) if $\psi(z)$ belongs to class C and the limit exists. This will occur, for example, if $\psi(z)$ is homogeneous (i.e. $\psi(\lambda z) = \lambda \psi(z)$ for all $\lambda \geq 0$ and all complex z). We then have

$$\sum_{\nu=1}^{n} \psi(\beta_\nu) \leqq \sum_{\nu=1}^{n} \psi(a_\nu) + \psi\left(-\frac{1}{b}\right) \cdot \left(\sum_{\nu=1}^{n} t_\nu \right) \quad \ldots \quad (22)$$

The most important applications are $\psi(z) = |Im\, z|$ and $\psi(z) = |z|$.

For the function

$$\varphi(z) = -az + b + \sum_{\nu=1}^{n} \frac{t_\nu}{z-a_\nu} \qquad (a > 0,\ t_\nu > 0,\ n \geqq 0) \quad . \quad (23)$$

whose zeros be denoted by $\beta_1, \ldots, \beta_{n+1}$, we have $\varphi(z) = \lim_{T \to +\infty} \varphi_T(z)$, where

$$\varphi_T(z) = \frac{T^2}{z+pT-q} + \frac{T^2}{z-pT-q} + \sum_{\nu=1}^{n} \frac{t_\nu}{z-\alpha_\nu} \qquad \left(p = \sqrt{\frac{2}{a}},\ q = \frac{b}{a}\right).$$

Application of theorem 6 to $\varphi_T(z)$ yields

$$\sum_{\nu=1}^{n+1} \psi(\beta_\nu) \leqslant \sum_{\nu=1}^{n} \psi(\alpha_\nu) + \lim_{T \to +\infty} \frac{T^2 + \sum_{\nu=1}^{n} t_\nu}{2T^2 + \sum_{\nu=1}^{n} t_\nu} \cdot \{\psi(pT+q) + \psi(-pT+q)\}.$$

This proves

Theorem 7. *If $\beta_1, \ldots, \beta_{n+1}$ are the zeros of* (23) *($a > 0$, $t_\nu > 0$, $n \geq 0$), and $\psi(z)$ is a convex function of Im z* (Cf. *Theorem* 2), *then we have*

$$\sum_{\nu=1}^{n+1} \psi(\beta_\nu) \leqslant \sum_{\nu=1}^{n} \psi(\alpha_\nu) + \psi\left(\frac{b}{a}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Mathematisch Instituut der Technische Hogeschool, Delft.

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Mathematics. — *On the theory of linear integral equations.* VIII. By A. C. ZAANEN. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of April 26, 1947.)

§ 1. *Introduction.*

This paper is an immediate sequel to the paper VII, bearing the same title [1]). For a general introduction into the theory of systems of linear integral equations with "symmetrisable matrix-kernels" and for the meaning of the used symbols and notations we refer to that paper, in particular to § 1. In the present paper a knowledge of the contents of the papers I, II and VII, which were published in these Proceedings [2]) under the same title, is assumed.

We shall discuss here the system of linear integral equations

$$\Sigma_{j=1}^{n} \int_{\Delta} K_{ij}(x,y)\, f^j(y)\, dy - \lambda f^i(x) = g^i(x) \qquad (i=1,\ldots,n), \qquad (1)$$

where $f^i(x)$ and $g^i(x) \in L_2^{(m)}(\Delta)$, λ is complex, and the matrix-kernel $\|K_{ij}(x,y)\|$ is the matrix-product of the matrix-kernel $\|A_{ij}(x,y)\|$ and the matrix $\|h_{ij}(y)\|$, the following conditions being satisfied:

a. All $h_{ij}(y)$ are bounded and measurable in Δ,

b. $h_{ij}(y) = \overline{h_{ji}(y)}$ and $\Sigma_{i,j=1}^{n} h_{ij}(y)\, \bar{a}_i\, a_j \geqq 0$ for any system of complex numbers $a_1,\ldots,a_n$ and every $y \in \Delta$; the matrix $\|h_{ij}(y)\|$ is therefore Hermitian and of positive type,

c. All $A_{ij}(x,y) \in L_2^{(2m)}(\Delta)$,

d. $\|h_{ij}(x)\| \cdot \|K_{ij}(x,y)\| = \|K_{ij}^*(x,y)\| \cdot \|h_{ij}(y)\|$,

in other words,

$$\|h_{ij}(x)\| \cdot \|A_{ij}(x,y)\| \cdot \|h_{ij}(y)\| = \|h_{ij}(x)\| \cdot \|A_{ij}^*(x,y)\| \cdot \|h_{ij}(y)\|.$$

This last condition is automatically satisfied if $\|A_{ij}(x,y)\|$ is Hermitian.

As we have already remarked in VII, § 1, the system (1) reduces for $n=1$ to the integral equation which we have discussed in III [3]). It will, however, be possible to read the present paper without any knowledge of the contents of III. The case $n \geqq 1$ was considered by J. ERNEST

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[1]) A. C. ZAANEN, On the theory of linear integral equations VII, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **50**, 357—368 (1947) (Indagationes Mathematicae IX, ✻).

[2]) A. C. ZAANEN, On the theory of linear integral equations, Proc. Kon. Ned. Akad. van Wetensch., Amsterdam, **49**, I, 194—204, II, 205—212 (1946) (Indagationes Mathematicae VIII, I, 91—101, II, 102—109).

[3]) A. C. ZAANEN, On the theory of linear integral equations III, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **49**, 292—301 (1946) (Indagationes Mathematicae VIII, 161—170).

WILKINS in 1944 [4]) (he generalized earlier results of W. T. REID [5]), who, however, supposed, instead of condition *a.*, all $h_{ij}(y)$ to be continuous on Δ, and, instead of condition *c.*, all $A_{ij}(x,y)$ to be bounded in $\Delta \times \Delta$ with their discontinuities "regularly distributed". We shall free ourselves of these restrictions, and, moreover, prove a great deal more.

After some preliminary lemma's in § 2 (one of these, Lemma 1, is of interest in itself), we shall show in § 3, amongst other things, that the characteristic values of (1), if any exist, are real; that under very general conditions this system possesses at least one characteristic value $\neq 0$, and that the characteristic values have some remarkable extremizing properties. Furthermore we shall prove expansion theorems for functions of the form $\sum_{j=1}^{n} \int_{\Delta} K_{ij}(x,y) f^j(y)\, dy$, as well as for the elements of the matrix-kernel $\|K_{ij}(x,y)\|$ themselves. In § 4 we shall consider the iterated kernels of $\|K_{ij}(x,y)\|$, and in § 5 we shall show that, by strengthening the conditions *a.* and *c.*, the results of the expansion theorems can be improved (replacement of convergence in mean by uniform convergence). Finally, in § 6, we shall compare our results with those obtained by WILKINS.

§ 2. *Some preliminary lemma's.*

Lemma 1. 1°. *If* $\|h_{ij}\|$ $(i,j=1,\ldots,n)$ *is an Hermitian matrix of positive type (that is,* $h_{ij}=\overline{h_{ji}}$ *and* $\sum_{i,j=1}^{n} h_{ij}\,\bar{a}_i\,a_j \geqq 0$ *for any sequence* $a_1,\ldots,a_n$ *of complex numbers), there exists a uniquely determined matrix* $\|h_{ij}^{(1/2)}\|$, *which is also Hermitian and of positive type, such that*

$$\|h_{ij}^{(1/2)}\| \cdot \|h_{ij}^{(1/2)}\| = \|h_{ij}\|.$$

2°. *If now the numbers* h_{ij} *depend upon the parameter* x, *running through the interval* Δ, *and all* $h_{ij}(x)$ *are bounded, measurable functions of* x *for* $x \in \Delta$, *the functions* $h_{ij}^{(1/2)}(x)$ *are also bounded and measurable in* Δ.

3°. *Finally, if the functions* $h_{ij}(x)$ *are continuous in* Δ, *the same holds for the functions* $h_{ij}^{(1/2)}(x)$.

Proof. 1°. Considering an n-dimensional unitary space with orthonormal base $\varphi_1,\ldots,\varphi_n$, the linear transformation A in this space, determined by $A\varphi_i = \sum_{j=1}^{n} h_{ij}\varphi_j$, is self-adjoint and of positive type. As well-known, there exist an orthonormal system $\psi_1,\ldots,\psi_n$ and non-negative numbers $\lambda_1,\ldots,\lambda_n$ such that $A\psi_i = \lambda_i \psi_i$ $(i=1,\ldots,n)$. The numbers λ_i are therefore the characteristic values of A and the elements

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[4]) J. ERNEST WILKINS, Definitely self-conjugate adjoint integral equations, Duke Mathematical Journal **11**, 155—166 (1944).

[5]) W. T. REID, Expansion problems associated with a system of linear integral equations, Transactions of the American Math. Soc. **33**, 475—485 (1931).

ψ_i are corresponding characteristic elements. Defining now the transformation $A^{1/2}$ by $A^{1/2}\psi_i = \lambda_i^{1/2}\psi_i$, this transformation is clearly self-adjoint and of positive type, and satisfies $(A^{1/2})^2 = A$. Moreover, $A^{1/2}$ is the only self-adjoint transformation of positive type, having this property. Writing $(A^{1/2}\varphi_i, \varphi_j) = h_{ij}^{(1/2)}$ $(i, j = 1, \ldots, n)$, the matrix $\| h_{ij}^{(1/2)} \|$ is therefore uniquely determined, Hermitian and of positive type, while moreover

$$h_{ij} = (A\varphi_i, \varphi_j) = (A^{1/2}\varphi_i, A^{1/2}\varphi_j) =$$

$$\Sigma_{k=1}^n (A^{1/2}\varphi_i, \varphi_k)\,\overline{(A^{1/2}\varphi_j, \varphi_k)} = \Sigma_{k=1}^n h_{ik}^{(1/2)}\, h_{kj}^{(1/2)},$$

so that

$$\| h_{ij}^{(1/2)} \| \cdot \| h_{ij}^{(1/2)} \| = \| h_{ij} \|.$$

2°. If $\psi_1, \ldots, \psi_n$ are the characteristic elements of A, introduced in 1°., the elements $\varphi_1, \ldots, \varphi_n$ can be represented by $\varphi_i = \Sigma_{j=1}^n b_{ij}\psi_j$, where $\Sigma_{j=1}^n |b_{ij}|^2 = 1$ $(i = 1, \ldots, n)$. Then it follows easily that for every integer $l \geqq 0$ we have

$$(A^l\varphi_i, \varphi_j) = \Sigma_{k=1}^n (A^l\varphi_i, \psi_k)\,\overline{(\varphi_j, \psi_k)} = \Sigma_{k=1}^n \lambda_k^l\, b_{ik}\,\overline{b_{jk}};$$

hence, if $P(\lambda)$ is an arbitrary polynomium,

$$(P(A)\varphi_i, \varphi_j) = \Sigma_{k=1}^n P(\lambda_k)\, b_{ik}\,\overline{b_{jk}}.$$

Furthermore, in the same way,

$$h_{ij}^{(1/2)} = (A^{1/2}\varphi_i, \varphi_j) = \Sigma_{k=1}^n \lambda_k^{1/2}\, b_{ik}\,\overline{b_{jk}}.$$

Let now all $h_{ij} = (A\varphi_i, \varphi_j)$ be bounded and measurable functions of x on Δ. Then the transformation A, depending on x, is also bounded, uniformly in x. The same holds therefore for the characteristic values of A, so that there exists an absolute constant $M > 0$, having the property that, for all $x \in \Delta$, all characteristic values of A are contained in $0 \leqq \lambda \leqq M$. Let now $\varepsilon > 0$ be given. Then there exists a polynomium $P(\lambda)$ such that $|P(\lambda) - \lambda^{1/2}| < \varepsilon$ for $0 \leqq \lambda \leqq M$. Writing $p_{ij} = (P(A)\varphi_i, \varphi_j)$, we have therefore

$$\left.\begin{aligned} |p_{ij} - h_{ij}^{(1/2)}| = |\Sigma_{k=1}^n \{P(\lambda_k) - \lambda_k^{1/2}\}\, b_{ik}\,\overline{b_{jk}}| \leqq \\ \varepsilon\, \Sigma_{k=1}^n |b_{ik}\,\overline{b_{jk}}| \leqq \varepsilon\, (\Sigma_{k=1}^n |b_{ik}|^2)^{1/2} (\Sigma_{k=1}^n |b_{jk}|^2)^{1/2} = \varepsilon, \end{aligned}\right\} \quad . \quad . \quad (2)$$

uniformly for $x \in \Delta$. This implies that, since the functions $p_{ij}(x)$ are evidently measurable on Δ, the same holds for the functions $h_{ij}^{(1/2)}(x)$. The boundedness of $h_{ij}^{(1/2)}(x)$ follows immediately from

$$|h_{ij}^{(1/2)}(x)| = |\Sigma_{k=1}^n \lambda_k^{1/2}\, b_{ik}\,\overline{b_{jk}}| \leqq M^{1/2}\, \Sigma_{k=1}^n |b_{ik}\,\overline{b_{jk}}| \leqq M^{1/2}.$$

3°. If the functions $h_{ij}(x)$ are continuous in Δ, the same holds for the functions $p_{ij}(x)$; hence, by (2), also for the functions $h_{ij}^{(1/2)}(x)$.

Before stating our Lemma 2, we recall that in I, § 5, we have considered symmetrisable transformations K in a HILBERT space R, having the form $K=AH$, where A is self-adjoint, H is self-adjoint and of positive type, and at least one of the transformations A and H is completely continuous. We have proved in that place an expansion theorem for elements of the form Kf, another one for every solution f of $Kf-\lambda f=g$, and finally that the self-adjoint transformation $\widetilde{K}=H^{1/2}AH^{1/2}$ has the same sequence of characteristic values $\neq 0$ as $K=AH$. Here we shall show that these results are also valid in a slightly more general case.

Lemma 2. *All results, proved in* I, § 5, *remain valid when* A *is not necessarily self-adjoint, but satisfies* $HAH=HA^*H$.

Proof. $(HKf,g)=(HAHf,g)=(f,HA^*Hg)=(f,HAHg)=(f,HKg)$; the transformation HK is therefore self-adjoint, in other words, K is symmetrisable (relative to H). The proofs of the expansion theorems in I, § 5, remain unaltered; furthermore it is easily seen that $H^{1/2}AH^{1/2}=H^{1/2}A^*H^{1/2}$, so that $\widetilde{K}=H^{1/2}AH^{1/2}$ is self-adjoint. As a consequence, the proof that $\widetilde{K}$ has the same sequence of characteristic values $\neq 0$ as K undergoes no change.

§ 3. *Properties of the system of linear integral equations* (1).

We shall discuss now the system of linear integral equations (1) with matrix-kernel $\|K_{ij}(x,y)\|$ satisfying the conditions *a.*, *b.*, *c.* and *d.*, mentioned in § 1. By VII, Theorem 4, the linear transformation $\{g\}=H\{f\}$ in the HILBERT space $[L_2^{(m)}(\triangle)]^n$, defined by

$$g^i(x)=\Sigma_{j=1}^n h_{ij}(x)\,f^j(x) \qquad (i=1,\ldots,n),$$

is bounded, self-adjoint and of positive type, while by VII, Theorem 5, the linear transformation $\{g\}=A\{f\}$ in this space, defined by

$$g^i(x)=\Sigma_{j=1}^n \int_{\triangle} A_{ij}(x,y)\,f^j(y)\,dy \qquad (i=1,\ldots,n),$$

is completely continuous. Moreover, by condition *d.*, we have

$$HAH=HA^*H.$$

The transformation $K=AH$, given by $\{g\}=K\{f\}$, where

$$g^i(x)=\Sigma_{j=1}^n \int_{\triangle} K_{ij}(x,y)\,f^j(y)\,dy \qquad (i=1,\ldots,n),$$

is therefore completely continuous, and, by Lemma 2, symmetrisable with respect to the transformation H. Supposing that at least one of the functions $h_{ij}(x)\neq 0$ on a set of positive measure, so that $H\neq O$ (cf. VII, Theorem 4), all theorems in I, and in particular (by Lemma 2)

those in I, § 5, may therefore be applied to the equation $K\{f\}-\lambda\{f\}=\{g\}$ in $[L_2]^n$ or, which is equivalent, to the system of equations (1) in L_2. Before stating results, we mention a theorem about H and its "positive square root" $H^{1/2}$.

Theorem 1. *The uniquely determined, bounded, self-adjoint transformation* $\{g\}=H^{1/2}\{f\}$ *of positive type in* $[L_2]^n$ *is determined by*

$$g^i(x)=\Sigma_{j=1}^n h_{ij}^{(1/2)}(x)\,f^j(x) \qquad (i=1,\ldots,n),$$

where the functions $h_{ij}^{(1/2)}(x)$ *are those determined in Lemma* 1.

Proof. Follows immediately from Lemma 1 and the fact that $H^{1/2}$ is uniquely determined.

For the system (1) the following theorems hold:

Theorem 2. *The characteristic values of* (1) *are real, and characteristic functionsets, belonging to different characteristic values, are H-orthogonal, that is, if* $\psi_k^i(x)$ *belongs to the characteristic value* λ_k, $\psi_l^i(x)$ *to* λ_l, *and* $\lambda_k\neq\lambda_l$, *then*

$$\Sigma_{i,j=1}^n \int_\Delta h_{ij}(x)\,\overline{\psi_l^i(x)}\,\psi_k^j(x)\,dx=0.$$

Proof. Follows from I, Theorem 5.

Theorem 3. *If* $\lambda\neq 0$ *is a characteristic value of* (1), *this system of equations has, for a given* $\{g\}\in[L_2]^n$, *a solution* $\{f\}\in[L_2]^n$ *for those and only those* $\{g\}\in[L_2]^n$ *that are H-orthogonal to all characteristic functionsets belonging to the characteristic value* λ. *If* $\lambda\neq 0$ *is no characteristic value of* (1), *the system* (1) *has a uniquely determined solution* $\{f\}\in[L_2]^n$ *for every* $\{g\}\in[L_2]^n$.

Proof. Follows from I, Theorem 12 and I, Theorem 3.

Theorem 4. *If*

$$\Sigma_{l=1}^n \int_\Delta K_{ll}^{(2)}(x,x)\,dx\neq 0,$$

where

$$K_{lj}^{(2)}(x,y)=\Sigma_{l=1}^n \int_\Delta K_{il}(x,z)\,K_{lj}(z,y)\,dz,$$

the system (1) *has a characteristic value* $\neq 0$.

Proof. Since, by I, Theorem 18, K has the same characteristic values $\neq 0$ as the self-adjoint transformation $\widetilde{K}=H^{1/2}AH^{1/2}$ with Hermitian matrix-kernel $\|\widetilde{K}_{ij}(x,y)\|$, where $\widetilde{K}_{ij}(x,y)=\Sigma_{k,l=1}^n h_{ik}^{(1/2)}(x)\,A_{kl}(x,y)\,h_{lj}^{(1/2)}(y)$, our system (1) has a characteristic value $\neq 0$ if only

$$\Sigma_{i,j=1}^n \int_{\Delta\times\Delta} |\widetilde{K}_{ij}(x,y)|^2\,dx\,dy\neq 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

(cf. VII, Theorem 9). An easy computation gives

$$\Sigma_{i,j=1}^{n} |\widetilde{K}_{ij}(x,y)|^2 = \Sigma_{i,j=1}^{n} K_{ij}(x,y)\, K_{ji}(y,x);$$

the inequality (3) is therefore equivalent with

$$\Sigma_{i=1}^{n} \int_{\Delta} K_{ii}^{(2)}(x,x)\, dx \neq 0.$$

Let now λ_k ($|\lambda_1| \geqq |\lambda_2| \geqq \ldots$) be the sequence of all characteristic values $\neq 0$, each of them occurring as many times as denoted by its multiplicity, and $\psi_k^i(x)$ a corresponding H-orthonormal sequence of characteristic functionsets. These functionsets satisfy therefore the relations

$$\Sigma_{i,j=1}^{n} \int_{\Delta} h_{ij}(x)\, \overline{\psi_k^i(x)}\, \psi_k^j(x)\, dx = 1,$$

$$\Sigma_{i,j=1}^{n} \int_{\Delta} h_{ij}(x)\, \overline{\psi_k^i(x)}\, \psi_l^j(x)\, dx = 0 \text{ for } k \neq l,$$

or, writing $\Sigma_{j=1}^{n} h_{ij}(x)\, \psi_k^j(x) = \chi_k^i(x)$ (so that $H\{\psi_k\} = \{\chi_k\}$),

$$\Sigma_{i=1}^{n} \int_{\Delta} \psi_k^i(x)\, \overline{\chi_l^i(x)}\, dx = \begin{cases} 1 \text{ for } k = l, \\ 0 \text{ for } k \neq l. \end{cases} \quad \ldots \quad (4)$$

Theorem 5 (*Expansion Theorem*). *If*

$$a_k = (\{f\}, \{\chi_k\}) = \Sigma_{i=1}^{n} \int_{\Delta} f^i(x)\, \overline{\chi_k^i(x)}\, dx,$$

then

$$\Sigma_{j=1}^{n} \int_{\Delta} K_{ij}(x,y)\, f^j(y)\, dy \backsim \Sigma_k \lambda_k a_k \psi_k^i(x) + p^i(x) \qquad (i = 1, \ldots, n) \quad (5)$$

for any $\{f\} \in [L_2]^n$. *where* $\{p\} = \{p^1(x), \ldots, p^n(x)\}$ *satisfies* $H\{p\} = \{0\}$, *that is,*

$$\Sigma_{j=1}^{n} h_{ij}(x)\, p^j(x) = 0 \qquad (i = 1, \ldots, n).$$

Furthermore, if $P = HK$ *has the matrix-kernel* $\|P_{ij}(x,y)\|$,

$$\Sigma_{i,j=1}^{n} \int_{\Delta \times \Delta} P_{ij}(x,y)\, \overline{f^i(x)}\, f^j(y)\, dx\, dy = \Sigma_k \lambda_k |a_k|^2. \quad \ldots \quad (6)$$

Proof. Follows from I, Theorem 15 and I, Theorem 9.

Corollary. *If* $a_k = (\{f\}, \{\chi_k\}) = 0$ *for* $k = 1, 2, \ldots$, *then* $HK\{f\} = \{0\}$ *and* $(H\{f\}, K\{g\}) = 0$ *for every* $\{g\} \in [L_2]^n$.

Proof. Since $a_k = 0$ for $k = 1, 2, \ldots$, we have $K\{f\} = \{p\}$ with $H\{p\} = \{0\}$, hence $HK\{f\} = \{0\}$. This implies

$$(H\{f\}, K\{g\}) = (\{f\}, HK\{g\}) = (HK\{f\}, \{g\}) = 0$$

for every $\{g\} \in [L_2]^n$.

Theorem 6. *Let λ_{k_l} $(l=1,2,\ldots)$ be the subsequence of all positive characteristic values, where $\lambda_{k_1} \geqq \lambda_{k_2} \geqq \ldots$, and let $P=HK$ have the matrix-kernel $\|P_{ij}(x,y)\|$. Then*

$$\lambda_{k_l} = \max \Sigma_{i,j=1}^{n} \int_{\Delta\times\Delta} P_{ij}(x,y)\,\overline{f^i(x)}\,f^j(y)\,dx\,dy \Big/ \Sigma_{i,j=1}^{n} \int_{\Delta} h_{ij}(x)\,\overline{f^i(x)}\,f^j(x)\,dx$$

for all $\{f\} \in [L_2]^n$ satisfying the conditions $\Sigma_{i,j=1}^{n} \int_{\Delta} h_{ij}(x)\,\overline{f^i(x)}\,f^j(x)\,dx \neq 0$ and $(\{f\},\{\chi_{k_1}\}) = \ldots = (\{f\},\{\chi_{k_{l-1}}\}) = 0$. For $\{f\} = \{\chi_{k_l}\}$ the maximum is attained.

A similar statement holds for the subsequence of all negative characteristic values.

Proof. Follows from I, Theorem 10.

Theorem 7. *Let the functionsets $p_1^i(x),\ldots,p_{l-1}^i(x)$ $(i=1,\ldots,n;$ all $p_r^i(x) \in L_2^{(m)}(\Delta))$ be arbitrarily given, and let*

$$\mu_l = \text{upper bound } \Sigma_{i,j=1}^{n} \int_{\Delta\times\Delta} P_{ij}(x,y)\,\overline{f^i(x)}\,f^j(y)\,dx\,dy \Big/ \Sigma_{i,j=1}^{n} \int_{\Delta} h_{ij}(x)\,\overline{f^i(x)}\,f^j(x)\,dx$$

for all $\{f\} \in [L_2]^n$ satisfying $\Sigma_{i,j=1}^{n} \int_{\Delta} h_{ij}(x)\,\overline{f^i(x)}\,f^j(x)\,dx \neq 0$ and $(H\{f\},\{p_1\}) = \ldots = (H\{f\},\{p_{l-1}\}) = 0$. The number μ_l depends on $\{p_1\},\ldots,\{p_{l-1}\}$. Letting now these elements run through the whole space $[L_2]^n$, we have

$$\lambda_{k_l} = \min \mu_l.$$

A similar statement holds for the negative characteristic values.

Proof. Follows from I, Theorem 11.

Theorem 8. *Let $\lambda \neq 0$, and let $\{g\} \in [L_2]^n$ be H-orthogonal to all characteristic functionsets of (1), belonging to the characteristic value λ (If λ is no characteristic value, $\{g\}$ is therefore arbitrary in $[L_2]^n$). Then every solution $\{f\}$ of (1) satisfies a relation of the form*

$$f^i(x) \backsim -\frac{g^i(x)}{\lambda} + \Sigma_k' \frac{\lambda_k}{\lambda(\lambda_k-\lambda)}\,a_k\,\psi_k^i(x) + q^i(x) \qquad (i=1,\ldots,n),$$

where $a_k = \Sigma_{i=1}^{n} \int_{\Delta} g^i(x)\,\overline{\chi_k^i(x)}\,dx$ for $\lambda_k \neq \lambda$, $H\{q\} = \{0\}$, and where Σ_k' denotes that for those values of k for which $\lambda_k = \lambda$ the coefficient of

$\psi_k^i(x)$ *has the value* $\Sigma_{l=1}^n \int_\Delta f^l(x)\,\overline{\chi_k^l(x)}\,dx$. *For every set of arbitrarily prescribed values of the latter coefficients there exists a solution of* (1).

Proof. Follows from I, Theorem 17.

Theorem 9. *We have*

$$\Sigma_{l=1}^n \int_\Delta K_{ll}^{(2)}(x,x)\,dx = \Sigma_k \lambda_k^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Proof. In the proof of Theorem 4 we have already seen that

$$\Sigma_{l=1}^n \int_\Delta K_{ll}^{(2)}(x,x)\,dx = \Sigma_{i,j=1}^n \int_{\Delta\times\Delta} |\widetilde{K}_{ij}(x,y)|^2\,dx\,dy,$$

so that, since by VII, Theorem 13, the expression on the right is equal to $\Sigma_k \lambda_k^2$, formula (7) follows.

Theorem 10 (*Expansion Theorem for the elements of the matrix-kernel*). *We have*

$$K_{ij}(x,y) - p_{ij}(x,y) \backsim \Sigma_k \lambda_k \psi_k^i(x)\,\overline{\chi_k^j(y)} \qquad (i,j=1,\ldots,n), \quad . \quad (8)$$

where $p_{ij}(x,y) \in L_2^{(2m)}(\Delta)$ $(i,j=1,\ldots,n)$ *satisfies*

$$\Sigma_{q=1}^n h_{iq}(x)\,p_{qj}(x,y) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Proof. We observe first that, by I, Theorem 18, $\widetilde{K} = H^{1/2} A H^{1/2}$ has the same sequence λ_k of characteristic values $\neq 0$ as $K = AH$, and that $\{\Psi_k\} = H^{1/2}\{\psi_k\}$ is a corresponding orthonormal sequence of characteristic elements. Hence, by VII, Theorem 13 (11),

$$\widetilde{K}_{iq}(x,y) \backsim \Sigma_k \lambda_k \Psi_k^i(x)\,\overline{\Psi_k^q(y)} \qquad (i,q=1,\ldots,n)$$

with $\Psi_k^i(x) = \Sigma_{j=1}^n h_{ij}^{(1/2)}(x)\,\psi_k^j(x)$, so that

$$\Sigma_{r=1}^n h_{ir}^{(1/2)}(x)\,K_{rj}(x,y) = \Sigma_{r,s=1}^n h_{ir}^{(1/2)}(x)\,A_{rs}(x,y)\,h_{sj}(y) =$$

$$\Sigma_{q=1}^n \widetilde{K}_{iq}(x,y)\,h_{qj}^{(1/2)}(y) \backsim \Sigma_k \lambda_k \Psi_k^i(x)\,\overline{(\Sigma_{q=1}^n \Psi_k^q(y)\,h_{jq}^{(1/2)}(y))}$$

or

$$\Sigma_{r=1}^n h_{ir}^{(1/2)}(x)\,K_{rj}(x,y) \backsim \Sigma_k \lambda_k \Psi_k^i(x)\,\overline{\chi_k^j(y)}. \quad . \quad . \quad . \quad (10)$$

Let us consider now the matrix-kernel $\|D_{ij}(x,y)\|$, belonging to the transformation $D = AH^{1/2}$. Then, for every $i\,(i=1,\ldots,n)$ and for almost every $x \in \Delta$, $\{d_i\} = \{d_i^1(y),\ldots,d_i^n(y)\}$, where $d_i^j(y) = D_{ij}(x,y)$, belongs to $[L_2]^n$. Hence, by BESSEL's inequality (the system of functionsets $\overline{\Psi_k^i(x)}$ is orthonormal),

$$\Sigma_k |(\{d_i\}, \{\overline{\Psi}_k\})|^2 \leqq \|\{d_i\}\|^2$$

or, since

$$(\{d_i\}, \{\overline{\Psi}_k\}) = \Sigma_{j=1}^n \int_\Delta d_i^j(y)\, \Psi_k^j(y)\, dy = \Sigma_{j=1}^n \int_\Delta D_{ij}(x,y)\, \Psi_k^j(y)\, dy =$$

$$\Sigma_{j=1}^n \int_\Delta \Sigma_{q,r=1}^n A_{iq}(x,y)\, h_{qj}^{(1/2)}(y)\, h_{jr}^{(1/2)}(y)\, \psi_k^r(y)\, dy =$$

$$\Sigma_{r=1}^n \int_\Delta K_{ir}(x,y)\, \psi_k^r(y)\, dy = \lambda_k\, \psi_k^i(x),$$

$$\Sigma_k\, \lambda_k^2\, |\psi_k^i(x)|^2 \leqq \Sigma_{j=1}^n \int_\Delta |D_{ij}(x,y)|^2\, dy \quad . \quad . \quad . \quad . \quad (11)$$

for almost every $x \in \Delta$.

Then we observe that in the HILBERT space $[L_2^{(2m)}(\Delta)]^{n^2}$ the elements

$$\{f_k\} \equiv \{f_k^{ij}(x,y) = \lambda_k\, \psi_k^i(x)\, \overline{\Psi_k^j(y)}\} \qquad (i,j = 1, \ldots, n)$$

are orthogonal on account of the orthogonality of the system $\{\overline{\Psi}_k\}$ in $[L_2^{(m)}(\Delta)]^n$. Furthermore, by (11),

$$\Sigma_k \|\{f_k\}\|^2 = \Sigma_k\, \lambda_k^2\, (\Sigma_{i,j=1}^n \int_{\Delta\times\Delta} |\psi_k^i(x)|^2 \cdot |\Psi_k^j(y)|^2\, dx\, dy) =$$

$$\Sigma_k\, \lambda_k^2\, (\Sigma_{i=1}^n \int_\Delta |\psi_k^i(x)|^2\, dx) \leqq \Sigma_{i,j=1}^n \int_{\Delta\times\Delta} |D_{ij}(x,y)|^2\, dx\, dy < \infty,$$

which shows, since

$$\|\Sigma_{k=p}^q \{f_k\}\|^2 = \Sigma_{k=p}^q \|\{f_k\}\|^2$$

by the orthogonality, that $\Sigma_k \{f_k\}$ converges in $[L_2^{(2m)}(\Delta)]^{n^2}$. This implies that, for $i,j = 1, \ldots, n$, the series $\Sigma_k\, \lambda_k\, \psi_k^i(x)\, \overline{\Psi_k^j(y)}$ converges in mean. Denoting the sumfunction by $f_{ij}(x,y)$, we have therefore

$$f_{ij}(x,y) \backsim \Sigma_k\, \lambda_k\, \psi_k^i(x)\, \overline{\Psi_k^j(y)},$$

so that, writing

$$\Sigma_{q=1}^n f_{iq}(x,y)\, h_{qj}^{(1/2)}(y) = K_{ij}(x,y) - p_{ij}(x,y),$$

we have also

$$K_{ij}(x,y) - p_{ij}(x,y) \backsim \Sigma_k\, \lambda_k\, \psi_k^i(x)\, \overline{\chi_k^j(y)} \qquad (i,j = 1, \ldots, n). \qquad (12)$$

The only thing that remains to be proved is $\Sigma_{q=1}^n h_{iq}(x)\, p_{qj}(x,y) = 0$. From (12) we deduce

$$\Sigma_{r=1}^n h_{ir}^{(1/2)}(x)\, K_{rj}(x,y) - \Sigma_{r=1}^n h_{ir}^{(1/2)}(x)\, p_{rj}(x,y) \backsim \Sigma_k\, \lambda_k\, \Psi_k^i(x)\, \overline{\chi_k^j(y)},$$

hence, comparing this with (10),

$$\Sigma_{r=1}^n h_{ir}^{(1/2)}(x)\, p_{rj}(x,y) = 0 \text{ or } \Sigma_{r=1}^n h_{ir}(x)\, p_{rj}(x,y) = 0.$$

This completes the proof.

Mathematics. — *Ueber Definitionen von Perron-Integralen.* II. By J. RIDDER. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of March 29, 1947.)

KAPITEL III.

§ 6. **Definition IV.** Die in $[a, b]$ definierte Funktion $f(x)$ ist P_3-*integrierbar über* $[a, b]$, wenn es zu willkürlich positivem ε ein Paar $[\psi_r(x), \varphi_r(x)]$ von rechtsseitigen Ober- und Unterfunktionen gibt, beide in $[a, b]$ zu $f(x)$ adjungiert, deren Differenz in jedem Punkte von $[a, b]$, absolut genommen, kleiner als ε ist.

Die Differenz $\psi_r(x) - \varphi_r(x)$ ist monoton nicht-abnehmend in $[a, b]$. Ist $f(x)$ P_3-integrierbar über $[a, b]$, so gibt es eine Funktion $F_3(x)$, welche gleich der unteren Schranke aller $\psi_r(x)$ und der oberen Schranke aller $\varphi_r(x)$ ist. Wir definieren nun:

$$(P_3)\int_a^b f(x)\,dx \doteq F_3(b).$$

Definition. Eine in $[a, b]$ definierte, stetige Funktion $\Psi(x)$ ist *unterhalb totalstetig* auf der perfekten Teilmenge P von* $[a, b]$, wenn: 1° $\Psi(x)$ unterhalb totalstetig ist auf P, 2° die Oszillationen von $\Psi(x)$ in den zu P komplementären Teilintervallen von $[a, b]$ eine (absolut) konvergente Reihe bilden.

Definition. Eine in $[a, b]$ definierte, stetige Funktion $\Phi(x)$ ist *oberhalb totalstetig* auf der perfekten Teilmenge P von* $[a, b[$, wenn: 1° $\Phi(x)$ oberhalb totalstetig ist auf P, 2° die Oszillationen von $\Phi(x)$ in den zu P komplementären Teilintervallen von $[a, b]$ eine (absolut) konvergente Reihe bilden.

Definition. Eine in $[a, b]$ definierte, stetige Funktion $\Psi(x)$ ist *in* $[a, b]$ *unterhalb totalstetig* im verallgemeinerten Sinne* [abgekürzt: *UTV**], wenn $[a, b]$ mit Ausnahme von höchstens abzählbar unendlich vielen Punkten sich überdecken lässt durch abzählbar viele perfekte Mengen $\{P_k\}$, derartig, dass $\Psi(x)$ unterhalb totalstetig* ist auf jeder Menge P_k.

Die Definition der Funktionen, welche *in* $[a, b]$ *oberhalb totalstetig* im verallgemeinerten Sinne* [abgekürzt: OTV*] sind, liegt hiernach auf der Hand.

Satz 7. *Ist zu der in* $[a, b]$ *definierten Funktion* $f(x)$ *daselbst sowohl eine Oberfunktion* $\psi_r(x)$ *wie eine Unterfunktion* $\varphi_r(x)$ *adjungiert, so ist* $\psi_r(x)$ *UTV*,* $\varphi_r(x)$ *OTV* in* $[a, b]$.

Beweis. In den Punkten von $[a, b]$, abzählbar viele ausgenommen, ist $D_+\psi_r \neq -\infty$ und $D^+\varphi_r \neq +\infty$; es sei E die von diesen Punkten gebildete Menge. Dann ist E Summe von abzählbar vielen Mengen $\{E^i{}_n\}$, wobei i eine ganze, n eine natürliche Zahl ist, welche in abgeschlossenen Intervallen $[\frac{i}{n}, \frac{i+1}{n}]$ enthalten sind, und die Eigenschaft haben, dass für jeden Punkt x eines E^i_n.

$$\left.\begin{aligned} &\text{aus } 0 < x' - x \leqq \frac{1}{n} \text{ folgt } \psi_r(x') - \psi_r(x) \geqq -n \cdot (x' - x) \\ &\qquad\qquad \text{und } \varphi_r(x') - \varphi_r(x) \leqq +n \cdot (x' - x). \end{aligned}\right\} \quad . \quad . \quad . \quad (5)$$

Es sei H^i_n die abgeschlossene Hülle von E^i_n. Dann folgt aus (5) und der Stetigkeit von ψ_r und von φ_r, dass für jedes Paar (x, x') von Punkten von H^i_n mit $x < x'$

$$\psi_r(x') - \psi_r(x) \geqq -n \cdot (x' - x), \text{ und } \varphi_r(x') - \varphi_r(x) \leqq +n \cdot (x' - x)$$

ist. ψ_r ist somit unterhalb- und φ_r oberhalb totalstetig auf dem perfekten Kerne K^i_n eines H^i_n, falls dieser Kern nicht leer, also H^i_n nicht abzählbar ist.

$\{(a^{i;p}_n, b^{i;p}_n)\}$ seien die zu K^i_n komplementären Intervalle des kleinsten, K^i_n enthaltenden, abgeschlossenen Intervalls $(p = 1, 2, \ldots)$. Die Oszillationen von $\psi_r(x)$ in $(a^{i;p}_n, b^{i;p}_n)$ sei $\omega^{i;p}_n = \psi_r(\bar{\xi}^{i;p}_n) - \psi_r(\xi^{i;p}_n)$, mit $a^{i;p}_n \leqq \bar{\xi}^{i;p}_n \leqq b^{i;p}_n$ und $a^{i;p}_n \leqq \xi^{i;p}_n \leqq b^{i;p}_n$. Dann ist wegen (5)

$$\psi_r(\xi^{i;p}_n) > \psi_r(a^{i;p}_n) - n \cdot (\xi^{i;p}_n - a^{i;p}_n) \quad . \quad . \quad . \quad . \quad (6)$$

und

$$\varphi_r(\bar{\xi}^{i;p}_n) < \varphi_r(a^{i;p}_n) + n \cdot (\bar{\xi}^{i;p}_n - a^{i;p}_n),$$

also auch

$$\psi_r(\bar{\xi}^{i;p}_n) < \psi_r(a^{i;p}_n) + n \cdot (\bar{\xi}^{i;p}_n - a^{i;p}_n) + \{\psi_r(\bar{\xi}^{i;p}_n) - \varphi_r(\bar{\xi}^{i;p}_n)\} - \\ - \{\psi_r(a^{i;p}_n) - \varphi_r(a^{i;p}_n)\},$$

oder, wenn wir die nicht-negative und nicht-abnehmende Funktion $\psi_r - c_r \equiv \sigma_r$ setzen,

$$\psi_r(\bar{\xi}^{i;p}_n) < \psi_r(a^{i;p}_n) + n \cdot (\bar{\xi}^{i;p}_n - a^{i;p}_n) + \{\sigma_r(\bar{\xi}^{i;p}_n) - \sigma_r(a^{i;p}_n)\}. \quad . \quad (7)$$

Aus (6) und (7) folgt:

$$\omega^{i;p}_n < n \cdot (\bar{\xi}^{i;p}_n + \xi^{i;p}_n - 2a^{i;p}_n) + \{\sigma_r(\bar{\xi}^{i;p}_n) - \sigma_r(a^{i;p}_n)\},$$

oder, umsomehr,

$$\omega^{i;p}_n < 2n\,(b^{i;p}_n - a^{i;p}_n) + \{\sigma_r(b^{i;p}_n) - \sigma_r(a^{i;p}_n)\}.$$

Aber daraus leitet man sofort ab, dass die Oszillationen von ψ_r in den Intervallen $\{(a_n^{i;p}, b_n^{i;p})\}$ eine konvergente Reihe liefern.

ψ_r ist unterhalb totalstetig* auf jedem K_n^i, somit UTV^* in $[a, b]$.

In analoger Weise findet man, dass φ_r OTV^* ist in $[a, b]$.

Satz 8. *Die Definition IV ist aequivalent mit der Definition II (nach Perron und Bauer).*

Beweis. Dass aus Integrierbarkeit nach Def. II Integrierbarkeit gemäss Def. IV folgt, mit gleichem Integralwert, ist evident.

Nehmen wir, umgekehrt, Integrierbarkeit von $f(x)$ über $[a, b]$ nach Def. IV an. Aus Satz 7 folgt dann, dass jede zu $f(x)$ in $[a, b]$ adjungierte Oberfunktion ψ_r in $[a, b]$ UTV^*, und jede zu $f(x)$ in $[a, b]$ adjungierte Unterfunktion φ_r dort OTV^* ist. Jede Funktion ψ_r und jede Funktion φ_r sind dadurch fast überall differenzierbar in $[a, b]$ [11]), wobei in fast allen Punkten von $[a, b]$ gilt:

$$D\psi_r \geqq f(x) \geqq D\varphi_r.$$

Aus diesen Eigenschaften der Ober- und Unterfunktionen $\{\psi_r\}$ und $\{\varphi_r\}$, und der Eigenschaft: untere Schranke aller $\psi_r(b) =$ obere Schranke aller $\varphi_r(b)$, welche wegen der Integrierbarkeit nach Def. IV auch erfüllt ist, folgt, dass $f(x)$ ein spezielles Denjoysches Integral [12]), und somit ebenfalls ein Perron–Bauersches Integral über $[a, b]$ hat, gleich $(P_3)\int_a^b f(x)\,dx$.

Der Satz ist damit bewiesen [13]).

§ 6bis. **Definition IVbis**, aus Def. IV hervorgehend durch Ersetzung der Funktionen ψ_r, φ_r durch Funktionen ψ_l, φ_l, führt zu *einem P_3^*-Integration, welche ebenfalls gleichwertig ist mit der speziellen Denjoyschen Integration.*

KAPITEL IV.

§ 7. **Definition.** $f(x)$ sei eine im abgeschlossenen Intervall $[a, b]$ definierte Funktion. Eine in $[a, b]$ stetige Funktion $\Psi_l(x)$ $[\Psi_r(x)]$ heisst *eine zu $f(x)$ in $[a, b]$ adjungierte allgemeine linksseitige [rechtsseitige] Oberfunktion*, wenn: *a)* $\Psi_l(a) = 0$ $[\Psi_r(a) = 0]$ ist; *b)* $[a, b]$ sich überdecken lässt durch eine abzählbare Menge E und abzählbar viele perfekte Mengen (P_j), derartig, dass in den Punkten eines jeden P_j, abzählbar viele ausgenommen, die „in bezug auf dieses P_j" genommene linksseitige untere Derivierte von Ψ_l, $D_-^{Pj}\Psi_l$, $\neq -\infty$ und $\geqq f(x)$ [die „in bezug auf dieses P_j" genommene rechtsseitige untere Derivierte von Ψ_r, $D_+^{Pj}\Psi_r$, $\neq -\infty$ und $\geqq f(x)$] ist.

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[11]) Siehe J. RIDDER, Math. Ztschr. **34** (1931), S. 252 (Satz 18).

[12]) Siehe RIDDER, loc. cit. 11), S. 255 (Satz E).

[13]) Man hätte auch das Verfahren, das zu Satz 3 führte, anwenden können. Vergl. J. RIDDER, Math. Ztschr. **41** (1936), S. 194—196 (Beweis von Satz α_2).

Die Definition *der zu $f(x)$ in $[a, b]$ adjungierten allgemeinen linksseitigen [rechtsseitigen] Oberfunktionen $\Phi_l(x)$ $[\Phi_r(x)]$* liegt hiernach auf der Hand (vergl. auch die Definitionen in § 1).

Neben der MC SHANEschen Integraldefinition von § 1 haben wir nun hier:

Definition V. Die in $[a, b]$ definierte Funktion $f(x)$ ist *P_4-integrierbar über $[a, b]$*, wenn es zu willkürlich positivem ε ein Quadrupel $[\Psi_l(x), \Psi_r(x), \Phi_l(x), \Phi_r(x)]$ von allgemeinen Ober- und Unterfunktionen gibt, in $[a, b]$ zu $f(x)$ adjungiert, deren je zwei in jedem Punkte von $[a, b]$ eine Differenz haben, welche, absolut genommen, kleiner als ε ist.

Die Differenz einer jeden allgemeinen (links- oder rechtsseitigen) Oberfunktion und einer jeden allgemeinen (links- oder rechtsseitigen) Unterfunktion ist monoton nicht-abnehmend in $[a, b]$ [14]). Ist $f(x)$ P_4-integrierbar über $[a, b]$, so gibt es eine Funktion $F_4(x)$, welche gleich der unteren Schranke aller $\Psi_l(x)$, der unteren Schranke aller $\Psi_r(x)$, der oberen Schranke aller $\Phi_l(x)$ und der oberen Schranke aller $\Phi_r(x)$ ist. Wir definieren nun:

$$(P_4)\int_a^b f(x)\,dx = F_4(b).$$

§ 8. **Definition.** $f(x)$ sei definiert in $[a, b]$. Eine in $[a, b]$ stetige Funktion $\Psi(x)$ heisst *eine zu $f(x)$ adjungierte allgemeine Perron-Oberfunktion*, wenn: *a*) $\Psi(a) = 0$ ist; *b*) $[a, b]$ sich überdecken lässt durch eine abzählbare Menge E und abzählbar viele perfekte Mengen (P_j), derartig, dass in den Punkten eines jeden P_j, abzählbar viele ausgenommen, die „in bezug auf dieses P_j" genommene allgemeine untere Derivierte von Ψ_l, $\underline{D}^{Pj}\Psi$, $\neq -\infty$ und $\geq f(x)$ ist.

Definition. Ebenso soll eine in $[a, b]$ stetige Funktion $\Phi(x)$ *eine zu $f(x)$ adjungierte allgemeine Perron-Unterfunktion* sein, wenn: *a*) $\Phi(a) = 0$ ist; *b*) $[a, b]$ sich überdecken lässt durch eine abzählbare Menge E und abzählbar viele perfekte Mengen (P_j), derartig, dass in den Punkten eines jeden P_j, abzählbar viele ausgenommen, die „in bezug auf dieses P_j" genommene allgemeine obere Derivierte von Φ, $\overline{D}^{Pj}\Phi$, $\neq +\infty$ und $\leq f(x)$ ist.

Definition VI. Die in $[a, b]$ definierte Funktion $f(x)$ ist *$\mathfrak{P}$-integrierbar* (integrierbar im verallgemeinerten Perronschen Sinne) *über* $[a, b]$, wenn

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[14]) Man beweist dies mittels transfiniter Induktion. Vergleiche RIDDER, loc. cit. 11), S. 247—249 (Beweis von Satz 14); siehe dabei auch MC SHANE, loc cit. 1), S. 314, 315 (Nr. 57. 4). Ein zweiter Beweis verläuft wie folgt. Man zeige erstens dass jede allgemeine Oberfunktion *UTV*-, jede allgemeine Unterfunktion *OTV* in $[a, b]$ ist, zweitens dass jede allgemeine Oberfunktion fast überall in $[a, b]$ eine approximative Ableitung $\geq f(x)$, jede allgemeine Unterfunktion fast überall in $[a, b]$ eine approximative Ableitung $\leq f(x)$ hat, und wende darauf RIDDER, Fund. math. 21 (1933), S. 4 (Satz) an. Schliesslich erhält man einen dritten Beweis durch Anwendung des beim Beweise des eben zitierten Satzes angewandten, indirekten Verfahrens.

die untere Schranke aller (existierend anzunehmenden) Oberfunktionen $\Psi(x)$ und die obere Schranke aller (ebenfalls existierend anzunehmenden) Unterfunktionen $\Phi(x)$ in jedem Punkte von $[a, b]$ zusammenfallen. Ist $\mathfrak{F}(x)$ der gemeinsame Wert beider Schranken, so sei das verallgemeinerte Perron-Integral

$$(\mathfrak{B})\int_a^b f(x)\,dx = \mathfrak{F}(b).$$

Bekanntlich reicht diese Definition genau ebenso weit wie die allgemeine Denjoysche Integraldefinition [15]).

Satz 9. *Die Definitionen V und VI sind aequivalent.*

Beweis. Aus Integrierbarkeit gemäss Def. VI folgt sofort Integrierbarkeit gemäss Def. V; die Integralwerte sind dieselben.

Nehmen wir, umgekehrt, Integrierbarkeit von $f(x)$ über $[a, b]$ gemäss Def. V an. Ist $(P_\delta)\int_a^x f(x)dx \equiv F_4(x)$ in $[a, b]$, so gibt es zu willkürlich positivem ε eine allgemeine linksseitige Oberfunktion $\Psi_l(x)$, eine allgemeine rechtsseitige Oberfunktion $\Psi_r(x)$, eine allgemeine linksseitige Unterfunktion $\Phi_l(x)$ und eine allgemeine rechtsseitige Unterfunktion $\Phi_r(x)$ mit

$$0 \leqq \Psi_l(x) - F_4(x) < \varepsilon/4,\quad 0 \leqq \Psi_r(x) - F_4(x) < \varepsilon/4,$$
$$0 \leqq F_4(x) - \Phi_l(x) < \varepsilon/4,\quad 0 \leqq F_4(x) - \Phi_r(x) < \varepsilon/4$$

in jedem Punkte x von $[a, b]$.

Die Differenzen $\Sigma_1(x) \equiv \Psi_l(x) - F_4(x)$, $\Sigma_2(x) \equiv \Psi_r(x) - F_4(x)$, $\Sigma_3(x) \equiv F_4(x) - \Phi_l(x)$ und $\Sigma_4(x) \equiv F_4(x) - \Phi_r(x)$ sind monoton nicht-abnehmend. Aus den Definitionen von $\Psi_l(x)$ und $\Psi_r(x)$ folgt, dass $[a, b]$ Summe ist von einer abzählbaren Menge und abzählbar vielen perfekten Mengen (Q_j), auf deren jeder, ausgenommen in den Punkten einer höchstens abzählbaren Teilmenge, gleichzeitig

$$\underline{D}^{Qj}\,\Psi_l(x) \neq -\infty \text{ und } \geqq f(x),$$
$$\underline{D}^{Qj}_{+}\,\Psi_r(x) \neq -\infty \text{ und } \geqq f(x)$$

ist. Daraus folgt für die Funktion $\Psi(x) \equiv F_4(x) + \Sigma_1(x) + \Sigma_2(x)$ in denselben Punkten von Q_j:

$$\underline{D}^{Qj}\,\Psi(x) \neq -\infty \text{ und } \geqq f(x).$$

$\Psi(x)$ ist somit eine zu $f(x)$ in $[a, b]$ adjungierte allgemeine Perron-Ober-

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[15]) Siehe RIDDER, loc. cit. 11), S. 263 (Satz G) u. 235. An der zitierten Stelle treten die hier in den Definitionen zugelassenen abzählbaren Ausnahmemengen in den P_j nicht auf; der Umfang der verallgemeinerten Perron-Integration wird dadurch nicht beeinträchtigt. Vergl. auch loc. cit. S. 257, 258 (Fussn. 43). — Auch zeigt der zitierte Satz G, dass in den Definitionen der allgemeinen Perron-Ober- und Unterfunktionen die Bedingungen $\underline{D}^{Pj}\,\Psi \geqq f(x)$ und $\overline{D}^{Pj}\,\Phi \leqq f(x)$ in den Punkten einer Teilmenge vom Masse Null eines jeden zugehörigen P_j nicht erfüllt zu sein brauchen, ohne dass dies den Umfang der Definition VI ändert.

funktion. Ebenso ist $\Phi(x) \equiv F_1(x) - \Sigma_2(x) - \Sigma_3(x)$ eine zu $f(x)$ in $[a, b]$ adjungierte allgemeine Perron-Unterfunktion. In jedem Punkte von $[a, b]$ ist $\Psi - \Phi < \varepsilon$, und weicht eine jede von ihnen um weniger als $\varepsilon/2$ von $F_4(x)$ ab. Somit existiert $(\mathfrak{P})\int_a^b f dx$ und ist gleich $F_4(b)$.

§ 9. **Definition VII.** Die in $[a, b]$ definierte Funktion $f(x)$ ist P_5-*integrierbar über* $[a, b]$, wenn es zu willkürlich positivem ε ein Paar $[\Psi_r(x), \Phi_l(x)]$ von allgemeiner rechtsseitiger Oberfunktion und allgemeiner linksseitiger Unterfunktion gibt, beide in $[a, b]$ zu $f(x)$ adjungiert, deren Differenz in jedem Punkte von $[a, b]$, absolut genommen, kleiner als ε ist.

Die Differenz $\Psi_r(x) - \Phi_l(x)$ ist monoton nicht-abnehmend in $[a, b]$ [14]). Ist $f(x)$ P_5-integrierbar über $[a, b]$, so gibt es eine Funktion $F_5(x)$, welche gleich der unteren Schranke aller $\Psi_l(x)$ und der oberen Schranke aller $\Phi_l(x)$ ist. Wir definieren nun:

$$(P_5)\int_a^b f(x)\,dx = F_5(b).$$

Satz 10. *In $[a, b]$ zu einer Funktion $f(x)$ adjungierte allgemeine links- oder rechtsseitige Oberfunktionen sind daselbst UTV; in $[a, b]$ zu $f(x)$ adjungierte allgemeine links- oder rechtsseitige Unterfunktionen sind daselbst OTV.*

Beweis. Betrachten wir nur den Fall einer allgemeinen rechtsseitigen Oberfunktion $\Psi_r(x)$. Ist P_j eine der in der Definition von $\Psi_r(x)$ eingeführten perfekten Mengen, so sei $\Psi_r^j(x)$ die Funktion, welche in den Punkten von P_j und in a und b mit $\Psi_r(x)$ zusammenfällt, und sich linear ändert in den zu P_j komplementären, abgeschlossenen Teilintervallen von $[a, b]$. Dann ist in den Punkten von P_j, diejenigen einer abzählbaren Menge ausgenommen, $D_+\Psi_r^j(x) = D_+^{P_j}\Psi_r(x)$, also auch $D_+\Psi_r^j(x) \neq -\infty$.

Nach dem Lemma von § 3 lässt P_j sich, ausgenommen in den Punkten einer (ev. leeren) abzählbaren Teilmenge, überdecken durch abzählbar viele perfekte Teilmengen $\{Q_{j;k}\}$ von P_j, auf deren jeder $\Psi_r^j(x)$, und somit auch $\Psi_r(x)$, unterhalb totalstetig ist. Daraus folgt der Satz für eine allgemeine Oberfunktion $\Psi_r(x)$.

Satz 11. *Die Definitionen VI und VII sind aequivalent.*

Beweis. Aus Integrierbarkeit gemäss Definition VI folgt sofort Integrierbarkeit gemäss Definition VII; die Integralwerte sind dieselben.

Nehmen wir, umgekehrt, Integrierbarkeit von $f(x)$ über $[a, b]$ gemäss Definition VII an.

Jede allgemeine rechtsseitige Oberfunktion $\Psi_r(x)$, welche in $[a, b]$ zu $f(x)$ adjungiert ist, ist, nach Satz 10, *UTV* in $[a, b]$. Auf einer jeden von

abzählbar vielen perfekten Mengen (P_j), welche $[a, b]$ bis auf den Punkten einer abzählbaren Teilmenge überdecken, ist $D_+^{P_j}\Psi_r \neq -\infty$, die Punkte einer abzählbaren Teilmenge ausgenommen. Daraus folgt, nach einem bekannten Denjoyschen Satze [7]), dass in fast allen Punkten von P_j $\Psi_r^{P_j}$ [16]), und somit auch Ψ_r selbst, eine endliche approximative Ableitung hat, welche gleich $D_+^{P_j}\Psi_r$, somit $\geqq f(x)$ ist.

Ψ_r ist eine in $[a, b]$ stetige Funktion mit den Eigenschaften: 1° $\Psi_r(a) = 0$, 2° Ψ_r ist UTV in $[a, b]$, 3° in fast allen Punkten von $[a, b]$ gibt es eine approximative Ableitung $D_{\text{appr.}}\Psi_r \geqq f(x)$.

Ebenso hat jede allgemeine linksseitige Unterfunktion Φ_l, welche in $[a, b]$ zu $f(x)$ adjungiert ist, die Eigenschaften: 1° $\Phi_l(a) = 0$, 2° Φ_l ist OTV in $[a, b]$, 3° in fast allen Punkten von $[a, b]$ gibt es eine approximative Ableitung $D_{\text{appr.}}\Phi_l \leqq f(x)$.

Ausserdem ist die Funktion ${}_{(P_5)}\int_a^x f(x)dx$ stetig in $[a, b]$.

Daraus folgt [17]), dass $f(x)$ auch ein allgemeines Denjoysches Integral über $[a, b]$ hat, welches gleich ${}_{(P_5)}\int_a^b f dx$ ist; somit [18]) existiert ebenfalls ein $\mathfrak{P}$-Integral von f über $[a, b]$ mit demselben Wert. —

Satz 3 ist ein Korollar zu Satz 11.

§ 9bis. **Definition VIIbis.** Die in $[a, b]$ definierte Funktion $f(x)$ ist P_5^**-integrierbar über* $[a, b]$, wenn es zu willkürlich positivem ε ein Paar $[\Psi_l(x), \Phi_r(x)]$ von allgemeiner linksseitiger Oberfunktion und allgemeiner rechtsseitiger Unterfunktion gibt, beide in $[a, b]$ zu $f(x)$ adjungiert, deren Differenz in jedem Punkte von $[a, b]$, absolut genommen, kleiner als ε ist.

Satz 11bis. *Die Definitionen VI and VIIbis sind aequivalent.*

§ 10. Schliesslich hätte man noch P_6*- und* $P^*{}_6$*-Integrationen* einführen können, unter ausschliesslicher Anwendung von allgemeinen rechtsseitigen Ober- und allgemeinen rechtsseitigen Unterfunktionen bzw. von allgemeinen linksseitigen Ober- und allgemeinen linksseitigen Unterfunktionen. *Beide Integrationen zeigen sich völlig aequivalent mit der allgemeinen Denjoyschen Integration.* Der Beweis verläuft wie der des Satzes 11.

Die Relationen (4^a) lassen sich nun durch Anwendung der Resultate von Kap. III und diesem Kapitel erweitern zu:

$$P_1 = \mathfrak{P}^* = D^* = P_3 = P_3^* \subset P_2 \subset D = \mathfrak{P} = P_4 = P_5 = P_5^* = P_6 = P_6^* .$$

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[16]) $\Psi_r^{P_j}(x)$ ist die auf P_j mit $\Psi_r(x)$ zusammenfallende Funktion.

[17]) Man wende hier RIDDER, loc. cit. 14), S. 6 (kursiv gedruckter Satz) an.

[18]) Siehe den Text bei Fussn. 15.

KAPITEL V.

Integration durch Substitution, und partielle Integration.

§ 11. Den bei Mc SHANE, loc. cit. 1), S. 329 (Nr. 64.1) vorkommenden Satz über Integration durch Substitution beim P_1-, und somit auch beim $\mathfrak{P}^*$- und D^*-Integral, wollen wir hier auf das $\mathfrak{P}$-, und damit auch auf das D-Integral übertragen.

Satz A. *$f(x)$ sei $\mathfrak{P}$-integrierbar über $[a, b]$; $g(y)$ sei stetig und monoton nicht-abnehmend in $[\alpha, \beta]$, mit $a = g(\alpha)$, $b = g(\beta)$, und $\overline{D}g(y) \neq +\infty$ in den Punkten von $[\alpha, \beta]$, höchstens diejenigen einer abzählbar unendlichen Menge ausgenommen. Dann ist $f[g(y)] \cdot \overline{D}g(y)$ $\mathfrak{P}$-integrierbar über $[\alpha, \beta]$, und es ist*

$$(\mathfrak{P})\int_a^b f(x)\,dx = (\mathfrak{P})\int_\alpha^\beta f[g(y)] \cdot \overline{D}\,g(y)\,dy. \quad . \quad . \quad . \quad . \quad (8)$$

Beweis [19]). Ist $\Psi(x)$ eine in $[a, b]$ zu $f(x)$ adjungierte allgemeine Perron-Oberfunktion [20a]), so wollen wir zeigen, dass $\Psi[g(y)]$ eine in $[\alpha, \beta]$ zu $f[g(y)] \cdot \overline{D}g(y)$ adjungierte allgemeine Perron-Oberfunktion ist [20b]).

Diejenigen Punkte von $[\alpha, \beta]$, welche sich mittels $x = g(y)$ auf die Punkte der in der Definition von $\Psi(x)$ vorkommenden abzählbaren Menge E abbilden, liefern eine abzählbare Menge F von einzelnen Punkten und eine abzählbare Menge von abgeschlossenen Teilintervallen $(\bar{i}_n)$ von $[\alpha, \beta]$, in deren jedem $g(y)$ konstant ist; die Punkte von F gehören somit zu keinem abgeschlossenen Intervall, in welchem $g(y)$ konstant ist. Jedes abgeschlossene Intervall $\bar{i}_n$ ist als perfekte Menge aufzufassen, auf welcher $\Psi[g(y)]$ konstant, und somit $\underline{D}\Psi[g(y)] = 0 = f[g(y)] \cdot \overline{D}g(y) \neq -\infty$ ist, die beiden Endpunkte ausgenommen. Neben den Intervallen $(\bar{i}_n)$ kann es noch höchstens abzählbar unendlich viele abgeschlossene Teilintervalle $(\bar{u}_n)$ geben, deren jedes die Teilmenge von $[\alpha, \beta]$ ist, die sich auf einen einzelnen, bestimmten Punkt von $[a, b]$ abbildet; auch hier ist $\underline{D}\Psi[g(y)] = 0 = f[g(y)] \cdot \overline{D}g(y) \neq -\infty$, die Endpunkte von $\bar{u}_n$ ausgenommen.

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[19]) Man erhält einen zweiten Beweis, ausgehend von der deskriptiven Definition des allgemeinen D-Integrals. Beachtet man, dass $(D)\int_a^x f(x)dx$ in $[a, b]$ TV ist, so folgt daraus und aus der Eigenschaft von $g(y)$ in $[\alpha, \beta]$ totalstetig zu sein, dass $(D)\int_a^x f(x)dx$, als Funktion von y betrachtet mittels der Substitution $x = g(y)$, in $[\alpha, \beta]$ TV ist; ausserdem hat diese Funktion fast überall in $[\alpha, \beta]$ eine approximative Ableitung gleich $f[g(y)] \cdot g'(y)$. Daraus folgt die Relation (8).

[20a]) Siehe die im Texte gegebene Definition.

[20b]) Dabei wollen wir die Definition dieser Oberfunktionen uns abgeändert denken gemäss der letzten Bemerkung in Fussn. 15.

Jede in der Definition von $\Psi(x)$ auftretende, perfekte Menge P_j ist, mittels $x = g(y)$, Bild einer perfekten Teilmenge Q_j von $[\alpha, \beta]$; lassen wir aus Q_j diejenigen Punkte fort, welche zu einem der offenen Intervalle (u_n) gehören, so entsteht eine perfekte Teilmenge R_j von Q_j; die Abbildung von R_j auf P_j ist ein-eindeutig, ausgenommen in Paaren von Punkten von R_j, welche beide Endpunkt eines $\bar{u}_n$ sind (hier ist die Abbildung zwei-eindeutig).

In den Punkten von P_j, abzählbar viele ausgenommen, ist

$$\underline{D}^{Pj}\Psi(x) \neq -\infty;$$

in den Punkten von R_j, abzählbar viele ausgenommen, ist $\overline{D}g(y) \neq +\infty$. Daraus folgt, dass in den Punkten von R_j, abzählbar viele ausgenommen, $\underline{D}^{Rj}\ \Psi[g(y)] \neq -\infty$ ist.

In den Punkten von P_j, abzählbar viele ausgenommen, ist

$$\underline{D}^{Pj}\Psi(x) \geqq f(x);$$

in fast allen Punkten von R_j existiert eine endliche Ableitung $g'(y) \geqq 0$. Daraus folgt, dass in fast allen Punkten von R_j:

$$\underline{D}^{Rj}\Psi[g(y)] = \underline{D}^{Pj}\ \Psi(x) \cdot g'(y) \geqq f(x) \cdot g'(y) = f[g(y)] \cdot \overline{D}g(y)$$

ist.

Jede zu $f(x)$ in $[a, b]$ adjungierte allgemeine Perron-Oberfunktion $\Psi(x)$ liefert somit eine zu $f[g(y)] \cdot \overline{D}g(y)$ in $[\alpha, \beta]$ adjungierte allgemeine Perron-Oberfunktion $\Psi[g(y)]$ im Sinne von Fussn. 15 (letzte Bemerkung).

Ebenso liefert jede zu $f(x)$ in $[a, b]$ adjungierte allgemeine Perron-Unterfunktion $\Phi(x)$ eine zu $f[g(y)]. \overline{D}g(y)$ in $[\alpha, \beta]$ adjungierte allgemeine Perron-Unterfunktion $\Psi[g(y)]$ im Sinne von Fussn. 15 (letzte Bemerkung).

Aber daraus folgt, dass $f[g(y)] \cdot \overline{D}g(y)$ über $[\alpha, \beta]$ ein $\mathfrak{P}$-Integral hat, mit

$$(\mathfrak{P})\int_{\alpha}^{\beta} f[g(y)] \cdot \overline{D}\, g(y)\, dy = (\mathfrak{P})\int_{a}^{b} f(x)\, dx.$$

§ 12. **Satz B.** *Hat $f(x)$ ein $\mathfrak{P}$-Integral über $[a, b]$, und ist $g(x)$ in $[a, b]$ von beschränkter Variation, so ist auch $f(x) \cdot g(x)$ $\mathfrak{P}$-integrierbar über $[a, b]$; dabei ist*

$$(\mathfrak{P})\int_{a}^{b} f(x) \cdot g(x)\, dx = F(b) \cdot g(b) - \int_{a}^{b} F(x)\, d\, g(x),$$

mit $F(x) \equiv (\mathfrak{P})\int_{a}^{x} f(x)dx$, und $\int_{a}^{b} F(x)dg(x)$ ein Stieltjessches Integral.

Zu diesem Satz [21]) machen wir einige Bemerkungen:

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[21]) Einen allgemeinen Satz über partielle Integration bei den Perron-Stieltjesschen, und den mit diesen aequivalenten Denjoy-Stieltjesschen Integralen, welcher Satz *B* als Spezialfall umfasst, findet man in unserer Arbeit: Math. Ann. **116** (1938), S. 76—103, insbes. S. 101, 102.

1. Für den Fall des $\mathfrak{P}^*$-Integrals findet man den Satz bei Mc SHANE, loc. cit. 1), S. 332 (Nr. 65. 1). Gerade um diesen Satz beweisen zu können führte Mc SHANE die hier unter Definition I vorkommende Perron-Integration ein.

2. Da wir uns hier auf die Gleichwertigkeit von $\mathfrak{P}$-Integration (Def. VI) und P_4-Integration berufen dürfen, können wir das Beweisverfahren von Mc SHANE auch benutzen beim Beweise des obigen Satzes *B*. Nur müssen wir dabei nicht mit allgemeinen extremen Derivierten, sondern mit extremen Derivierten in bezug auf die in den Definitionen der allgemeinen rechts- oder linksseitigen Ober- und Unterfunktionen auftretenden, perfekten Mengen (P_j) arbeiten (§ 7). Uebrigens treten keine wesentlichen Aenderungen im Beweisverfahren auf.

3. Satz *B* ist natürlich aequivalent mit dem Satze, der entsteht durch Aenderung von $\mathfrak{P}$ in *D*. Daraus folgt, dass der Satz wohl auch bewiesen werden kann, ausgehend vom allgemeinen Denjoyschen Integralbegriff. Ein auf der deskriptiven Definition dieses Integrals beruhendes Beweisverfahren enthält SAKS, loc. cit. 7), S. 201—203, und ein auf der konstruktiven Definition des Integrals beruhendes HOBSON, loc. cit. 9), S. 711—715.

Mathematics. — *Non-homogeneous binary quadratic forms.* III. By H. DAVENPORT. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of February 22, 1947.)

1. The papers of this series [1]) are concerned with the determination, for particular indefinite binary quadratic forms $f(x, y)$, of the least number M such that, for any real x_1, y_1, we can satisfy

$$|f(x+x_1, y+y_1)| \leqslant M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in integers x, y. In the first paper I mentioned that I had not then been able to solve this problem for the third form of MARKOFF's series, the form $f(x, y) = 5x^2 - 11xy - 5y^2$. The present note gives a solution, which is obtained by a modification of the methods used in the second paper of this series for the form $x^2 + xy - y^2$. There is, however, an appreciable difference in the treatment, as will be seen in § 2.

Writing $f(x, y) = 5(x - \theta y)(x - \theta' y)$, where

$$\theta = \tfrac{1}{10}(11 + \sqrt{221}), \quad \theta' = \tfrac{1}{10}(11 - \sqrt{221}), \quad . \quad . \quad . \quad . \quad (2)$$

the inequality (1) takes the form

$$|(x - \theta y - a)(x - \theta' y - b)| \leqslant \tfrac{1}{5} M,$$

where a, b are real numbers determined by x_1, y_1. We prove:

Theorem 1. *For any real a, b there exist integers x, y such that*

$$|(x - \theta y - a)(x - \theta' y - b)| \leqslant \tfrac{1}{4}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and the constant $\frac{1}{4}$ on the right is the best possible.

The last clause follows at once from the case $a = b = \frac{1}{2}$. For

$$(x - \theta y - \tfrac{1}{2})(x - \theta' y - \tfrac{1}{2}) = \tfrac{1}{20}(5X^2 - 11XY - 5Y^2),$$

with $X = 2x - 1$, $Y = 2y$, and 5 is the minimum of $|5X^2 - 11XY - 5Y^2|$, by the work of MARKOFF. The question arises whether a better result holds if special values of a, b are excluded (values equivalent, in a certain sense, to $\frac{1}{2}$, $\frac{1}{2}$). I indicate briefly how the work can be extended to show that this is in fact the case. The result established is:

Theorem 2. *Suppose a, b are not of the form*

$$a = x_0 - \theta y_0 \pm \tfrac{1}{2}\tau^n, \quad b = x_0 - \theta' y_0 \pm \tfrac{1}{2}\tau^{-n}, \quad . \quad . \quad . \quad (4)$$

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[1]) Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **49**, 815—821 (1946) and **50**, 378—389 (1947).

nor of the form

$$a = x_0 - \theta y_0 \pm \tfrac{1}{5}\tau^n \theta, \quad b = x_0 - \theta' y_0 \pm \tfrac{1}{5}\tau^{-n}\theta' \quad . \quad . \quad (5)$$

where x_0, y_0 *are integers and*

$$\tau = 2 + 5\theta = \tfrac{1}{2}(15 + \sqrt{221}). \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Then there exist integers x, y *such that*

$$|(x - \theta y - a)(x - \theta' y - b)| < \frac{1}{4.00001}. \quad . \quad . \quad . \quad . \quad (7)$$

2. We denote by M the lower bound of the product on the left of (3), and can suppose $M > 0$, as if $M = 0$ there is nothing to prove. For an arbitrarily small $\varepsilon_0 > 0$ there exist integers x_0, y_0 for which

$$|(x_0 - \theta y_0 - a)(x_0 - \theta' y_0 - b)| = \frac{M}{1-\varepsilon}, \text{ where } 0 \leqq \varepsilon < \varepsilon_0.$$

We define real numbers α, β by

$$\alpha = (a - x_0 + \theta y_0)^{-1}, \quad \beta = (b - x_0 + \theta' y_0)^{-1}, \quad . \quad . \quad . \quad (8)$$

so that

$$|\alpha\beta| = \frac{1-\varepsilon}{M} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and have

$$|\{(x - \theta y)\alpha - 1\}\{(x - \theta' y)\beta - 1\}| \geqq 1 - \varepsilon \quad . \quad . \quad . \quad (10)$$

for all integers x, y.

Our next step is to derive from α, β two other numbers which also satisfy (9) and (10), and satisfy in addition certain simple inequalities. An important difference between the present case and the previous case, where θ was $\frac{1}{2}(1 + \sqrt{5})$, now arises, in that we are not entitled to interchange α and β. This was permissible in the previous case because either $x - \theta y$ or $x - \theta' y$ represented the general integer of $k(\theta)$, and the significance of (10) was unaltered when we interchanged α and β. In the present case, $x - \theta y$ and $x - \theta' y$ do not represent the same elements of $k(\theta)$. We can, however, prove the following Lemma.

Lemma 1. *If numbers* α, β *exist satisfying* (9) *and* (10), *then numbers exist satisfying both* (9), (10), *and*

$$\alpha > 0, \quad \theta\tau^{-1}\alpha \leqq |\beta| \leqq \theta\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Proof. We consider first the effect of the integral unimodular substitution

$$x = 13X + 5Y, \quad y = 5X + 2Y$$

on the linear forms $x - \theta y$, $x - \theta' y$. We have

$$x - \theta y = (13 - 5\theta)X + (5 - 2\theta)Y = (13 - 5\theta)(X - \theta Y),$$

since $5\theta^2 = 11\theta + 5$, and similarly with θ' in place of θ. Now

$13-5\theta' = 13-(11-5\theta) = 2+5\theta = \tau$, and $13-5\theta = 2+5\theta' = \tau^{-1}$.

Hence the effect of the above substitution on (10) is to change it into a similar inequality with α, β replaced by $\alpha\tau^{-1}$, $\beta\tau$. If $\alpha = \alpha_1\tau^n$, $\beta = \beta_1\tau^{-n}$ where n is any integer, then α_1, β_1 also satisfy (9) and (10), and we can choose n so that

$$\theta\tau^{-1} \leqq \left|\frac{\beta_1}{\alpha_1}\right| \leqq \theta\tau.$$

If $|\beta_1| \leq \theta|\alpha_1|$, this suffices to prove (11), since we can always replace α, β by $-\alpha$, $-\beta$ to ensure that $\alpha > 0$. Now suppose that $|\beta_1| > \theta|\alpha_1|$. By the substitution $x = Y$, $y = -X$, we have

$$x-\theta y = Y+\theta X = \theta(X-\theta' Y),$$

and similarly with θ, θ' interchanged. Hence (10) remains valid if we replace β by $\alpha\theta$ and α by $\beta\theta'$. Writing $\beta_1 = \alpha_2\theta$, $\alpha_1 = \beta_2\theta'$, we have (since $\theta\theta' = -1$),

$$\left|\frac{\beta_2}{\alpha_2}\right| = \theta^2\left|\frac{\alpha_1}{\beta_1}\right| \geqq \theta^2(\theta\tau)^{-1} = \theta\tau^{-1},$$

$$\left|\frac{\beta_2}{\alpha_2}\right| = \theta^2\left|\frac{\alpha_1}{\beta_1}\right| < \theta^2\theta^{-1} = \theta.$$

Hence, in this case the set α_2, β_2 (or $-\alpha_2$, $-\beta_2$) satisfies (11), and this establishes the Lemma.

3. We now follow the same general plan as before, by investigating the intervals which are possible for α and β in virtue of (10) and (11), and deducing the possible values of M from (9). We record first some numerical values:

$$\begin{aligned} \theta &= 2.5866069\ldots, & \theta' &= -0.3866069\ldots, \\ \tau &= 14.9330344\ldots, & \tau^{-1} &= 0.0669656\ldots, \\ \tau\theta^{-1} &= 5.7732137\ldots, & \tau^{-1}\theta &= 0.1732137\ldots. \end{aligned}$$

Lemma 2. *It is impossible that* $\alpha < 1$.

Proof. By (10), $|(\alpha-1)(\beta-1)| \geq 1-\varepsilon$. By (11), this implies

$$(1-\alpha)(\theta\alpha+1) \geqq 1-\varepsilon,$$

or $\theta\alpha^2 \leq (\theta-1)\alpha + \varepsilon$. Since $1-\theta^{-1} = 0.6133\ldots$, we must have $\alpha < 0.614$. Again by (10), $|(2\alpha-1)(2\beta-1)| \geq 1-\varepsilon$, whence

$$|2\alpha-1|(2\theta\alpha+1) \geqq 1-\varepsilon.$$

It is impossible that $2\alpha > 1$, since

$$\{2(0.614)-1\}\{2\theta(0.614)+1\} < (0.228)(4.2) < 0.96.$$

Hence $2\alpha < 1$, and in the same way as before we obtain

$$4\theta\alpha^2 \leq 2(\theta - 1)\alpha + \varepsilon, \text{ whence } \alpha < 0.307.$$

Let m be the largest positive integer for which $m\alpha < 1$, then $m\alpha \geq 1 - \alpha$. By (10) and (11),

$$(1 - m\alpha)(m\theta\alpha + 1) \geq 1 - \varepsilon,$$

whence

$$m\alpha(m\theta\alpha - \theta + 1) \leqq \varepsilon.$$

This is impossible, since

$$m\alpha \geqq 1 - \alpha > 0.693, \quad m\theta\alpha - \theta + 1 \geqq 1 - \alpha\theta > 1 - (0.307)(2.6).$$

Lemma 3. *It is impossible that* $\alpha > 1$ *and* $0 < \beta < 1$.

Proof. By (10), we have in this case $(\alpha - 1)(1 - \beta) \geq 1 - \varepsilon$. Since $\beta \geq \theta\tau^{-1}\alpha > 0.173\,\alpha$, this implies

$$(\alpha - 1)(1 - 0.173\,\alpha) \geqq 1 - \varepsilon,$$

or $0.173\,\alpha^2 - 1.173\,\alpha + 2 - \varepsilon \leq 0$. But this quadratic expression is always positive, since $(1.173)^2 < 4(0.173)(2 - \varepsilon)$.

Lemma 4. *It is impossible that* $\alpha > 1$ *and* $-1 < \beta < 0$.

Proof. We use three particular cases of (10), namely

$$(\alpha + 1)(\beta + 1) \geqq 1 - \varepsilon,$$

$$|\{(\theta - 2)\alpha - 1\}\{(\theta' - 2)\beta - 1\}| \geqq 1 - \varepsilon,$$

$$|\{(\theta - 1)\alpha - 1\}\{(\theta' - 1)\beta - 1\}| \geqq 1 - \varepsilon.$$

The nature of the argument will be such that we can neglect ε. Writing $\beta = -\beta'$, where $\beta' > 0$, and neglecting ε, the three inequalities become, on noting that $(\theta - 2)(\theta' - 2) = -\frac{7}{5}$ and $(\theta - 1)(\theta' - 1) = -\frac{11}{5}$,

$$(\alpha + 1)(1 - \beta') \geqq 1, \quad \ldots\ldots\ldots \quad (12)$$

$$|(\alpha - 1.7047\ldots)(\beta' - 0.4190\ldots)| \geqq \tfrac{5}{7}, \quad \ldots\ldots \quad (13)$$

$$|(\alpha - 0.6303\ldots)(\beta' - 0.7212\ldots)| \geqq \tfrac{5}{11} \quad \ldots\ldots \quad (14)$$

If $\beta' < 0.4190\ldots$, (11) tells us that

$$\alpha \leqq \tau\theta^{-1}\beta' < (5.774)(0.42) < 2.5,$$

and we would have an obvious contradiction in (13). If $\beta' > 0.4190\ldots$, we must have $\alpha > 1.7047\ldots$, otherwise (13) is violated, and using $\alpha < 5.774\,\beta'$ we obtain

$$(5.774\,\beta' - 1.7047\ldots)(\beta' - 0.4190\ldots) \geqq \tfrac{5}{7}.$$

This gives

$$5.774\,\beta' \geqq 1.7047\ldots + (5.774)(0.4190\ldots);$$

whence $\beta' > 0.712$. It is now plain that we must have $\beta' > 0.7212\ldots$ to avoid a contradiction in (14).

Applying the same argument to (14), we obtain

$$(5.774\,\beta' - 0.6303\ldots)(\beta' - 0.7212\ldots) \geqq \tfrac{5}{11}.$$

This gives $5.774\,\beta' \geq 0.6303\ldots + (5.774)(0.7212\ldots)$, whence $\beta' > 0.829$. On the other hand, the same argument applied to (12) gives

$$(5.774\,\beta' + 1)(1 - \beta') \geqq 1,$$

whence $\beta' \leq 4.774/5.774 = 0.8268\ldots$. This is a contradiction [2]).

Lemma 5. *If $\alpha > 1$ and $\beta < -1$ then $\alpha\,|\beta| > 4.1$.*

Proof. We use the following cases of (10):

$$|(\theta\alpha - 1)(\theta'\beta - 1)| \geqq 1 - \varepsilon,$$
$$|\{(\theta - 2)\alpha - 1\}\{(\theta' - 2)\beta - 1\}| \geqq 1 - \varepsilon,$$
$$|\{(3 - \theta)\alpha - 1\}\{(3 - \theta')\beta - 1\}| \geqq 1 - \varepsilon,$$
$$|\{(4\theta - 10)\alpha - 1\}\{(4\theta' - 10)\beta - 1\}| \geqq 1 - \varepsilon,$$
$$|(2\,\theta\alpha - 1)(2\,\theta'\beta - 1)| \geqq 1 - \varepsilon.$$

We write $\beta = -\beta'$, where $\beta' > 1$. Neglecting ε, and noting that $(3 - \theta)(3 - \theta') = \frac{7}{5}$ and $(4\,\theta - 10)(4\,\theta' - 10) = -4$, we can write these inequalities as

$$(\alpha - 0.3866\ldots)\,|\beta' - 2.5866\ldots| \geqq 1, \quad \ldots\ldots \quad (15)$$
$$|\alpha - 1.7047\ldots|\,(\beta' - 0.4190\ldots) \geqq \tfrac{5}{7}, \quad \ldots\ldots \quad (16)$$
$$|\alpha - 2.4190\ldots|\,(\beta' + 0.2953\ldots) \geqq \tfrac{5}{7}, \quad \ldots\ldots \quad (17)$$
$$|\alpha - 2.8866\ldots|\,(\beta' - 0.0866\ldots) \geqq \tfrac{1}{4}, \quad \ldots\ldots \quad (18)$$
$$(\alpha - 0.1933\ldots)\,|\beta' - 1.2933\ldots| \geqq \tfrac{1}{4}. \quad \ldots\ldots \quad (19)$$

We may also suppose, in proving this Lemma, that

$$\alpha\beta' \leqq 4.1. \quad \ldots\ldots\ldots\ldots \quad (20)$$

Case 1. Suppose $\beta' > 2.5866\ldots$. By (20), $\beta' < 4.1$. By (15),

$$\alpha \geqq 0.3866\ldots + (\beta' - 2.5866\ldots)^{-1},$$
$$\alpha\beta' \geqq (0.3866\ldots)\,\beta' + 1 + (2.5866\ldots)(\beta' - 2\,5866\ldots)^{-1}.$$

This last expression decreases as β' increases in the range under consideration, hence

$$\alpha\beta' \geqq 4.1\,(0.3866\ldots) + 1 + (2.5866\ldots)(4.1 - 2.5866\ldots)^{-1} > 4.2.$$

---

[2]) The contradiction just obtained really arises from the fact that the lower bound found for β' is $\frac{5}{11}(7 - 2\theta)$, and the upper bound is $6 - 2\theta$. That the latter is the smaller is equivalent to $\theta > \frac{23}{11} = 2.5833\ldots\ldots$

Case 2. Suppose $2 \leq \beta' < 2.5866\ldots$. Using (15) again, we obtain

$$\alpha\beta' \geqq (0.3866\ldots)\beta' - 1 + (2.5866\ldots)(2.5866\ldots - \beta')^{-1},$$

and since this plainly increases with β' we have

$$\alpha\beta' \geqq 2(0.3866\ldots) - 1 + (2.5866\ldots)(0.5866\ldots)^{-1} > 4.18.$$

Case 3. Suppose $1 < \beta' < 2$. In the first place $(\alpha - 1)(\beta' + 1) \geq 1$, whence $\alpha > 1.33$. Next, by (16),

$$|\alpha - 1.7047\ldots| \geqq \frac{5}{7(1.5810\ldots)} > 0.450,$$

whence $\alpha > 2.154$. Further, by (17),

$$|\alpha - 2.4190\ldots| \geqq \frac{5}{7(2.2953\ldots)} > 0.311,$$

whence

$$\alpha > 2.729. \quad \ldots\ldots\ldots \quad (21)$$

By (20),

$$\beta' < \frac{4.1}{2.729} < 1.503.$$

Using this in (18), we obtain

$$|\alpha - 2.8866\ldots| \geqq \frac{1}{4(1.4164\ldots)} > 0.176.$$

In view of (21), this gives

$$\alpha > 3.062,$$

and (20) gives

$$\beta' < \frac{4.1}{3.062} < 1.34.$$

It is now plain from (19) that we cannot have $\beta' > 1.2933\ldots$. For then $\beta' - 1.2933\ldots < 0.0468$, and $\alpha < 4.1$ by (20), giving a product considerably less than $\frac{1}{4}$. Hence

$$\beta' < 1.2933\ldots \quad \ldots\ldots\ldots \quad (22)$$

We now use the inequality

$$(\alpha + 1)(\beta' - 1) \geqq 1, \quad \ldots\ldots\ldots \quad (23)$$

a special case of (10), apart from ε. In view of (20), this implies

$$4.1 + \beta' - \alpha - 1 \geqq 1,$$

whence $\alpha \leq 2.1 + \beta' < 3.394$ by (22). Now (19) gives

$$\beta' < 1.2933\ldots - \frac{1}{4(3.201)} < 1.216,$$

and (23) gives

$$\beta' > 1 + \frac{1}{4.394} > 1.227,$$

a contradiction. This completes the proof of Lemma 5.

Lemma 6. *If $\alpha>1$ and $\beta>1$ then $\alpha\beta\geq 4-3\varepsilon$.*

Proof. By (10), $(\alpha-1)(\beta-1)\geq 1-\varepsilon$. Hence, by a well known inequality, $(\alpha\beta)^{\frac{1}{2}}\geq 1+\{(\alpha-1)(\beta-1)\}^{\frac{1}{2}}\geq 1+(1-\varepsilon)^{\frac{1}{2}}$, and

$$\alpha\beta\geq 1+1-\varepsilon+2(1-\varepsilon)^{\frac{1}{2}}\geq 4-3\varepsilon.$$

4. Proof of Theorem 1. By Lemmas 1 to 6, we have numbers α, β with $|\alpha\beta|\geq 4-3\varepsilon$, and, by (9),

$$|\alpha\beta|=\frac{1-\varepsilon}{M}.$$

Hence either $M<\frac{1}{4}$ or $M=\frac{1}{4}$ and $\varepsilon=0$. In view of the definitions of M and ε, this suffices to prove Theorem 1.

5. Outline of the proof of Theorem 2. The first step is to prove that if $\alpha|\beta|\leq 4.00001$, then

$$|\alpha-2|<0.0034,\quad |\beta-2|<0.0034. \quad . \quad . \quad . \quad . \quad (24)$$

From Lemmas 2 to 6, we can restrict ourselves to the case $\alpha>1$, $\beta>1$, and (24) is easily deduced from $(\alpha-1)(\beta-1)\geq 1-\varepsilon$.

For we have

$$\alpha\beta\geq\beta+1-\varepsilon+\frac{1-\varepsilon}{\beta-1},$$

and if $\beta\geq 2.0034$ this would give

$$\alpha\beta\geq 2.0034+1-\varepsilon+\frac{1-\varepsilon}{1.0034}>4.00001.$$

Similarly if $\beta\leq 1.9966$, and again for α.

The next step is to improve (24) to

$$|\alpha-2|\leq K\varepsilon,\quad |\beta-2|\leq K\varepsilon, \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where K is an absolute constant. This can be done by using three particular cases of (10). The first two are those used for (16) and (17), namely

$$(\alpha-1.7047\ldots)(\beta+0.4190\ldots)\geq\tfrac{5}{7}(1-\varepsilon),$$
$$(2.4190\ldots-\alpha)(\beta-0.2953\ldots)\geq\tfrac{5}{7}(1-\varepsilon).$$

These are satisfied with equality when $\alpha=\beta=2$ and $\varepsilon=0$, as the reader will easily verify on going back to their origin in (10). The third is

$$|\{(38+97\theta)\alpha-1\}\{(38+97\theta')\beta-1\}|\geq 1-\varepsilon.$$

This is also satisfied with equality when $\alpha=\beta=2$ and $\varepsilon=0$, since

$$(75+194\theta)(75+194\theta')=75^2+(15)(194)(11)-194^2=-1.$$

Writing $\alpha=2+\alpha_1$, $\beta=2+\beta_1$ and simplifying, the three inequalities become

$$(1+3.386\ldots\alpha_1)(1+0.413\ldots\beta_1)\geq 1-\varepsilon, \quad . \quad . \quad . \quad (26)$$

$$(1-2.386\ldots\alpha_1)(1+0.586\ldots\beta_1)\geqq 1-\varepsilon, \quad . \quad . \quad . \quad (27)$$

$$(1+0.501\ldots\alpha_1)(1-287.9\ldots\beta_1)\geqq 1-\varepsilon. \quad . \quad . \quad . \quad (28)$$

A little thought now shows that (26), (27), (28) imply (25); e.g. if $\beta_1 < -K\varepsilon$, (26) and (27) are inconsistent, and so on.

From (25) we shall deduce that $\alpha = \beta = 2$. We easily find that if n is any integer (positive or negative), then

$$\pm\tau^{3n} = 2(x-\theta y)-1,$$

where x, y are integers. Giving x, y these values in (10), we obtain

$$|\{1+\tfrac{1}{2}(1+T)\alpha_1\}\{1+\tfrac{1}{2}(1+T^{-1})\beta_1\}|\geqq 1-\varepsilon,$$

where $T = \pm\tau^{-3n}$. The argument proceeds as in the proof of Lemma 3 in the preceding paper. If $\alpha_1 > 0$, we get a contradiction by taking $T = -\tau^{3n}$ with a suitable positive integral value of n. If $\alpha_1 < 0$, we take $T = \tau^{3n}$. If $\beta_1 > 0$ we take $T = -\tau^{-3n}$, and if $\beta_1 < 0$ we take $T = \tau^{-3n}$.

In this way we prove that either $\alpha\,|\,\beta\,| > 4.00001$ or $\alpha = \beta = 2$. In the former case, (7) follows from (9). In the latter case, we have to examine the possible values of a, b. Retracing the argument of Lemma 1, the original numbers α, β are either of the form

$$\alpha = \pm 2\tau^n, \quad \beta = \pm 2\tau^{-n},$$

or

$$\alpha = \pm 2\tau^n\theta^{-1}, \quad \beta = \pm 2\tau^{-n}(\theta')^{-1}.$$

By (8), a and b are either of the form (4) or of the form (5).

It may be observed that the form (5) is actually included under the form (4), since

$$\tfrac{1}{2}\tau^n\theta = \tfrac{1}{2}\tau^n(\tau-2-4\theta) = \tfrac{1}{2}\tau^{n+1} + x - \theta y,$$

where x, y are integers.

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Mathematics. — *Inequalities for the coefficients of trigonometric polynomials.* By R. P. BOAS Jr. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of March 29, 1947.)

1. Let $F(t) = \sum_{-n}^{n} a_j e^{ijt}$ be a trigonometric polynomial; for simplicity of statement we assume that $F(t)$ is real, i.e. $a_{-k} = \bar{a}_k$.

The following inequalities are known [1]:

$$2|a_k| \leqq \max |F(t)|, \qquad k > \frac{n}{3} \quad . \quad . \quad . \quad . \quad (1)$$

$$|a_0| + 2|a_k| \leqq \max |F(t)|, \qquad k > \frac{n}{2} \quad . \quad . \quad . \quad . \quad (2)$$

$$|a_0| + \tfrac{2}{3}|a_k| \leqq \tfrac{1}{4}\int_0^{2\pi} |F(t)|\,dt, \qquad k > \frac{n}{2} \quad . \quad . \quad . \quad . \quad (3)$$

If in addition $F(t) \geqq 0$, then [2]

$$|a_k| \leqq a_0 \cos \frac{\pi}{\left[\frac{n}{k}\right] + 2}, \qquad k > 0 \quad . \quad . \quad . \quad . \quad (4)$$

The proofs given by VAN DER CORPUT and VISSER for (1)–(3) are so simple that it seems worth while to point out that (4) can be obtained from their work by a device used for another purpose by BOAS and KAC [3]. The same idea allows us to generalize (1), (2) to inequalities analogous to (4), applying to all values of k. As to (3), it will appear that the constant $\frac{1}{4}$ on the right can be replaced by $\frac{1+\frac{1}{3}\sqrt{2}}{2\pi} = .234\ldots.$

Let us introduce the notation $M = \max\ |F(t)|$, $I = \frac{1}{2\pi}\int_0^{2\pi} |F(t)|\,dt$.

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[1]) J. G. VAN DER CORPUT and C. VISSER, Inequalities concerning polynomials and trigonometric polynomials, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **49**, 383—392 (1946).

[2]) E. v. EGERVÁRY and O. SZÁSZ, Einige Extremalprobleme im Bereiche der trigonometrischen Polynome, Math. Zeitschr., **27**, 641—652 (1928); G. SZEGÖ, Koeffizientenabschätzungen bei ebenen und räumlichen harmonischen Entwicklungen, Math. Annalen, **96**, 601—632 (1927).

[3]) R. P. BOAS Jr. and M. KAC, Inequalities for Fourier transforms of positive functions, Duke Math. J., **12**, 189—206 (1945).

Our results are the following:

$$|a_k| \leqq M \cos \frac{\pi}{p+2}, \qquad k>0,\ p=2\left[\frac{n-k}{2k}\right]+1 \quad . \quad . \quad (1')$$

(p is the largest odd integer with $pk \leqq n$),

$$|a_0|+|a_k| \sec \frac{\pi}{\left[\frac{n}{k}\right]+2} \leqq M, \qquad k \geqq 0, \quad . \quad . \quad (2')$$

$$|a_0|+\tfrac{3}{5}|a_k| \leqq (1+\tfrac{1}{5}\sqrt{2})\, I, \qquad k>\frac{n}{2}. \quad . \quad . \quad (3')$$

Other inequalities analogous to (3) can be obtained, for example

$$|a_0|+2|a_k| \leqq (1+\sqrt{2})\, I, \qquad k>\frac{n}{2}. \quad . \quad . \quad . \quad (3'')$$

Results extending (3) to all k could be obtained by the same method, but would be more complicated. We also notice an inequality analogous to (2) and (3''), but involving $M_2 = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 dx\right\}^{\frac{1}{2}}$, namely .

$$|a_0|+2|a_k| \leqq \sqrt{3}\, M_2, \qquad k>\frac{n}{2}. \quad . \quad . \quad . \quad (5)$$

2. We begin with the case where $f(x) \geqq 0$ and prove (4). Let $p=\left[\frac{n}{k}\right]$, $0 \leqq \gamma < \frac{1}{2}\sec\frac{\pi}{p+2}$. Then [4]) there is a nondecreasing function $\alpha(t)$ in $[0, 2\pi]$, whose FOURIER–STIELTJES coefficients c_j satisfy $c_0 = 1$, $c_1 = c_{-1} = \gamma$, $c_j = c_{-j} = 0$, $j = 2, 3, \ldots, p$. Define $\alpha(t)$ in $[2k\pi, (2k+2)\pi]$ by $\alpha(t) = k + \alpha(t-2\pi k)$. Then the convolution

$$f(x) \cdot \frac{1}{k}\alpha(kx) = \frac{1}{2\pi k}\int_0^{2\pi} f(x-t)\, d\alpha(kt) \quad . \quad . \quad . \quad . \quad (6)$$

is a nonnegative trigonometric polynomial $g(x)$ whose coefficients b_j are zero if j is not a multiple of k, while $b_{rk} = a_{rk}c_r$. Thus $b_0 = a_0$; $b_{\pm k} = \gamma a_{\pm k}$; $b_{rk} = 0$ if $|r| > 1$, since $k(p+1) > n$. That is,

$$g(x) = a_0 + \gamma\, a_k\, e^{ikx} + \gamma\, \bar{a}_k\, e^{-ikx} \geqq 0. \quad . \quad . \quad . \quad . \quad (7)$$

Choose x so that $a_k e^{ikx} = -|a_k|$; then (7) gives

$$2\gamma |a_k| \leqq a_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Let $\gamma \to \frac{1}{2}\sec\frac{\pi}{p+2}$; then (8) reduces to (4).

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[4]) This follows from the theory of the trigometric moment problem; cf. BOAS and KAC, loc. cit., p. 194—195.

3. Next we consider a general $f(x)$ and prove (2'). Construct $g(x)$ as in § 2; $g(x)$ is not necessarily nonnegative, but from (6)

$$|g(x)| \leqq M \cdot \frac{1}{2\pi k} \int_0^{2\pi} d\alpha(kt) \leqq M \cdot \frac{1}{2\pi k} \int_0^{2\pi k} d\alpha(t) = M.$$

Instead of (7) we now have

$$|a_0 + \gamma a_k e^{ikx} + \gamma \bar{a}_k e^{-ikx}| \leqq M.$$

Choose x so that $a_k e^{ikx} = |a_k| \operatorname{sgn} a_0$; then

$$|a_0| + 2\gamma |a_k| \leqq M.$$

If $\gamma \to \frac{1}{2} \sec \frac{\pi}{p+2}$, (2') follows.

4. Our method leads to a particularly simple proof of (1). We have $k > \frac{n}{3}$; let $p = 1$ and construct $\alpha(t)$ as in § 2. Now consider

$$g(x) = \{e^{ikx} f(x)\} * \frac{1}{2k} \alpha(2kx),$$

which is a trigonometric polynomial whose coefficients b_j satisfy $b_0 = a_{-k}$, $b_{2k} = a_k \gamma$, $b_{-2k} = a_{3k} \gamma = 0$ since $3k > n$; all the other b_j are also zero. Thus

$$|g(x)| = |a_{-k} + a_k \gamma e^{2ikx}| \leqq M,$$

and (1) follows by choosing x appropriately and letting $\gamma \to 1$.

5. To establish (1'), we let p be the largest odd integer such that $pk \leqq n$ and construct $\alpha(t)$ as in § 2. Then $g(x) = f(x) * \frac{1}{k} \alpha(kx)$ is a trigonometric polynomial whose coefficients b_j satisfy $b_{\pm k} = \gamma a_{\pm k}$; $b_{(2r+1)k} = 0$ if $r > 0$ or $r < -1$, since $b_{(2r+1)k} = a_{(2r+1)k}\, c_{2r+1}$; for $2r+1 \leqq p$, $c_{2r+1} = 0$, while when $2r+1 > p$, $(2r+1)k > n$ and $a_{(2r+1)k} = 0$. We now use one of the two fundamental formulas of VAN DER CORPUT and VISSER, namely

$$\max |g(x)| \geqq \max \Big| \sum_{j \equiv m \pmod s} b_j e^{ijx} \Big|;$$

taking $m = k$, $s = 2k$, we then have

$$\max |g(x)| \geqq \max |b_{-k} e^{-ikt} + b_k e^{ikt}| = 2|b_k| = 2\gamma |a_k|;$$

letting $\gamma \to \frac{1}{2} \sec \frac{\pi}{p+2}$, we obtain (1').

6. We now turn to (3) and its generalizations. We begin by observing that the trigonometric polynomial

$$h(x) = 1 + \tfrac{8}{3} \cos x - a \cos 2x$$

satisfies $|h(x)| \leqq 1 + a + \frac{1}{18a}$, and this maximum for $|h(x)|$ is smallest when $a = \frac{1}{3\sqrt{2}}$, when we have $|h(x)| \leqq 1 + \frac{1}{3}\sqrt{2}$. If $k > \frac{n}{2}$, let

$$g(x) = \frac{1}{2\pi} \int_0^{2\pi} F(x-t)\, h(kt)\, dt; \quad \ldots \ldots \quad (9)$$

then

$$|g(x)| = |a_0 + \sum (a_k e^{ikx} + \bar{a}_k e^{-ikx})| \leqq \max |h(x)| \cdot \frac{1}{2\pi} \int_0^{2\pi} |F(x)|\, dx \leqq (1 + \tfrac{1}{3}\sqrt{2}) I.$$

Choosing x appropriately, we have (3′).

More generally, let

$$h(x) = 1 + 2\gamma \cos x - a \cdot \cos 2x = 1 + a + \gamma \cos x - 2a \cos^2 x.$$

If $H(\gamma) = \min_a \max_x |h(x)|$, we have

$$|a_0| + 2\gamma |a_k| \leqq H(\gamma) I, \qquad k > \frac{n}{2}. \quad \ldots \quad (10)$$

For example, $H(1) = 1 + \sqrt{2}$ and (10) reduces to (3″).

7. In a similar way, let $h(x) = 1 + 2\gamma \cos x$ and estimate (9) by SCHWARZ's inequality. We obtain

$$|a_0| + 2\gamma |a_k| \leqq \left\{ \frac{1}{2\pi} \int_0^{2\pi} |F(x)|^2\, dx \right\}^{\frac{1}{2}} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |h(kt)|^2\, dt \right\}^{\frac{1}{2}}$$

$$= \sqrt{1 + 2\gamma^2}\, M_2, \qquad k > \frac{n}{2}.$$

If we take $\gamma = 1$, we have (5).

Similar inequalities involving other mean values of $F(x)$ are readily written down.

Mathematics. — *Distance geometry. Curvature in abstract metric spaces.* By J. HAANTJES. (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of April 26, 1947.)

The problem of treating the local properties of geometrical figures metrically, i.e. of developing a differential geometry without the use of coordinates, has recently received the attention of some mathematicians. The local properties of curves and surfaces have been studied in differential geometry almost entirely by means of analytic methods, which make it necessary to impose some conditions (e.g. conditions concerning differentiability) upon the entities involved. In a metrical theory the differential geometry can be freed from many of these restrictions, which are often geometrically unessential and serve merely to make the application of differential calculus possible [1]).

In this paper we are concerned with a metric treatment of the local property of curvature for arcs in abstract metric spaces. A metric definition of curvature has been given by MENGER [2]). A somewhat more general notion of curvature is due to ALT [3]). Both definitions impose, however, a euclidean notion of curvature upon general metric spaces (See § 2 and § 3). In the following a definition is given, which is free from this aesthetic imperfection. This definition of curvature proves to be more general than the notion of Menger curvature, though it can be shown that both definitions are equivalent for arcs in euclidean planes.

§ 1. *The abstract metric space.*

A metric space M is a set of abstract elements, called points, such that to each pair of elements p, q is attached a non-negative real number pq, called the *distance* of p and q, satisfying the conditions:

1. $pq = 0$ if and only if $p = q$.
2. $pq + qr \geq pr$ (triangle inequality).

An infinite sequence of points $\{p_\nu\}$ is said to have the limit p if and only if $\lim_{\nu \to \infty} p_\nu p = 0$. From the triangle inequality follows the continuity of the metric, which means that if $\{q_\nu\} \to q$ and $\{p_\nu\} \to p$ $\lim p_\nu q_\nu = pq$.

A neighbourhood $U(p, d)$ of the point p is defined as the set of points q for which $pq < d$.

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[1]) See L. M. BLUMENTHAL. Distance Geometries, A study of the development of abstract metrics, The University of Missouri studies, Vol. 13, nr 2 (1938).

[2]) K. MENGER. Untersuchungen über allgemeine Metrik, Vierte Untersuchung. Zur Metrik der Kurven. Math. Annalen **103**, 466—501 (1930). Referred to as: Menger IV.

[3]) F. ALT. Ueber eine metrische Definition der Krümmung einer Kurve, Vienna Dissertation (1931).

§ 2. *The Menger curvature.*

Because of the triangle inequality each set of three pairwise distinct points q, r, s in M is congruent with three points q', r', s' of the euclidean plane. The inverse of the radius of the circumscribed circle of the triangle with vertices q', r', s' can be expressed in terms of the distances of the points q', r', s', which equal the distances of the points q, r, s in M. This expression is according to a formula of elementary geometry

$$K(q,r,s) = \frac{\sqrt{(qr+rs+sq)\,(qr+rs-sq)\,(qr-rs+sq)\,(-qr+rs+sq)}}{qr \,.\, rs \,.\, sq} \quad (1)$$

and is called the curvature of q, r, s. Let K be a sub-set of M. Then MENGER defines the curvature at an accumulation point p of K as follows [4])

Definition. *The set K has, at an accumulation point p, a curvature $K(p) \geqq 0$ provided that, corresponding to each $\varepsilon > 0$ there is a $\delta > 0$, such that $|K(p) - K(q,r,s)| < \varepsilon$ for every triple q, r, s in the neighbourhood $U(p,\delta)$.*

This curvature is called the *Menger curvature*; throughout this paper it will be denoted by K_M. The relation of K_M with the classical curvature is expressed in [5])

Theorem 1. *If a curve in a euclidean plane has a Menger curvature K_M at a point p, then the classical curvature exists at p and is equal to K_M. The converse is not true.*

The second part of this theorem follows from the fact that the Menger curvature is a continuous function of p, which is not necessarily the case for the classical curvature.

PAUC [6]) answered the question which continua possess Menger curvature by proving

Theorem 2. *If a metric continuum K has a Menger curvature at one of its points p, then K is, in a neighbourhood of p, a rectifiable curve.*

§ 3. *The Alt curvature.*

The existence of a Menger curvature at p requires that $K(q,r,s)$ has a limit as the points q, r, s independently approach to p. ALT takes p as one of these points. This leads to the following

Definition [7]). *The set K has, at an accumulation point p, a curvature $K(p) \geqq 0$ provided that corresponding to each $\varepsilon > 0$ there is a $\delta > 0$ such that $|K(p) - K(p,r,s)| < \varepsilon$ for every pair of points r, s in the neighbourhood $U(p,\delta)$.*

This curvature is called the *Alt curvature* and is denoted by K_A. It can be shown

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[4]) MENGER IV, p. 480.

[5]) O. HAUPT and F. ALT. Zum Krümmungsbegriff, Ergebnisse eines mathematischen Kolloquiums (Wien) Hèft 3 (1932), 4—5.

[6]) C. PAUC. Courbure dans les espaces métriques, Atti Acad. di Lincei, Serie 6, 24 (1936) 109—115.

[7]) F. ALT. l.c. [3]).

Theorem 3. *For euclidean plane curves the existence of classical curvature implies the existence of Alt curvature and the two are equal. The converse is not true, at least when the classical curvature of the curve* $y = f(x)$ *is defined by the expression* $f''(1 + f'^2)^{-3/2}$ [8])

PAUC showed (see [6]))

Theorem 4. *A metric continuum K that has an Alt curvature at one of its points p is, in a neighbourhood of p, either the sum of finitely many rectifiable arcs, which have pairwise only the end-point p in common, or the sum of a denumerable infinity of such arcs, the diameters of which converge to zero.*

We see that both definitions impose a euclidean notion upon general metric spaces. This can however be avoided as will be shown in the next paragraph, where a third definition of curvature is given. This definition applies only to rectifiable arcs, though an extension to other sub-sets could be given. But for continua theorems 2 and 4 show that for Menger curvature and Alt curvature too we may confine ourselves to rectifiable arcs.

§ 4. *The curvature K.*

Let B be a rectifiable arc (topological image of a segment) in a metric space M. Then to each pair of points q, s of B are attached two numbers: the distance $d = qs$ and the length l of the arc qs, part of the arc B.

Definition. *The arc B has at a point p a finite curvature* $K(p) \geqq 0$, *provided that, corresponding to each* $\varepsilon > 0$ *there is a* $\delta > 0$, *such that*

$$\left| 4\,l \left(\frac{l-d}{l^3} \right)_{q,s} - K^2(p) \right| < \varepsilon. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

for every pair of points q, s of B in the neighbourhood $U(p, \delta)$.

In this paper this curvature will be denoted by K. It is natural to ask how this curvature, applied to arcs of spaces where a classical curvature can be defined (Riemannian spaces) compares with the classical curvature. The answer is given in

Theorem 5. *For rectifiable arcs in Riemannian spaces, which defining equations are differentiable to a sufficiently high order, the classical curvature is equal to K.*

In order to prove this theorem we introduce in the n-dimensional Riemannian space V_n polar coordinates with the point P (one of the points of the given arc) as pole. The coordinates of a point Q are determined by the geodesic distance z to P together with the unit vector i^x at P, which is tangent to the geodesic PQ. The polar coordinates of Q are defined as [9])

$$\xi^h = \delta^h_x \, i^x \, z.$$

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[8]) See K. MENGER. La géométrie des distances et ses relations avec les autres branches mathématiques. L'Enseignement Mathématique, Nos 5—6, 348—372 (1936).

[9]) J. A. SCHOUTEN. Einführung in die neueren Methoden der Differentialgeometrie I, Noordhoff, Groningen 1935, p. 101.

In a neighbourhood of P the arc is given by

$$\xi^h = a_1^h s + \frac{1}{2!} a_2^h s^2 + \frac{1}{3!} a_3^h s^3 + \dots$$

where s is the arc length. The classical curvature ϱ is defined as the length of the vector

$$\frac{d^2 \xi^h}{ds^2} + \Gamma_{ij}^h \frac{d\xi^i}{ds} \cdot \frac{d\xi^j}{ds}$$

at P. As a result of the vanishing of Γ_{ij}^h at P (see SCHOUTEN p. 103) this vector is a_2^k. The distance z to P is given by

$$z^2 = (g_{hi})_P \xi^h \xi^i = s^2 + (a_1 a_2) s^3 + \{\tfrac{1}{3}(a_1 a_3) + \tfrac{1}{4}(a_2 a_2)\} s^4 + \dots \quad (3)$$

where (uv) stands for $(g_{hi})_P u^h v^i$. Now

$$\frac{d\xi^h}{ds} = a_1^h + a_2^h s + \frac{1}{2!} a_3^h s^3 + \dots$$

is a unit vector. Using the formula (SCHOUTEN p. 138)

$$g_{hi} = (g_{hi})_P + \tfrac{1}{3} \xi^k \xi^j K_{kijh}$$

it follows

$$(a_1 a_2) = 0; (a_1 a_3) + (a_2 a_2) = 0.$$

So equation (3) gets the form

$$z^2 = s^2 - \tfrac{1}{12} \varrho^2 s^4 + \dots$$

from which we have — in harmony with (2) —

$$\varrho^2 = 4! \lim_{s \to 0} \frac{s - z}{s^3}$$

We will not investigate here which curves in Riemannian spaces possess a curvature K. Theorem 5 has only been proved to show that the classical curvature in Riemannian spaces is connected in the same way with the same limit as K is; therefore, in Riemannian spaces our definition of curvature cannot lead to paradoxical results.

Theorem 6. *The curvature K is a continuous function of p wherever it exists.*

Let us denote the expression $4!(l - d)\, l^{-3}$ for the pair of points q, r by $K^2(q, r)$. According to the definition of curvature there corresponds to each ε a δ such that

$$|K(p) - K(r, s)| < \frac{\varepsilon}{2} \quad \text{for} \quad r, s \subset U(p, 2\delta). \quad (4)$$

Let q be a point of the arc B contained in $U(p, \delta)$. There exists a neighbourhood $U(q, \delta_1)$ such that for each pair of arc points r, s contained in this neighbourhood we have

$$|K(q) - K(r, s)| < \frac{\varepsilon}{2}. \quad (5)$$

The intersection of $U(q, \delta_1)$ and $U(p, 2\delta)$ contains an infinity of arc points, q being an accumulation point of B. For two of these points both the inequalities (4) and (5) hold, from which it follows

$$|K(p)-K(q)|<\varepsilon \quad \text{for} \quad q \subset U(p, \delta).$$

This means however that $K(p)$ is a continuous function.

It is natural to investigate how the curvature K compares with the curvatures K_A and K_M. One of the results is given by the following theorem:

Theorem 7. *If both K_A (or K_M) and K exist, these curvatures are equal.*

Consider three points p, q, r of the arc, for which $pq = qr = d$. The points q and r are supposed to lie on the same side of the fixed point p. If the distance pr is denoted by a the following expression for the curvature K_A (or K_M) at p is obtained from (1)

$$K_A^2=\lim_{d\to 0}\frac{(2d+a)\,a^2\,(2d-a)}{d^4\,a^2}=\lim_{d\to 0}\frac{2d+a}{d}\cdot\frac{2d-a}{d^3}=\lim_{d\to 0}\frac{4}{d^2}\left(1-\frac{a^2}{4d^2}\right)$$

It follows that $a \to 2d$ as d tends to zero and therefore

$$K_A^2=4\lim_{d\to 0}\frac{2d-a}{d^3}. \qquad \ldots\ldots\ldots \quad (6)$$

Let the lengths of the arcs pq and qr be denoted by l_1 and l_2 respectively. According to the assumption that the curvature K exists at p, we have

$$\lim_{d\to 0}\frac{l_1-d}{l_1^3}=\lim_{d\to 0}\frac{l_2-d}{l_2^3}=\lim_{d\to 0}\frac{l_1+l_2-a}{(l_1+l_2)^3}=\frac{K^2}{4!}. \qquad \ldots \quad (7)$$

Now

$$\frac{l_1+l_2-a}{(l_1+l_2)^3}=\frac{l_1-d}{l_1^3}\cdot\frac{l_1^3}{(l_1+l_2)^3}+\frac{l_2-d}{l_2^3}\cdot\frac{l_2^3}{(l_1+l_2)^3}+\frac{2d-a}{d^3}\cdot\frac{d^3}{(l_1+l_2)^3}. \quad (8)$$

If it is taken into consideration that $l_1/d \to 1$ and $l_2/d \to 1$ as $d \to 0$ it is seen from (7) and (8)

$$\tfrac{1}{4}K^2=\lim_{d\to 0}\frac{2d-a}{d^3},$$

which gives for K exactly the same expression as (6) gives for K_A. So K and K_A (or K_M) if both existing are equal.

Further investigation to the relations existing between the curvatures K_M, K_A and K leads to the following theorem:

Theorem 8. *If the curvature K_A of the arc at p exists, it equals* $\lim_{q \leftarrow p} K(p, q)$.

It is easily seen that the preceding theorem 7 is an immediate consequence of theorem 8. In order to prove this latter theorem we consider the triangle in the euclidean plane corresponding to the arc points p, q, r, i.e.

the triangle with the sides pq, qr and pr. Let $e(p, q, r)$ stand for the sum of the smallest two angles of this triangle. PAUC [10]) has shown that if K_A exists at p the quantity $e(p, q, r) \to 0$ as both q and r approach the point p. Thus there is a neighbourhood $U(p, \delta_1)$ such that for each pair q, r contained in this neighbourhood $e(p, q, r) < \frac{\pi}{2}$ which means that the triangle is obtuse-angled. If at the same time two of the sides are equal (b), the length of the third side will exceed $b\sqrt{2}$ (*property P*). By means of this property it can be shown that the distance d of the arc points q ($q \subset U(p, \delta_1)$) to p will increase steadily if q traverses the arc B starting from the point p. For d is a continuous function of q and thus of the parameter on the arc. Suppose this function is not monotonic. Then it has relative maxima or minima and it is possible to find two different points which have the same distance to p but a smaller mutual distance, contrary to property P. Hence this cannot occur. It is equally impossible that d remains constant for a while, so d will increase.

Another consequence of the existence of K_A at p follows immediately from the definition. There is a $\delta_2 > 0$ such that for each pair of arc points q, r in $U(p, \delta_2)$

$$K_A^2 (1-\eta) < K^2 (p, q, r) < K_A^2 (1+\eta). \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Consider a point q of B in the neighbourhood $U(p, \delta)$, where $\delta < \delta_1$ and $\delta < \delta_2$. From the proof of theorem 4 it follows that the arc pq (part of B) is rectifiable. Let l be its length and d the distance of the end points. A finite subset $(p_1, \ldots, p_N)$ of the arc pq (B') is called an ε-subset provided that each pair of two consecutive points has the distance ε and $pp_1 \leq \varepsilon$ and $p_N q \leq \varepsilon$. It has been proved by MENGER [11]) that corresponding to each positive ζ there is a positive number ε_0 such that for every ε-subset $(\varepsilon \leq \varepsilon_0)$

$$l \geqq p p_1 + \ldots + p_{N-1} p_N > l - \zeta. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The number N depends of course on ε.

Let r and s be two points of B' such that in traversing B' from p to q the point r is encountered first. Then as we have seen $ps > pr$. Suppose $pr \geq rs$. Putting for shortness $ps = a$, $pr = b$, $rs = c$, $K(p, r, s) = R^{-1}$ we obtain from some formulae of elementary geometry

$$a = b\sqrt{1 - \frac{c^2}{4R^2}} + c\sqrt{1 - \frac{b^2}{4R^2}}. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

By using the first of the following inequalities

$$\sqrt{1-\alpha} \leqq 1 - \tfrac{1}{2}\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$\sqrt{1-\alpha} \geqq 1 - \tfrac{1}{2}\alpha - \alpha^2 \text{ for } \alpha < \tfrac{1}{2} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$\sqrt{1-\alpha} \geqq 1 - \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

[10]) See footnote [6]).

[11]) MENGER IV, p. 469.

it is readily seen from (11)

$$a < b + c - \frac{1}{8R^2} cb^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

whereas the second inequality (13) leads to

$$a > b + c - \frac{1}{8R^2}(bc^2 + cb^2) - \frac{1}{8R^4} cb^4 . \quad . \quad . \quad . \quad (16)$$

This it is true is only right if b and c are small enough but we may choose δ smaller than $K_A(1+\eta)$ and then it is correct. Yet another inequality will be used. It is obtained from (11) and (14) and runs as follows

$$a > (b+c)\left(1 - \frac{dc}{4R^2}\right) . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

for b and c are smaller than d, d being the distance pq.

The inequalities (15), (16) and (17) are now applied succesively to the sets pp_ip_{i+1} $(i = 1, \ldots, N-1)$ where the subset p_i is supposed to be an ε-subset (with $pp_1 = \varepsilon$) satisfying (10). Let us start with the third inequality. It will be proved that

$$pp_j > j\,\varepsilon\{1 - \tfrac{1}{8}(j+1) K_A^2 (1+\eta)\, d\,\varepsilon\}. \quad . \quad . \quad . \quad . \quad (18)$$

This is true for $j = 1$. Suppose it is true for $j = m$ and let us prove it for $j = m+1$. From (17) it follows

$$\begin{aligned} pp_{m+1} &> (pp_m + \varepsilon)\{1 - \tfrac{1}{4} d\,\varepsilon\, K_A^2 (1+\eta)\} \\ &> (m+1)\,\varepsilon\{1 - \tfrac{1}{8} m\,\varepsilon\, d\, K_A^2 (1+\eta)\}\{1 - \tfrac{1}{4}\varepsilon\, d\, K_A^2 (1+\eta)\} \\ &> (m+1)\,\varepsilon\{1 - \tfrac{1}{8}(m+2)\,\varepsilon\, d\, K_A^2 (1+\eta)\} \end{aligned}$$

which is exactly the inequality (18) for $j = m+1$.

In a similar way it can be proved from (15)

$$pp_m \leqslant m\,\varepsilon - \tfrac{1}{8}\varepsilon^3 K_A^2 (1-\eta) \sum_{\nu=1}^{m-1} \nu^2 + \tfrac{1}{64} K_A^4 (1+\eta)^2\, d\,\varepsilon^4 f(m) \quad . \quad (19)$$

with $f(m) = \frac{1}{2}m^4 - \frac{1}{3}m^3 - \frac{1}{2}m^2 + \frac{1}{3}m$.

For we know it is true for $m = 1$. Suppose it holds for m. Then it follows from (15) under consideration of the inequality (18)

$$\begin{aligned} pp_{m+1} &< (m+1)\,\varepsilon - \tfrac{1}{8}\varepsilon^3 K_A^2 (1-\eta) \sum_{\nu=1}^{m-1} \nu^2 + \tfrac{1}{64} K_A^4 (1+\eta)^2\, d\,\varepsilon^4 f(m) \\ &\quad - \tfrac{1}{8} K_A^2 (1-\eta)\,\varepsilon^3 m^2 \{1 - \tfrac{1}{8}(m+1) K_A^2 (1+\eta)\, d\,\varepsilon\}^2 \\ &< (m+1)\,\varepsilon - \tfrac{1}{8}\varepsilon^3 K_A^2 (1-\eta) \sum_{\nu=1}^{m} \nu^2 + \tfrac{1}{64} K_A^4 (1+\eta)^2\, d\,\varepsilon^4 f(m+1) \end{aligned}$$

since

$$f(m+1) = f(m) + 2m^2(m+1).$$

Application of the inequality (16) to the sets pp_ip_{i+1} leads in much the same way to

$$pp_m \geqslant m\,\varepsilon - \tfrac{1}{8} K_A^2 (1+\eta)\,\varepsilon^3 \sum_{\nu=1}^{m-1} (\nu^2 + \nu) - \tfrac{1}{8} K_A^4 (1+\eta)^2\,\varepsilon^5 \sum_{\nu=1}^{m-1} \nu^4 \quad (20)$$

Again it is true for $m = 1$. If it is true for m we have as a result from (16)

$$pp_{m+1} \geqq (m+1)\,\varepsilon - \tfrac{1}{8} K_A^2 (1+\eta)\,\varepsilon^3 \sum_{\nu=1}^{m-1} (\nu^2+\nu) - \tfrac{1}{8} K_A^4 (1+\eta)^2 \varepsilon^5 \sum_{\nu=1}^{m-1} \nu^4 +$$
$$- \tfrac{1}{8} K_A^2 (1+\eta)(\varepsilon^3 m + \varepsilon^3 m^2) - \tfrac{1}{8} K_A^4 (1+\eta)^2 \varepsilon^5 m^4$$

since

$$pp_m \leqq m\,\varepsilon$$

according to the triangle inequality. But this shows that (20) holds still if m is replaced by $m+1$.

We turn back to (19). Putting $m = N$ it gives

$$pp_N \leqq N\varepsilon - \tfrac{1}{24} K_A^2 (1-\eta)(N\varepsilon)^3 \left\{1 - \frac{3}{2N} + \frac{1}{2N^2}\right\} +$$
$$+ 2^{-7} K_A^4 (1+\eta)^2 d\,(N\varepsilon)^4 \left\{1 - \frac{2}{3N} - \frac{1}{N^2} + \frac{1}{3N^3}\right\}$$

So we have in connection with (10)

$$d \leqq pp_N + \varepsilon \leqq \varepsilon + l - \tfrac{1}{24} K_A^2 (1-\eta)(l-\zeta)^3 \left\{1 - \frac{3}{2N} + \frac{1}{2N^2}\right\} +$$
$$+ 2^{-7} K_A^4 (1+\eta)^2 d l^4 \left\{1 - \frac{2}{3N} - \frac{1}{N^2} + \frac{1}{3N^3}\right\}$$

Now after fixing the point q, the quantities ζ and ε can be chosen arbitrarily small. Thus we have

$$d \leqq l - \tfrac{1}{24} K_A^2 (1-\eta)\, l^3 + 2^{-7} K_A^4 (1+\eta)^2 d l^4$$

from which it follows

$$K^2(p,q) = 4!\,\frac{l-d}{l^3} \geqq K_A^2 (1-\eta) - 2^{-7}\, 4!\, K_A^4 (1+\eta)^2 d l. \quad . \quad (21)$$

Another inequality is obtained from (20) by putting $m = N$. Since

$$\sum_{\nu=1}^{N-1} \nu^4 < N^5 \text{ and } pp_N \leqq d$$

we have again in connection with (10)

$$d \geqq l - \zeta - \tfrac{1}{24} K_A^2 (1+\eta)\, l^3 \left(1 - \frac{1}{N^2}\right) - \tfrac{1}{8} K_A^4 (1+\eta)^2 l^5$$

which leads to

$$d \geqq l - \tfrac{1}{24} K_A^2 (1+\eta)\, l^3 - \tfrac{1}{8} K_A^4 (1+\eta)^2 l^5$$

or

$$K^2(p,q) \leqq K_A^2 (1+\eta) + 3\, K_A^4 (1+\eta)^2 l^2. \quad . \quad . \quad . \quad . \quad (22)$$

From (21) and (22) together it follows that $|K^2(p,q) - K_A^2|$ is smaller than an arbitrarily given ε' provided that pq is sufficiently small ($d < \delta'$), in other words

$$K_A = \lim_{q \to p} K(p,q) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

which proves theorem 8.

So the existence of the Alt curvature implies the existence of the limit (23). The existence of a curvature K at a point p requires however that the function $K(q, r)$ has a limit as the two points q, r independently approach the point p. This is more than the existence of the limit (23). A curve may possess Alt curvature at a point p without having a curvature K. For according to theorem (6) the curvature $K(p)$ is a continuous function, which is not the case for the Alt curvature. E.g. the plane curve $(y = x^4 \sin\frac{1}{x},\ x \neq 0;\ y = 0, x = 0)$ in the euclidean plane has an Alt curvature, but not a curvature K at the origin.

If however the Menger curvature exists at p the inequalities (9) are not only true for $K^2(p, q, r)$ but even for $K^2(q, r, s)$, providing that the triple q, r, s is contained in $U(p, \delta_2)$. As a consequence of this the inequalities (21) and (22) are true not only for the pair p, q, but for every pair of points q, r contained in $U(p, \delta)$. So we see that the limit of $K(q, r)$ exists if q and r approach p, which means that the curvature K exists and

$$K_M = \lim_{\substack{q \to p \\ r \to p}} K\,(q, r) = K. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Hence we have the theorem

Theorem 9. *The existence of the Menger curvature of an arc implies the existence of the curvature K and the two are equal.*

§ 5. *Further investigations as to how the curvature K compares with K_M.*

In § 4 it has been shown that the existence of K_M implies the existence of the curvature K so far as arcs in metric spaces are concerned.

The converse of this theorem is not necessarily true. A curve may possess a curvature K at a point without having a Menger curvature or even an Alt curvature. We give an example.

Consider a space M formed by the points of the interval $0 \leqq x \leqq 1$, where the distance xy of two points x, y is defined by

$$x\,y = t - \frac{1}{3!}\,t^3 + \frac{1}{4!}\,t^4 \sin\frac{1}{t} \equiv f(t) \qquad t = |x - y|. \quad . \quad . \quad (25)$$

The first thing to show is that M is indeed a metric space, thus that the triangle inequality is satisfied. Let x, y, z be three points of M and $y - x = u,\ z - y = t,\ z - x = u + t\ (u > 0,\ t > 0)$.

Since

$$\frac{df(t)}{dt} = 1 - \tfrac{1}{2}\,t^2 + \tfrac{1}{6}\,t^3 \sin\frac{1}{t} - \tfrac{1}{24}\,t^2 \cos\frac{1}{t} > 1 - \tfrac{1}{2} - \tfrac{1}{6} - \tfrac{1}{24} > 0$$

the distance will increase with t. So $xz > xy$ and $xz > yz$ and it is sufficient to show that $xy + yz \geq xz$. According to (25)

$$xy + yz - xz = \tfrac{1}{2}\,ut(u + t) - \frac{1}{4!}\{\varphi(u + t) - \varphi(u) - \varphi(t)\};\ \varphi(u) = u^4 \sin\frac{1}{u} \quad (26)$$

Suppose $u \leq t$ (this may be supposed because the expression is symmetrical in u and t). Then

$$|\varphi(u+t)-\varphi(t)-\varphi(u)| = |u\varphi'(t+\theta u)-\varphi(u)|$$
$$< 4u(u+t)^3 + u(u+t)^2 + u^4 < (8+2+1)\,ut(u+t)$$

from which it follows that the expression (26) is positive.

The metric of M is topologically equivalent to the euclidean metric, hence M is an arc.

Next it will be shown that the arc is rectifiable. Let E' ($p_0 = x$, $p_1, \ldots, p_N = y$) be a finite subset of the arc with end points x and y and $|p_{i+1} - p_i| = \varepsilon = t/N$, $t = |x-y|$. Then

$$\lambda(E') \equiv \sum_{i=1}^{N} p_{i-1}p_i = t\left(1 - \frac{1}{3!}\varepsilon^2 + \frac{1}{4!}\varepsilon^3 \sin\frac{1}{\varepsilon}\right)$$

which expression converges to t if $\varepsilon \to 0$. Now the length $l(x, y)$ of the arc is defined as the least upper bound of the numbers $\lambda(E)$ if E describes all finite subsets of the arc. So we see $l(x, y) \geq t$ for $\lambda(E') \to t$, but from (25) it is clear that for any finite subset E the number $\lambda(E) < t$. Hence

$$l(x, y) = t.$$

If d denotes the distance of x and y, we have therefore

$$K^2(x, y) \equiv 4!\,\frac{l-d}{l^3} = 4 - t\sin\frac{1}{t}.$$

Hence

$$|K(x, y) - 2| < \eta \quad \text{for} \quad |x-y| < 2\eta.$$

So the curvature K exists at every point of the arc and is equal to 2. The arc has constant curvature.

In order to show that the Alt curvature does not exist we consider again the points x, y and z ($y-x=u$, $z-y=t$, $z-x=u+t$) and study the function $K(x, y, z)$ as x and z approach the fixed point y. According to (1)

$$K^2(x, y, z) = \frac{(d+d_1+d_2)(d-d_1+d_2)(d+d_1-d_2)(d_1+d_2-d)}{d^2 d_1^2 d_2^2}$$

where

$$d_1 = f(u), \quad d_2 = f(t), \quad d = f(u+t).$$

Put

$$p = \frac{d+d_1+d_2}{u+t} = 2 - \frac{1}{3!}\left\{(u+t)^2 + \frac{u^3}{u+t} + \frac{t^3}{u+t}\right\} + \frac{1}{4!}\left\{\frac{\varphi(u+t)+\varphi(u)+\varphi(t)}{u+t}\right\}$$

$$q = \frac{d-d_1+d_2}{t} = 2 - \frac{1}{3!}\{2t^2 + 3u(u+t)\} + \frac{1}{4!}\left\{t^3\sin\frac{1}{t} + \varphi'(u+\theta_1 t)\right\}$$

$$r = \frac{d+d_1-d_2}{u} = 2 - \frac{1}{3!}\{2u^2 + 3t(u+t)\} + \frac{1}{4!}\left\{u^3\sin\frac{1}{u} + \varphi'(t+\theta_2 u)\right\}$$

$$s = \frac{d_1+d_2-d}{ut(u+t)} = \tfrac{1}{2} - \frac{1}{4!}\left\{\frac{\varphi(u+t)-\varphi(u)-\varphi(t)}{ut(u+t)}\right\} = \tfrac{1}{2} - \frac{1}{4!}\psi(u, t).$$

It is easily seen that

$$\lim p = \lim q = \lim r = 2$$

as x and z independently approach to y. Thus the Alt curvature will exist if and only if $\lim s$ exists.

For $u = t$

$$\psi(u, u) = \frac{\varphi(2u) - 2\varphi(u)}{2u^3} = 8u \sin\frac{1}{2u} - u \sin\frac{1}{u}$$

which expression goes to zero if $u \to 0$. For $u^{-1} = 2n\pi$, $t^{-1} = 2n^3\pi$ however

$$\psi(u, t) = -\frac{(u+t)^3}{ut} \sin\frac{2\pi n}{n^2+1} = -\frac{(n^2+1)^3}{2\pi n^5} \sin\frac{2\pi n}{n^2+1}$$

and the limit of the expression for $n \to \infty$ is -1. Hence the limit of $\psi(u, t)$ as u and t independently approach zero does not exist from which it follows that the arc has no Alt curvature. The Menger curvature does not exist either because the existence of the Menger curvature implies the existence of the Alt curvature.

This example shows that the notion of curvature K is more general than the notion of Menger curvature. It may of course be possible that for certain metric spaces the definitions are equivalent. Without giving a set of necessary and sufficient conditions for such spaces it is shown in the following that for arcs in a euclidean plane both definitions are equivalent. Because of theorem (9) it will be sufficient to prove that in this space the existence of the curvature K implies the existence of the Menger curvature. As a first result the following theorem will be proved.

Theorem 10. *An arc in a euclidean space with a curvature K at a point p has a tangent in a neighbourhood of this point.*

Let q and r be two arc points lying on the same side of p such that the arcs pq, qr and pr have the lengths l, l_1 and $l + l_1$ respectively. Suppose further that q and r lie in a neighbourhood $U(p, \delta)$ with the property that for each pair of arc points s, t in $U(p, \delta)$

$$|K^2(r, s) - K^2| < 4l\varepsilon \quad \ldots\ldots\ldots \quad (27)$$

where ε is an arbitrarily chosen number. Putting $pq = c$, $qr = a$, $pr = b$, we have from (27)

$$\left.\begin{aligned} c &= l - \sigma l^3 + \eta_1 l^3, \qquad \sigma = \frac{K^2}{4l} \\ a &= l_1 - \sigma l_1^3 + \eta_2 l_1^3 \\ b &= l + l_1 - \sigma(l + l_1)^3 + \eta_3 (l + l_1)^3 \end{aligned}\right\} \quad \ldots\ldots \quad (28)$$

with $|\eta_i| < \varepsilon$. Let α denote the angle qpr. Then

$$4 \sin^2 \tfrac{1}{2}\alpha = \frac{(b + a - c)(c + a - b)}{bc} \quad \ldots\ldots \quad (29)$$

Suppose $l_1 \leq l$. In this case

$$b + a - c < 2l + 10\varepsilon l^3,$$
$$c + a - b < (6\sigma + 10\varepsilon) l^3.$$

Hence

$$4\sin^2\tfrac{1}{2}\alpha < \frac{(6\sigma+10\varepsilon)(2+10\varepsilon l^2)l^2}{(1-\sigma l^2-\varepsilon l^2)(1-8\sigma l^2-8\varepsilon l^2)}. \quad . \quad . \quad . \quad (30)$$

As the angle $\frac{1}{2}\alpha$ is acute

$$\tfrac{1}{2}\alpha < \frac{\pi}{2}\sin\tfrac{1}{2}\alpha$$

From this inequality and (30) it is seen that if l is sufficiently small $(l<\lambda)$

$$\alpha < A\,l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

where A is a fixed number. Consider the points q_l $(q_0 = q)$ on the arc pq for which the lengths of the arcs pq_l are equal to $2^{-l}l$. If the angle $q_l\,p\,q_{l-1}$ is denoted by α_l we have from (31)

$$\alpha_l < A\,l\,2^{-l}$$

as a result of which the angle $q_l pq$ $(=\varphi_l)$ satisfies the inequality

$$\varphi_l \leqq \alpha_1+\alpha_2+\ldots+\alpha_l < A\,l\left(\tfrac{1}{2}+\tfrac{1}{4}+\ldots+\frac{1}{2^l}\right) < A\,l.$$

Let s be an arbitrarily chosen point on the arc pq. Then s lies on one of the arcs q_lq_{l+1} and so we have

$$\angle s\,p\,q \leqq \varphi_l + \angle s\,p\,q_l < A\,l\left(\tfrac{1}{2}+\ldots+\frac{1}{2^l}+\frac{1}{2^l}\right) = A\,l.$$

Thus corresponding to each $\xi>0$ there is a $\varrho>0$ such that for every pair of points q, s of the arc on the same side of p for which each of the distances pq and ps is less than ϱ, the angle spq is smaller than ξ. In a similar way it can be shown that the angle $spq \to 2\pi$ if s and q approach p from different sides. Hence the tangent at p exists, but not only at p for in the above account we may replace p by any other point in the neighbourhood $U(p,\delta)$, which completes the proof of theorem 10.

In the following we confine ourselves to arcs in a euclidean *plane* and it will be proved

$$\varphi_l = \alpha_1+\ldots\alpha_l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

It is sufficient to prove that $\varphi_2 = \alpha_1+\alpha_2$. As will be shown later in this paper $\alpha_2<\alpha_1$. So there are only two possibilities $\varphi_2=\alpha_1+\alpha_2$ and $\varphi_2=\alpha_1-\alpha_2$. Suppose $\varphi_2=\alpha_1-\alpha_2$. If a point r traverses the arc from q to q_1, the angle $r\,p\,q$ is a continuous function of the parameter and this function runs from 0 to α_1. So it passes the value $\varphi_2=\alpha_1-\alpha_2$. (The angle cannot pass π because it is acute if l is small enough). Let r be the point on the arc $q\,q_1$ for which $\angle r\,p\,q=\varphi_2$. Then the points p_1q_2 and r lie on a straight line and we have

$$(p\,q_2+q_2\,r-p\,r)(p\,q_2-q_2\,r+p\,r)(-p\,q_2+q_2\,r+p\,r)=0. \quad . \quad (32)$$

Let u be the arc length of pq_2 and t the length of the arc q_2r. Thus $u=\frac{1}{4}l$, $\frac{1}{2}l<t<l$ and

$$\begin{aligned} p\,q_2+q_2\,r-p\,r &> 3\sigma\,u\,t\,(u+t)-\varepsilon\{u^3+t^3+(u+t)^3\} \\ &> 18\,\sigma\,u^3-190\,\varepsilon\,u^3>0 \end{aligned}$$

if ε has been chosen small enough. It is equally impossible that one of the other factors in (32) vanish. So $\varphi_2 = \alpha_1 - \alpha_2$ cannot occur and there remains only the other possibility $\varphi_2 = \alpha_1 + \alpha_2$.

By substituting the value of a, b and c for $l_1 = l$ given in (28) into the formula (29) it is seen that corresponding to a ε' there is a δ' such that

$$\left| \frac{\alpha^2}{l^2} - 6\sigma \right| < \varepsilon' \text{ for } l < \delta'$$

Hence

$$2^{-i} l \sqrt{6\sigma - \varepsilon'} < a_i < 2^{-i} l \sqrt{6\sigma + \varepsilon'}$$

from which it follows firstly that (again if ε' is small enough) $a_{i+1} < a_i$ (already used for proving (31)) and secondly that the angle φ between pq and the tangent at p satisfies the inequalities

$$l \sqrt{6\sigma - \varepsilon'} < \varphi < l \sqrt{6\sigma + \varepsilon'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

The same is true however for the angle between rs and the tangent at r providing that r and s are contained in $U(p, \delta)$ and the length of the arc rs is smaller than δ'. Let q, r, s be three points in a neighbourhood $U(p, d)$ where d is chosen smaller than δ and so that the length of the arc between two arc points in $U(p, d)$ is smaller than δ'. The lengths of the arcs qr, rs and qs denoting by l_1, l and $l + l_1$ respectively it follows from (33) that the angle $r\,q\,s$ (β) satisfies the inequalities

$$\beta < (l + l_1) \sqrt{6\sigma + \varepsilon'} - l_1 \sqrt{6\sigma - \varepsilon'} < l \sqrt{6\sigma + \varepsilon'} + l_1 \frac{\sqrt{2}\,\varepsilon'}{\sqrt{6\sigma}}$$

$$\beta > l \sqrt{6\sigma - \varepsilon'} - l_1 \frac{\sqrt{2}\,\varepsilon'}{\sqrt{6\sigma}}.$$

Now suppose $l_1 \leq l$ (For $l_1 \geq l$ one proceeds in the same way with the angle $r\,s\,q$).

The inverse of the radius of the circumscribed circle of the triangle with vertices q, r, s is given by

$$K(q, r, s) = \frac{2 \sin \beta}{r s} = \frac{2 \sin \beta}{l} \cdot \frac{l}{r s} \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Since $\beta > \sin \beta > \beta - \frac{\beta^3}{3!}$, $K = 2\sqrt{6\sigma}$ (see (28)) it is seen from (34) that

$$\lim K(q, r, s) = K$$

as q, r and s independently approach p. So the Menger curvature exists and equals K. The result is

Theorem 11. *For arcs in a euclidean plane the notion of curvature K is equivalent with the notion of Menger curvature.*

It should be observed that this proof is only valid for a euclidean plane, not for euclidean spaces of higher dimension, though it is natural to conjecture that theorem 11 will be true for every euclidean space, whatever its dimension may be.

Biology. — *Mathematics of pollen diagrams.* I. By J. WESTENBERG. (Communicated by Prof. M. W. WOERDEMAN.)

(Communicated at the meeting of March 29, 1947.)

In biological work, one of the most usual problems for statistical treatment is to state significant differences between two frequency distributions. In treating the question, the investigator used to believe, and is still inclined to suppose that the data will follow an idealized form, obtained by imposing certain properties on the frequency curves. In a general way, the trend of thought in this procedure may be represented as follows.

The probability, that a variate n should have a value in the range $(n, n + dn)$ is supposed to be expressible by a function $f(n)$, the so-called frequency function, in the form

$$dF = f(n)dn$$

specifying the distribution of n in the population, from which our series of data is regarded as a random sample. If in the frequency function of n, one or more parameters $\theta_1, \theta_2, \theta_3 \ldots$ are introduced, we have

$$dF = f(n, \theta_1, \theta_2, \theta_3 \ldots)dn$$

where f now specifies only the type of the frequency function.

The supposition usually made, implies the special case, in which the population is specified by the "normal" or Gaussian frequency distribution, in other words the probability that the variate should have a value in the range $(n, n + dn)$ is expressed by the formula

$$dF(n) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(\tilde{n}-n)^2}{2\sigma^2}} \cdot dn$$

In this frequency distribution two symbols appear as parameters viz. $\tilde{n}$ the (arithmetic) mean of the population and σ_n (or simply σ) the standard deviation of the variate, measuring in the same units the extent to which the individual values are scattered. In all populations (normal or not), σ^2 is defined as the mean square of all the deviations from $\tilde{n}$, occurring in the population.

Now, if a sample is taken from a normal population, the mean $\bar{n}$ of the sample is in its turn also distributed according to a normal distribution around the same origin $\tilde{n}$, and its standard deviation $\sigma_{\bar{n}}$ is determined by the relation

$$\sigma_{\bar{n}}^2 = \frac{\sigma^2}{k}$$

k representing the number of individuals included in our sample.

The difference between two means of samples taken from independent normal distributions is again distributed according to a normal distribution,

the square of its standard deviation being equal to the sum of the squared deviations of the two means of our samples:

$$\sigma^2_{\bar{n}'-\bar{n}} = \sigma^2_{\bar{n}'} + \sigma^2_{\bar{n}}$$

$\bar{n}'$ and $\bar{n}$ representing the (arithmetic) means of the two samples respectively.

Mathematicians have proved, that for any normal variate, and hence for $\bar{n}' - \bar{n}$, the probability of obtaining a + or — deviation greater than $2\frac{1}{2}$ times its own standard deviation, is as small as 0,0124. From such a deviation from zero, we may conclude, that the difference $\bar{n}' - \bar{n}$ is unlikely to have had its origin at zero, or in other words that it is unlikely that the two populations should have had equal means.

In testing the significance of a given + (or —) difference $\bar{n}' - \bar{n}$ we are interested in the chances of obtaining + (or —) deviations only. As the normal distribution is symmetrical, the + and — chances are equal, and so the probability of a + (or —) deviation greater than $2\frac{1}{2}$ times its own standard deviation is just half of 0,0124. From this consideration it appears to be still more unlikely, that a given difference $\bar{n}' - \bar{n}$ with a $|\bar{n}' - \bar{n}|$ exceeding $2\frac{1}{2}$ times its standard deviation, should have had its origin at zero. Hence we are led to conclude, that $\tilde{n}' > \tilde{n}$, in case $\bar{n}' - \bar{n}$ is + (or that $\tilde{n}' < \tilde{n}$ in case $\bar{n}' - \bar{n}$ is —), which implies the significance of $\bar{n}' - \bar{n}$.

A fundamental difficulty is experienced in applying the test described above, because we cannot know exactly the parameters involved in the formula. The parameters are characters of the population, and if we could know their exact values, we would know more than any sample from the population could tell us. Samples only permit us to make estimates of these values. Technically, these estimates are denoted as statistics. These statistics are of course calculated from the observations, and are in their turn distributed around the parameters. The larger the series of observations, the closer the statistics will be distributed around the parameters, these being the limits to which the statistics approach as the series of observations are extended to infinity [1]). For this reason the difficulty becomes especially inconvenient in the treatment of small series.

The parameters being the limits of the statistics, the best estimation of $\tilde{n}$ can be proved to be the (arithmetic) mean of the sample

$$\bar{n} = \frac{n_1 + n_2 + n_3 + \ldots n_k}{k} = \frac{\Sigma n}{k}$$

and s', the estimate of the standard deviation is defined by placing

$$s'^2 = \frac{\Sigma x^2}{k-1}$$

x being the deviation from the mean $\bar{n}$ of the sample.

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[1]) Here we properly deal with stochastic limits, which means, that even in the case of large series of data, great deviations may incidentally occur.

Apart from this, we think of the relations

$$\sigma^2_{\bar{n}} = \frac{\sigma^2_n}{k} \text{ and}$$

$$\sigma^2_{\bar{n}'-\bar{n}} = \sigma^2_{\bar{n}'} + \sigma^2_{\bar{n}}$$

and hence the quantity $s'_{\bar{n}'-\bar{n}}$ is defined by

$$s'^2_{\bar{n}'-\bar{n}} = \frac{\Sigma x'^2}{k'(k'-1)} + \frac{\Sigma x^2}{k(k-1)}$$

has the expectation value $\sigma_{\bar{n}'-\bar{n}}$, so that the test may finally be specified by the well known formula

$$|\bar{n}' - \bar{n}| > 2\tfrac{1}{2} \sqrt{\frac{\Sigma x'^2}{k'(k'-1)} + \frac{\Sigma x^2}{k(k-1)}}$$

x' and x representing the deviations from the (arithmetic) means of the two samples respectively,

k' and k representing the numbers of individuals, included in the two samples respectively.

In spite of the fact, that this test is only approximately valid for large normal samples, it has been applied successfully in many a case, and it does not mean depreciating the merits of early workers, to state that an essential progress has been made later on, by designing a special method for the treatment of small samples taken from normal populations. The application of this method resolves itself into the calculation of a statistic, indicated with the symbol t.

The statistic t belonging to some normal statistic, is defined as the ratio of this statistic to its own standard deviation, as estimated from observations. In our problem, we have to apply the calculation to the statistic $\bar{n}' - \bar{n}$, thus computing

$$t_{\bar{n}'-\bar{n}} = \frac{\bar{n}' - \bar{n}}{s'_{\bar{n}'-\bar{n}}}$$

It is clear, that any normal statistic, distributed around zero, will yield a t, that is likewise distributed around zero. From the frequency distribution of t, the limits have been calculated, beyond which t is unlikely to have had its origin at zero, the number of observations being taken into account. If this so-called t-test is applied to the difference in means of two samples from normal populations with equal variances, we are able to decide, whether it is unlikely, that the two populations should have had equal means.

The values, which a t is expected not to exceed in a certain number of cases out of 100, are tabulated to the third decimal. One of the entries of the t-table indicates the relative frequencies and the other entry is based on the number of available data (giving the so-called number of "degrees of freedom").

The condition for a difference, to be significant, is:

$$F(t) = \int_{-\infty}^{t} f(t) \cdot d(t) \geqslant 0{,}99$$

The practical meaning of this statement is: If we always use this criterion, the percentage of correct statements can in the long run be expected to be very close to 99 %. It goes without saying, that the choice of 0,99 is an arbitrary one.

It is a remarkable fact, that not a single parameter enters into the formula, specifying $f(t)$, and this constitutes the mathematical beauty of the *t*-test. When applying this method, one need not use any statistic as an approximating substitute for a parameter, and from this it is evident, that the *t*-test is a sound method, even for the treatment of small samples (taken from normal populations).

A serious restriction, however, is entailed by the assumption of the normal law, on which both of the given methods are based. Many authors have emphasized, that in biology this law can by no means be recognised as the leading principle in frequency distributions, since only few predictions can be made on the properties of distributions characterising biological phenomena (cf. E. F. DRION a.o.). For the treatment of such data special methods have been designed by K. PEARSON, F. Y. EDGEWORTH and by J. C. KAPTEYN & M. J. VAN UVEN. Still, in spite of these considerations, we must guard against an underestimation of the applicability of the methods based on the normal law, for in many cases, as for instance in counting bacteria, blood corpuscules, pollen grains or seeds, the obtained frequency distributions are mainly imposed by well defined sampling techniques. The law, that is fundamental for the distribution of such counting data is known as the binomial or Bernoullian frequency distribution, which may approach the normal law.

The binomial frequency distribution is expressed by the formula

$$P_n = \frac{N!}{n!\,(N-n)!} \cdot p^n \cdot (1-p)^{N-n}$$

P_n expresses the probability, that a certain event should happen exactly n times in N trials, provided the probability of the occurrence at a single trial be equal to p, and each happening be independent of previous events.

The fact, that in certain cases, the binomial frequency distribution approaches the normal law, greatly enhances the applicability of those statistical methods, based on this law. Nevertheless the application of the ordinary methods may be somewhat unsatisfactory.

The English statistician R. A. FISHER (1935) has given a special method for testing the significance of the difference in binomial variates. This method will prove to meet exactly our needs in many a case of biological routine work.

The probability, that a binomial variate is equal tho the integer $n + \Delta$ is:

$$P_{n+\Delta} = \frac{N_1!}{(n+\Delta)!\,(N_1-n-\Delta)!} \cdot p^{n+\Delta} \cdot (1-p)^{N_1-n-\Delta}$$

Similarly

$$P_{n-\Delta} = \frac{N_2!}{(n-\Delta)!\,(N_2-n+\Delta)!} \cdot p^{n-\Delta} \cdot (1-p)^{N_2-n+\Delta}$$

N_1 and N_2 being the values of N in two series of countings.

The probability of the simultaneous occurrence of these two events or rather of the occurrence of a certain difference 2Δ with fixed values of n, N_1, N_2 and p, will therefore be

$$P_{\Delta;n,N_1,N_2,p} = \frac{N_1!\,N_2!}{(n+\Delta)!\,(N_1-n-\Delta)!\,(n-\Delta)!\,(N_2-n+\Delta)!} \cdot p^{2n} \cdot (1-p)^{N_1+N_2-2n}$$

Since p is not known, this probability can not be computed.

In this formula, the powers of p are independant of Δ, and therefore the unknown factor

$$p^{2n} \cdot (1-p)^{N_1+N_2-2n}$$

is the same for all possibilities with fixed values of n, N_1, N_2 and p. If these values are kept constant, the probability of any value of Δ occurring is proportional to

$$F = \frac{N_1!\,N_2!}{(n+\Delta)!\,(N_1-n-\Delta)!\,(n-\Delta)!\,(N_2-n+\Delta)!}$$

The total probability of all possible values of Δ, with fixed values of n, N_1, N_2 and p is represended by

$$\sum_{\Delta} P_{\Delta;n,N_1,N_2,p} = \frac{N_1!\,N_2!}{(n+\Delta)!\,(N_1-n-\Delta)!\,(n-\Delta)!\,(N_2-n+\Delta)!} \cdot p^{2n} \cdot (1-p)^{N_1+N_2-2n}$$

which may also be written as

$$\sum_{\Delta} P_{\Delta;n,N_1,N_2,p} = p^{2n} \cdot (1-p)^{N_1+N_2-2n} \cdot \sum_{\Delta} F.$$

In this summation $|\Delta| \leq n$ and $\leq N_1 - n$ and $\leq N_2 - n$.

The probability of a certain Δ, the values of n, N_1 and N_2 being fixed, is given by

$$P_{\Delta;\,n,N_1,N_2} = \frac{P_{\Delta;n,N_1,N_2,p}}{\sum_{\Delta} P_{\Delta;n,N_1,N_2,p}} = \frac{F}{\sum_{\Delta} F}$$

whatever the value of p might be.

It will be noticed, that the unaccessible parameter p has disappeared from the formula.

In order to carry out the summation $\sum_{\Delta} F$, we follow the trend of thought given below:

According to the theory of combinations

$$\frac{(N_1+N_2)!}{A!B!C!D!} \text{ in which } A+B+C+D=N_1+N_2$$

represents the number of ways, in which N_1+N_2 objects may be grouped into four different groups of A, B, C and D objects respectively, the arrangements of the objects within the groups not being considered. In case the condition $A \neq B \neq C \neq D$ is not fulfilled, we have to keep in mind, that the four groups are not interchangeable.

The number of ways, in which N_1+N_2 objects may similarly be devided into two groups of $A+B$ and $C+D$ objects respectively, is given by

$$\frac{(N_1+N_2)!}{(A+B)!(C+D)!}.$$

The number of ways, in which these objects may be rearranged into 2 groups of $A+C$ and $B+D$ objects respectively, regardless of the previous division, is given by

$$\frac{(N_1+N_2)!}{(A+B)!(C+D)!} \cdot \frac{(N_1+N_2)!}{(A+C)!(B+D)!}.$$

This double choice completely specifies all the sets of A, B, C and D having fixed values of $A+B$, $C+D$, $A+C$, and $B+D$.

The same number of ways may be obtained directly from

$$\frac{(N_1+N_2)!}{A!B!C!D!}$$

by varying A, B, C and D according to the same condition. In order to vary the factorials in our formula

$$\frac{(N_1+N_2)!}{(n+\Delta)!(N_1-n-\Delta)!(n-\Delta)!(N_2-n+\Delta)!}$$

according to the aforesaid condition, we have only to vary Δ, as becomes clear from a glance at the following 2×2 table, since the marginal frequencies are independent of Δ.

$A+B=N_1$	$A=n+\Delta$	$B=N_1-n-\Delta$
$C+D=N_2$	$C=n-\Delta$	$D=N_2-n+\Delta$
	$A+C=2n$	$B+D=N_1+N_2-2n$

We are now able to carry out the summation of the series obtained by varying Δ.

$$\sum_{\Delta} F = \sum_{\Delta} \frac{N_1!\,N_2!}{(n+\Delta)!\,(N_1-n-\Delta)!\,(n-\Delta)!\,(N_2-n+\Delta)!} =$$

$$= \frac{N_1!\,N_2!}{(N_1+N_2)!} \cdot \sum_{\Delta} \frac{(N_1+N_2)!}{(n+\Delta)!\,(N_1-n-\Delta)!\,(n-\Delta)!\,(N_2-n+\Delta)!} =$$

$$= \frac{N_1!\,N_2!}{(N_1+N_2)!} \cdot \frac{(N_1+N_2)!}{N_1!\,N_2!} \cdot \frac{(N_1+N_2)!}{(2n)!\,(N_1+N_2-2n)!} =$$

$$= \frac{(N_1+N_2)!}{(2n)!\,(N_1+N_2-2n)!}.$$

Finally we may write

$$P_{\Delta;n,N_1,N_2} = \frac{F}{\sum_{\Delta} F} = \frac{\dfrac{N_1!\,N_2!}{(n+\Delta)!\,(N_1-n-\Delta)!\,(n-\Delta)!\,(N_2-n+\Delta)!}}{\dfrac{(N_1+N_2)!}{(2n)!\,(N_1+N_2-2n)!}}$$

The derivation of this formula has previously been published by FISHER in an extremely abridged form; for the complete demonstration I am indebted to DRION.

A shorter method for the solution of ΣF has been suggested by D. VAN DANTZIG (private communication). The same principle will be applied later on in the development of the test for extra pollen countings.

The method is based on the fact, that if two polynomials or power series are identically equal for all values of the variable (in a certain interval) the coefficients of equal powers of the variable are equal.

For instance:

$$a + bx + cx^2 \equiv A + Bx + Cx^2$$

implies $a = A$, $b = B$ and $c = C$.

According to NEWTON's binomial:

$$\sum_{0}^{N_1} \frac{N_1!}{k!\,(N_1-k)!}\,x^k \equiv (1+x)^{N_1}$$

and

$$\sum_{0}^{N_2} \frac{N_2!}{l!\,(N_2-l)!}\,x^l \equiv (1+x)^{N_2}$$

Multiplication yields

$$\sum_{0}^{N_1}\sum_{0}^{N_2} \frac{N_1!\,N_2!}{k!\,l!\,(N_1-k)!\,(N_2-l)!} \cdot x^{k+l} \equiv (1+x)^{N_1+N_2} \equiv$$

$$\equiv \sum_{0}^{N_1+N_2} \frac{(N_1+N_2)!}{h!\,(N_1+N_2-h)!} \cdot x^h.$$

Equating the coefficients of x^h in the first and the last part of this identity gives

$$\sum_{0}^{h}{}_{k} \frac{N_1!\,N_2!}{k!\,(h-k)!\,(N_1-k)!\,(N_2-h+k)!} = \frac{(N_1+N_2)!}{h!\,(N_1+N_2-h)!}$$

with $l = h-k$, $0 \leq k \leq h$, $h-N_2 \leq k \leq N_1$

If we put $h = n+m$, $k = n+\triangle$, and hence $h-k = m-\triangle$ we get

$$\sum_{-n}^{m}{}_{\triangle} \frac{N_1!\,N_2!}{(n+\triangle)!\,(m-\triangle)!\,(N_1-n-\triangle)!\,(N_2-m+\triangle)!} = \frac{(N_1+N_2)!}{(n+m)!\,(N_1+N_2-n-m)!}.$$

For $n = m$, we arrive at the same result as previously obtained.

The present form of the function $P_{\triangle;\,n,N_1,N_2}$ allows computation of the chances of specified values of $\triangle$, belonging to fixed values of $n \cdot N_1$ and N_2. It is therefore possible to calculate in what proportion of cases, any value of $\triangle$ will not be exceeded. A detailed tabulation of this function would enable an easy and correct treatment of the difference in means of samples taken from binomial populations.

For equal values of N, we arrive at the simplified form

$$P_{\triangle;\,n,N} = \frac{\dfrac{(N!)^2}{(n+\triangle)!\,(N-n-\triangle)!\,(n-\triangle)!\,(N-n+\triangle)!}}{\dfrac{(2N)!}{(2n)!\,(2N-2n)!}}$$

Tabulation for some given values of N is of special interest to certain fields of biological work. For exemple a table for $N = 150$ can be of practical use in pollen analysis.

In pollen analytical research, samples from different strata of peat bogs are examined in order to detect significant differences in the relative abundance of various kinds of pollen grains. This is done with the aim of reconstructing previous botanical aspects of the region.

In each sample, the individuals of marking species are scored up to a total number of 150. The number of grains of any species, counted in a sample may be expected to follow the binomial law, provided the sample has been thoroughly blended. If p be the chance, that the grain first counted should belong to a certain species, the chance that the second grain should likewise belong to this same species, should be p^2 &c. If 150 grains are taken into account, this reasoning leads to

$$P_n = \frac{150!}{n!\,(150-n)!} \cdot p^n \cdot (1-p)^{(150-n)}.$$

Eventually peat samples contain certain kinds of pollen in so large quantities, that the general view of other species would be confused, if these abundant species would be treated in the same way. For this reason counting up to 150 is restricted to a limited set of species, accepted as markers. Apart from this, grains of the remaining species, the so-called

extra pollen, are scored up as they accompany the 150 marker grains. It is clear, that the extra pollen should undergo a slightly different treatment, since

$$P_n^* = \frac{(150+n)!}{n!\,150!} \cdot p^{*n} \cdot (1-p^*)^{150}.$$

The comparison between samples from different strata is undertaken in the assumption, that each sample actually represents the composition of the entire stratum in the bog, or in other words that the stratum is homogeneous in horizontal direction. Some statistical work concerning this condition has been carried out by BARKLEY (1934) and by WOODHEAD & HODGSON (1935). For the rest one has to make sure that each sample is sufficiently mixed and that various kinds of pollen grains have been preserved to the same extent in strata of different age.

An example of graphical representation of a pollen analysis is given in fig. 1. The stratigraphical survey of the peat bog is shown at the left side

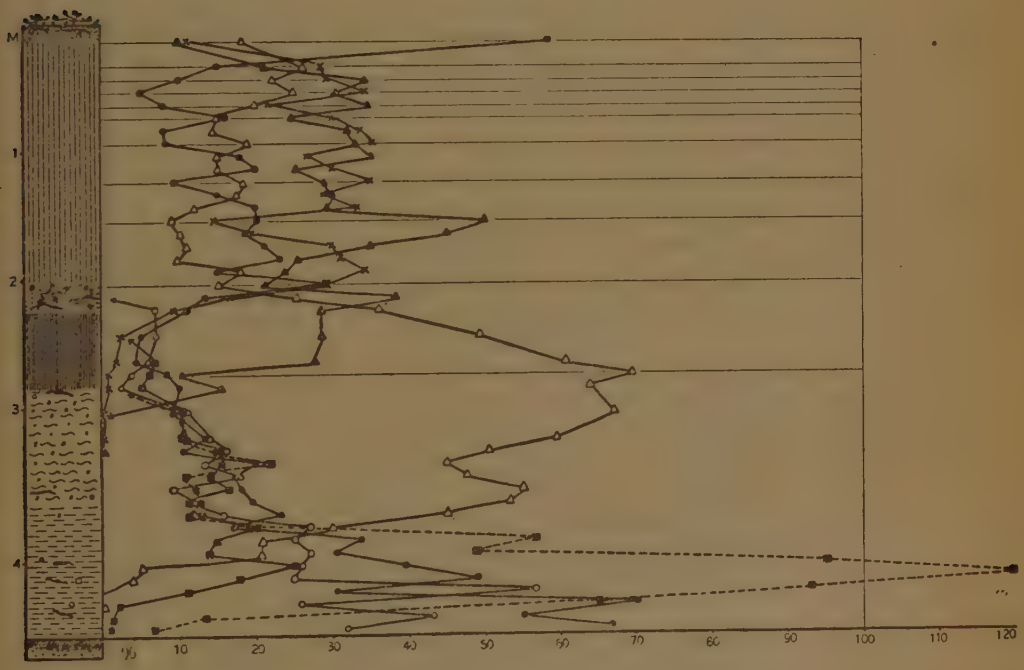


Fig. 1. A pollen diagram (reproduced after WALTER).

of the diagram. The countings of the corresponding samples are plotted on horizontal axes, each beginning on the left side. The frequencies of one sample are gathered on one axis. Special marks are used to indicate the different species. The scheme is completed by connecting marks of the same species. The marks of extra pollen countings are connected with dotted lines.

The binomial test can easily be adapted to this graphical representation. For this purpose we have to compute the values of $P_{\Delta;\,n,N}$ for $N = 150$, varying both Δ and n.

For $n = 75$, we obtain:

$\Delta = 16$	$P = 0{,}0001$	3	0,0724
15	0,0002	2	0,0827
14	0,0005	1	0,0895
13	0,0010	0	0,0919
12	0,0020	—1	0,0895
11	0,0037	—2	0,0827
10	0,0064	—3	0,0724
9	0,0107	—4	0,0601
8	0,0168	—5	0,0474
7	0,0250	.	.
6	0,0354	.	.
5	0,0474	.	.
4	0,0601	.	.

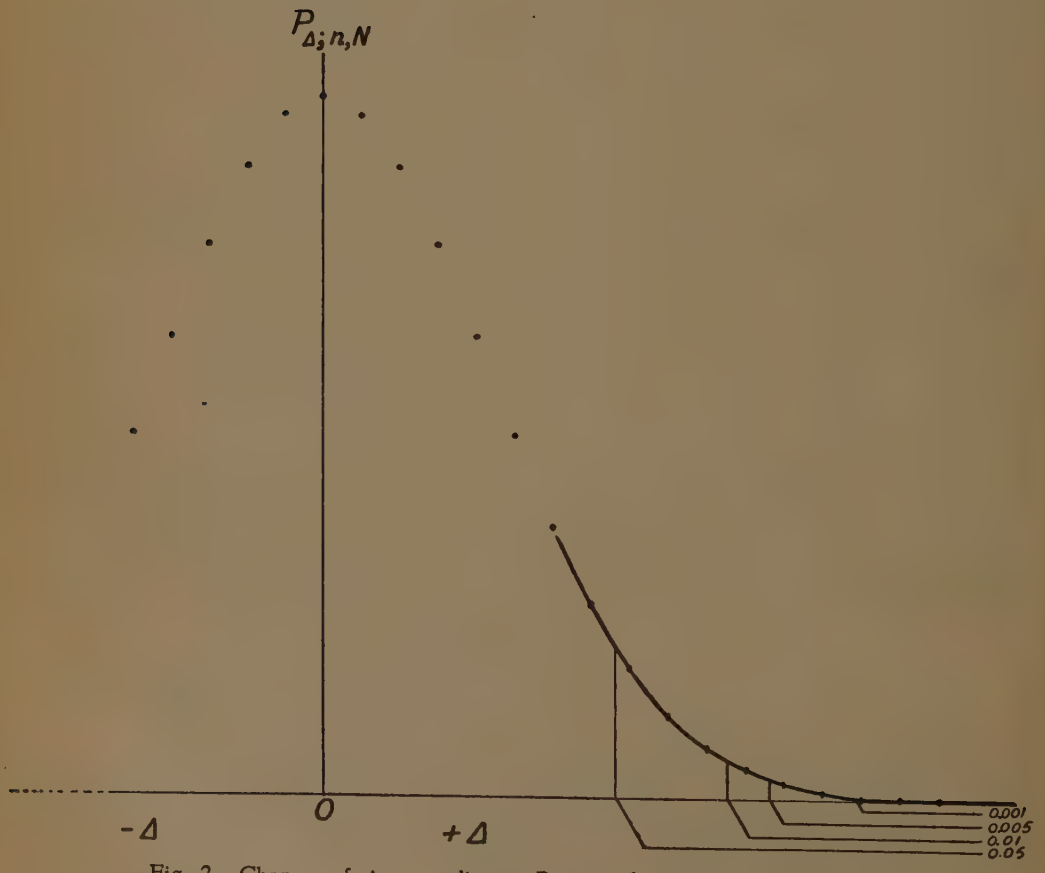


Fig. 2. Chances of Δ, according to $P_{\Delta;\,n,N}$ for $n = 75$ and $N = 150$.

These probabilities are represented in fig. 2. From this column we calculate the cumulative chances:

$\Delta = 16$	0,0001	8	0,0415
15	0,0004	7	0,0666
14	0,0009	6	0,1020
13	0,0019	5	0,1493
12	0,0039	.	.
11	0,0076	.	.
10	0,0140	.	.
9	0,0247		

represented in fig. 3.

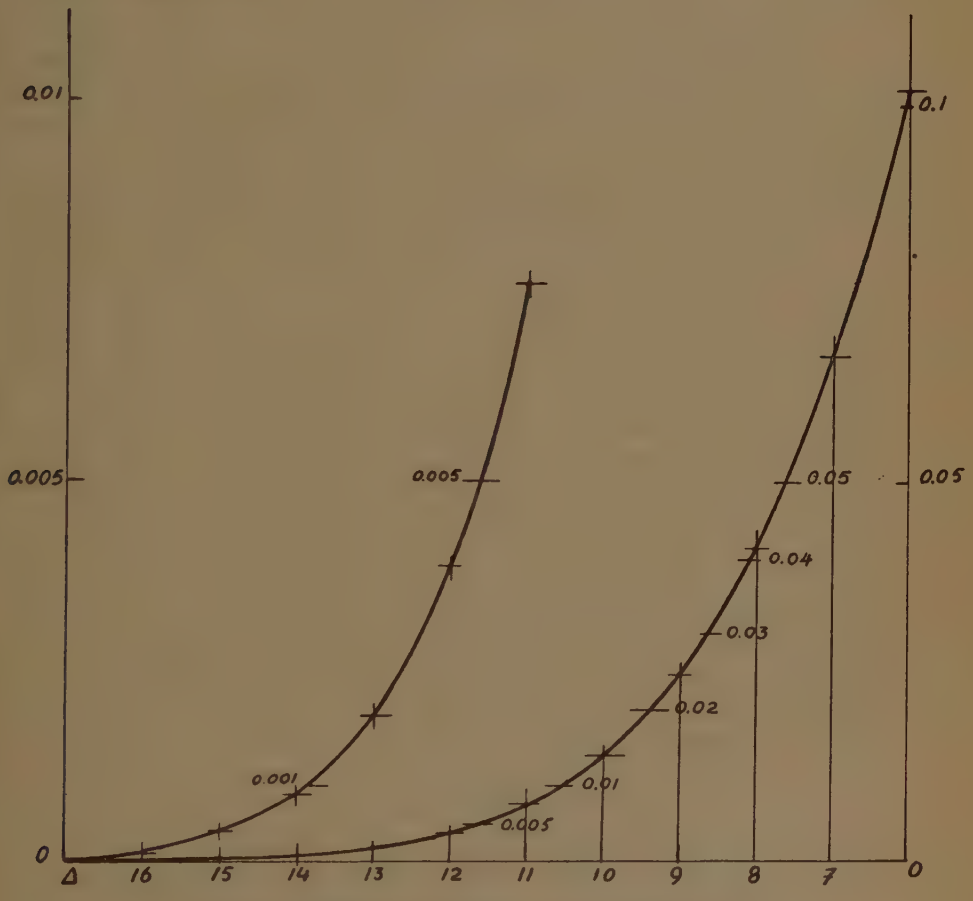


Fig. 3. Graphical interpolation of cumulative chances of Δ.

By means of graphical interpolation we find the values of Δ belonging to cumulative chances amounting to 0,001, 0,005, 0,01 &c. In carrying out this interpolation, we treat $P_{\Delta;n,N}$ as a continuous function. A mathematical significance can be attributed to this procedure by replacing the factorials in $P_{\Delta;n,N}$ by their corresponding Γ-functions for broken values of Δ. This will not affect our further reasoning.

These cumulative chances, the so-called (unilateral) tail errors, are represented in fig. 2 by the specially indicated areas.

The same is done for other values of n, and we obtain the following data

Tail error	$N = 150$ $n = 5$	10	20	30	40	50	60	75	90	100	110	120	130	140	145
$\frac{1}{10}$%	$\Delta =$ 4,9	6,9	9,4	11,1	12,2	13,0	13,5	13,8	13,5	13,0	12,2	11,1	9,4	6,9	4,9
$\frac{1}{2}$%	4,3	5,9	8,1	9,3	10,3	11,0	11,4	11,6	11,4	11,0	10,3	9,3	8,1	5,9	4,3
1%	4,0	5,4	7,3	8,5	9,3	10,0	10,4	10,5	10,4	10,0	9,3	8,5	7,3	5,4	4,0
2%	3,6	4,9	6,5	7,6	8,3	8,9	9,2	9,4	9,2	8,9	8,3	7,6	6,5	4,9	3,6
3%	3,3	4,5	6,0	7,0	7,7	8,2	8,4	8,6	8,4	8,2	7,7	7,0	6,0	4,5	3,3
4%	3,1	4,3	5,6	6,6	7,2	7,6	7,9	8,1	7,9	7,6	7,2	6,6	5,6	4,3	3,1
5%	3,0	4,0	5,3	6,2	6,8	7,2	7,5	7,6	7,5	7,2	6,8	6,2	5,3	4,0	3,0

These data can be used for the construction of a transparant graph of the same breadth as the pollen diagram. The values of n are plotted along an oblique line from zero to 150, and for a certain value of n, the corresponding values of Δ (e.g. for a tail error of $\frac{1}{2}$ %) are plotted along a horizontal axis to the right and left side of n (fig. 4). This is done for various values of n, and the points, thus obtained are connected by smooth lines, the so-called tail error lines.

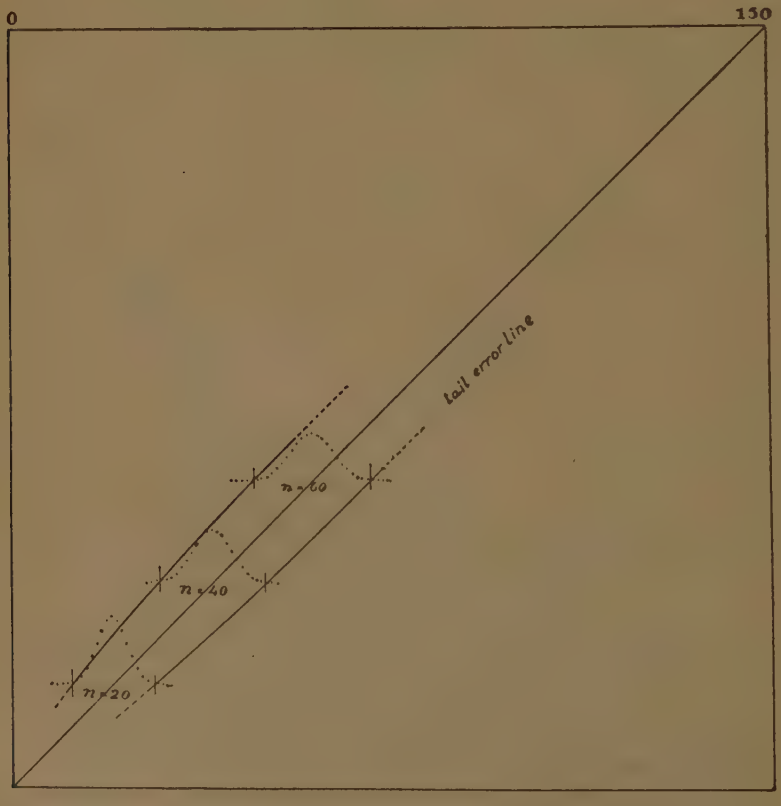


Fig. 4. Construction of tail error line.

Botany. — *Isolement et culture du plasmode de Licea flexuosa* PERS. By JOHANNA C. SOBELS. (Communicated by Prof. V. J. KONINGSBERGER.)

(Communicated at the meeting of April 26, 1947.)

Les travaux de COHEN 1939 sur les Myxomycètes nous ont incité à isoler et à cultiver *in vitro* le plasmode de l'un de ces organismes.

Cet auteur décrit une méthode qui permet de débarrasser le plasmode des microorganismes qui l'accompagnent et de l'obtenir en culture pure. Cet auteur a en outre réalisé des cultures associées de plasmode et d'un autre microorganisme (levure ou bactérie) sur les milieux généralement utilisés en bactériologie. En partant d'un sclérote récolté sur *Stereum purpureum* nous avons pu obtenir des cultures de plasmode d'une espèce, qui après fructifications a pu être déterminée comme étant *Licea flexuosa*.

Nous avons pu purifier le plasmode en le laissant s'étaler sur des plaques de gélose. Le milieu avait été ajusté à pH 6 et tamponné à l'aide du mélange de phosphate monopotassique et disodique de SÖRENSEN.

Dans les traces laissées par le plasmode sur la gélose nous avons isolé une levure ne provoquant pas la fermentation alcoolique, que nous avons déterminée comme étant le *Torulopsis Laurentii* nov. var. On a obtenu le développement optimum de cette levure à 15—20° centigrades.

Pour les cultures associées, nous avons utilisé, soit la gélose aux flocons d'avoine, soit de la gélose contenant 6 % de malt (6 % maltose), sur le premier de ces milieux les cultures se conservent 6 semaines, sur l'autre 3 à 4 mois (Voir les photos I et II).

Nous avons utilisé pour ces cultures de longs tubes dans lesquels nous faisons couler le milieu en couche inclinée, on obtient ainsi beaucoup moins d'infection que lorsque l'on emploie des boîtes de PÉTRI. On introduit à la surface de la gélose une suspension stérile de *Torulopsis Laurentii* obtenue par centrifugation et lavage à l'eau stérile. Nous avons cherché à savoir si le plasmode était capable de se nourrir non seulement avec de la levure tuée, mais avec les constituants de celle-ci.

Pour cela nous avons introduit à l'aide d'une pipette dans une boîte de PÉTRI à la surface de la gélose, soit un extrait alcoolique de levure (Bios extrait, E. C. WASSINK, 1934: 603), soit un extrait purifié (L'extrait purifié était deux fois moins concentré que l'extrait brut).

Nous avons constaté que le plasmode suit la trace de l'extrait brut plus rapidement que celle de l'extrait purifié. Par la suite nous avons uniquement employé l'extrait brut.

Il est important de ne pas mettre l'extrait de levure directement sur le milieu, il faut ajouter un substrat que le plasmode puisse phagocyter. Comme substrat nous utilisions une suspension de *Torulopsis Laurentii* préparée de la façon suivante. La levure était stérilisée à 120° C. avec cinq fois son

volume d'eau, puis agitée pendant une semaine et ensuite centrifugée pour séparer le liquide clair. Nous lavons le culot à l'eau stérile et on en fait une suspension. C'est cette suspension qui est utilisée comme substrat.

Nous avons comparé l'effet de l'extrait de *Torulopsis* obtenu par centrifugation suivant la méthode que nous venons de décrire et celui de l'extrait alcoolique de levure de boulanger et nous n'avons constaté aucune différence (voir les photos III, IV et V).

Il serait d'un grand intérêt de rechercher un milieu de culture synthétique dont tous les éléments seraient chimiquement définis. Le plasmode de *Licea flexuosa* ensemencé sur un milieu gélosé ajusté à pH 6 auquel on ajoute 10 % d'extrait alcoolique de levure donne quelques jours après une belle culture régulière avec une nervation dense et un contour bien développé de couleur jaune clair. Cette culture reste vivante plusieurs semaines (voir photo VI). D'après la photo, nous pouvons constater que le plasmode est capable de dissoudre la gélose.

Mlle. DALEUX en 1940 a démontré que le plasmode de *Fuligo septica* corrode notablement la gélose. Elle dit aussi „il s'agit d'une attaque par le plasmode . . . probablement due à la secrétion d'une diastase, la *gélase* dont SKUPIENSKI a supposé l'existence."

Nous avons démontré que le plasmode des Myxomycètes est capable de se développer en absorbant les substances nutritives d'un milieu gélosé sans phénomènes phagocytaires.

Nous avons ensuite cherché à déterminer quels étaient les constituants de l'extrait de levure nécessaire à la croissance du plasmode.

Pour cela nous avons déterminé la teneur en azote de l'extrait alcoolique par la méthode de KJELDAHL.

Cet extrait en referme une quantité correspondant à 2 % de protéines, si bien que le milieu préparé à l'aide de cet extrait n'en contient que 0,2 %.

Nous avons émis l'hypothèse que les protéines jouent un rôle capital dans la nutrition du plasmode et nous avons cherché à la vérifier en cultivant le *Licea flexuosa* sur des milieux glucosés à 0,5 % qui renferment 0,2 % de peptone soluble, et sur d'autres auxquels nous ajoutions en plus l'extrait de levure alcoolique. (Les plasmodes cultivés en présence de 0,5 % de glucose ont une nervation beaucoup moins dense).

Nous avons utilisé différentes sortes de peptones. Les milieux préparés avec la Peptone Witte, Peptone Rhône Poulenc, Protéose-Peptone Difco, Peptone Organon provoquent pendant quelques jours une très bonne croissance, mais celle-ci se ralentit très vite et le plasmode meurt au bout d'une semaine. Au contraire avec la Peptone Merck et la Bacto-Peptone Difco le plasmode se développe normalement et l'adjonction d'extrait alcoolique de levure n'apporte aucune modification (voir les photos VII et VIII).

On peut donc supposer que le plasmode est capable de se développer en présence de seuls facteurs de croissance apportés par les peptones.

Sur le milieu préparé avec la peptone Merck, le plasmode a vécu 3 semaines et sur la Bacto peptone Difco nous l'avons entretenu pendant deux mois, période après laquelle il a formé des sporanges. Nous avons ainsi démontré qu'un Myxomycète peut se développer normalement sur un milieu ne renfermant aucun microorganisme vivant ou tué, ni aucun extrait de microorganisme.

Etant donné qu'au cours des premiers jours qui suivent le repiquage le développement du plasmode est identique sur toutes les espèces de peptone que nous avons essayées, on peut se demander ce qui différencie la peptone Merck et la Bacto peptone Difco des autres qui sont impropres à la culture. Il se peut que ces dernières ne renferment pas les substances de croissance convenables, ou bien on peut imaginer qu'au cours de son développement en présence de ces peptones le plasmode élabore des produits de métabolisme toxiques (NH_3 par exemple).

Nous avons cherché à déterminer la concentration optimale en azote pour la croissance du plasmode. Pour cela nous avons incorporé au milieu de culture des quantités croissantes de peptone de manière à ce que ce dernier renferme 0,1 %, 0,2 %, 0,4 %, 0,6 % et 0,8 % d'azote.

Nous avons constaté qu'aux plus fortes concentrations utilisées 0,6 %, 0,8 % la croissance était médiocre, le plasmode devenait transparent et sa nervature était lâche. Cependant, au bout d'une période de 3 semaines, la végétation semble s'améliorer.

Toutefois après un mois de culture on peut constater que ce sont les milieux contenant 0,1 % d'azote qui fournissent les meilleurs résultats. La concentration optimale pour la peptone Merck nous paraît être voisine de 0,1 % ou 0,2 %. L'adjonction d'extrait de levure à des milieux renfermant cette quantité d'azote n'amène aucune amélioration.

Nous avons cherché ensuite à cultiver le plasmode sur un milieu de composition bien déterminée et pour cela nous avons réalisé une série d'expériences en employant comme substrat les milieux suivants:

(ensemencé le 3 mars 1944, gélose lavée pH 6 et)

1. Asparagine 0,27 % + Tréhalose 0,1 % + Extrait de levure 0,1 %
2. „ + Tréhalose 0,1 %
3. „ + Extrait de levure
4. „
5. Tréhalose 0,5 %

Nous avons obtenu le meilleur développement sur *asparagine-tréhalose-extrait de levure* (1). Le plasmode a vécu sur ce milieu pendant une période de 39 jours. Sa couleur était jaune clair, il se déplaçait assez rapidement en formant un contour très net.

Le développement du plasmode sur un milieu *asparagine-tréhalose* (2) diffère très peu du premier. Il reste en vie aussi pendant 39 jours. La vitesse de déplacement est au début plus petite, mais la différence disparaît. Les plasmodes sont plus petits, mais de forme régulière et avec un contour très net.

Nous avons supposé que la culture de l'organisme sur *asparagine-extrait de levure* (3) provoquerait un développement important. Ceci se produisit au début; les cultures se déplacaient rapidement, mais bientôt nous avons observé un changement. Le plasmode se divise en petits morceaux qui pénètrent profondément dans le milieu et ne se réunissent plus. Ces cultures restent en vie pendant 31 jours.

Les cultures sur *asparagine* (4) et sur *tréhalose* (5) se déplacent difficilement et nous avons bientôt observé sur le milieu contenant de l'*asparagine* (4) une division du plasmode en très petites parties qui pénètrent dans la gélose. Nous avons gardé vivantes ces cultures pendant 31 jours.

Sur les milieux contenant du *tréhalose* (5) les plasmodes ne se divisent pas, mais ils restent très petits, de couleur jaune orangé, avec une nervation lâche, sans contour, mais plus tard ils se contractent. Ces cultures dissolvent un peu la gélose et restent en vie pendant 31 jours.

Nous avons pu constater que sur le milieu *asparagine-tréhalose* (2) qui est entièrement synthétique, la croissance était excellente, presque comparable à celle que nous obtenions avec le milieu (1) asparagine-tréhalose qui renfermait lui, l'extrait de levure.

Les cultures sur *asparagine-tréhalose* (2) se développaient mieux que sur *asparagine et extrait de levure* (3).

A cette concentration, l'extrait de levure ne renferme pas suffisamment de glucides pour remplacer le tréhalose.

Avec le *tréhalose* (5) seul, la croissance était un peu meilleure qu'avec l'*asparagine* (4) seule qui n'est pas une source de carbone suffisante pour subvenir aux besoins du métabolisme du plasmode. Ceci explique que le développement soit mieux sur les milieux ne renfermant que du tréhalose, et que l'extrait de levure seul ne puisse remplacer cet effet. L'action de cet extrait semble être d'ordre hormonal. Nous nous proposons de vérifier cette hypothèse en faisant des cultures sur des milieux renfermant à la fois du tréhalose et de l'extrait de levure.

Historique.

Dès 1890, STANGE constata sur le *Chondrioderma* l'effet chimiotactique de l'asparagine utilisée à la concentration 1,2 %. En 1925, IWANOW isola de différents Myxomycètes, une diastase qui hydrolysait le tréhalose et qu'il nomma tréhalase.

En 1896, BEIJERINCK réussit à cultiver des amibes in vitro en les associant à des cultures de bactéries de vinaigre ou de *Bacterium zymophila*.

En 1899, NADSON fit des cultures associées d'amibes de *Dictyostelium mucoroïdes* et de *Bacillus fluorescens liquefaciens*. Il essaya de cultiver ces organismes sans bactéries sur des solutions de peptones et de phosphates; mais dans ces conditions le développement était très mauvais. En 1902, POTTS réussit à cultiver le *Dictyostelium mucoroïdes* en présence des cellules de *Bacterium fimbriatum*, tué par le chloroforme; mais il n'obtint aucun résultat en opérant de la même manière avec le *Bacillus megatherium*.

PINOY en 1907 réalisa des cultures de *Dictyostelium discoïdeum* en présence de différentes bactéries, entre autres *B. fluorescens liquefaciens*, mais il ne parvint pas à cultiver cet organisme en présence des bactéries tuées par un antiseptique. A la suite de ces expériences, l'auteur considérait le Myxomycète comme un parasite de la bactérie.

Cependant en 1898, MILLER avait déjà réussi la culture aseptique d'un myxomycète. En 1925—26, SCHUCKMANN put cultiver le *Dictyostelium* sur un bouillon gélosé. Il constata le développement normal des amibes et des pseudoplasmodes. Mais à la longue le myxomycète perdait le pouvoir de fructifier.

En 1937, RAPER reprit la question, il montra que le *Dictyostelium mucoroïdes* se développait très bien en présence des bactéries tuées par la chaleur ou les rayons ultraviolets. Mais sur l'extrait des pois gélosé ou sur la gélose à 1 % de lactose et 3 % de peptone, le développement était très mauvais. De même, il n'obtint aucun résultat en employant comme substrat un filtrat stérile de *B. coli*. Ces résultats confirment ceux de NADSON.

RAPER (1937) émit l'hypothèse que l'on pouvait cultiver les *Dictyosteliaceae* sur des milieux synthétiques. Les *Dictyosteliaceae* en effet forment des agrégats ou pseudoplasmodes, mais nous n'avons trouvé aucune indication bibliographique sur le développement des fusiplasmodes sur milieux stériles.

SMART 1938 a étudié la germination des spores de deux Myxomycètes. Il a constaté que des spores désinfectés germaient beaucoup plus facilement in vitro sur un milieu stérile qu'en présence des bactéries ou des champignons. Il a également remarqué que les solutions nutritives facilitaient beaucoup la fusion de Myxomycètes et la formation du plasmode.

MARTIN 1940 émit l'opinion que certains myxomycètes sont capables (sur la page 365): „... to utilize nutrient material in solution as well as to ingest concrete particles and digest them. It may be inferred that in nature both methods of securing food are utilized". Sur la page 379, il dit avec grand optimisme „... once bacteria-free cultures are secured, it should be possible to grow myxomycetes in pure culture from spore to fructification with no greater difficulty than is entailed in culturing some species of Phycomycetes".

Les expériences que nous avons relatées précédemment montrent que l'on peut cultiver un myxomycète sur un milieu gélosé à base d'asparagine et de tréhalose. Or, avant les travaux de COHEN (1939) on considérait que les microorganismes étaient indispensables à la nutrition des amibes et des myxomycètes.

La guerre nous a obligée à interrompre ces recherches en septembre 1944. Nous les avons repris en mai 1945 au „Botanisch Laboratorium Utrecht", mais les charges de l'enseignement ne nous ont pas permis de nous y consacrer entièrement et à cette époque par suite d'une infection due à un *Penicillium* antibiotique la souche de cultures associées de *Licea flexuosa* que nous entretenions depuis décembre 1940 fut détruite. A l'heure actuelle

grâce à une bourse du Gouvernement français nous avons repris ces recherches à l'Institut Pasteur sous la direction du Dr. A. LWOFF et du Dr. J. MAGROU.

Nous travaillons avec le plasmode d'une espèce encore indéterminée récoltée en Hollande et nous cherchons en ce moment une méthode de mesure qui nous permette de déterminer quantitativement la croissance du plasmode.

En terminant nous tenons à exprimer toute notre reconnaissance au Professeur V. J. KONINGSBERGER, qui a bien voulu s'intéresser à notre travail et n'a cessé de nous aider de ses conseils. Nous adressons aussi nos remerciements très sincères au Professeur A. J. KLUYVER qui nous a aimablement accueillie pendant quinze mois dans son laboratoire.

Paris le 21 mars 1947.

BIBLIOGRAPHIE.

1. M. W. BEYERINCK, Kultur Versuche mit Amoeben auf festem Substrate. Cent. Bakt. u. Parasitenkunde Jena, Abt. I, **19**, 257—267 (1896).
2. A. L. COHEN, Nutrition of the Myxomycetes I. Pure culture and two membered cultures of Myxomycete Plasmodia, Bot. Gaz., **101**, 243—276 (1939).
3. GENEVIÈVE DALLEUX, „Recherches sur les Plasmodes de deux Myxomycètes", Thèse: Fac. de Sc. de l'Univ. de Paris (1940).
4. N. N. IWANOW, Ueber die Trehalose und Trehalase bei Myxomyceten, Biochem. Zeitschr., **161—162**, 455 (1925).
5. A. LISTER, A Monograph of the Mycetozoa, London, 3rd Edition, G. Lister (1925).
6. G. W. MARTIN, The Myxomycetes, Bot. Review, **6**, 356—388 (1940).
7. C. O. MILLER, The asceptic cultivation of Mycetozoa, Quart. Journ. Micr. Sc., **41**, 43—71 (1899).
8. G. A. NADSON, Des cultures du Dictyostelium mucoroïdes Bref. et des cultures pures des Amibes en général (Russian with French résumé), Scripta Botanica (Horti Univ. Imp. Petropolitanae) **15**, 153—190 (1899).
9. E. PINOY, Rôle des Bactéries dans le développement de certains Myxomycètes, Ann. Inst. Pasteur, **21**, 622—656 et 686—700 (1907).
10. G. POTTS, Zur Physiologie des Dictyostelium mucoroïdes, Flora (Jena), **91**, 281—347 (1902).
11. K. B. RAPER, Growth and Development of Dictyostelium discoïdeum with different bacterial Associates, Journ. of Agricult. Res., **55**, 289—316 (1937).
12. ———, Influence of culture conditions upon the growth and development of Dictyostelium discoïdeum, Journ. of Agricult. Res., **58**, 157—198 (1939).
13. W. v. SCHUCKMANN, Zur Biologie von Dictyostelium mucoroïdes Bref, Centrbl. Bakt. (etc.) (I) **91**, 302—309 (1924).
14. ———, Zur Morphologie und Biologie von Dictyostelium mucoroïdes Bref., Arch. Protistenkunde, **51**, 495—529 (1925).
15. F. X. SKUPINSKI, Influence du milieu de culture sur le développement des Myxomycètes, Acta Soc. Bot. Poloniae, **10**, 113—127 (1933).
16. R. F. SMART, Influence of certain external factors on spore-germination in the Myxomycetes, Amer. Journ. Bot., **24**, 145—159 (1937).
17. ———, The reactions of the swarm-cells of Myxomycetes to nutrient materials, Mycologia, **30**, 254—264 (1938).
18. B. STANGE, Ueber chemotactische Reizbewegungen. Die Myxamoeben der Myxomyceten, Bot. Zeit., **48**, 155—159 et 161—166 (1890).
19. E. C. WASSINK, Begrenzende Bedingungen bei der Atmung von Phycomyces, Dissertatie: Utrecht (1934).

PHOTO I. — Culture associée du plasmode de *L. flexuosa* et de levure *T. Laurentii*. *Agée de 15 jours*. Couleur: jaune foncé.
Ensemencée le 30 juin 1943 sur gélose aux flocons d'avoine, pH 6.
Photographiée le 15 juillet 1943.

PHOTO II. — Culture associée du plasmode de *L. flexuosa* et de la levure *T. Laurentii*. *Agée de 16 jours*.
Ensemencée le 31 août 1943 sur gélose à 6 % de malt.
Photographiée le 16 septembre 1943.
On observe 2 plasmodes. A la périphérie, un plasmode jaune clair, qui s'est développé symétriquement et qui a atteint un diamètre de 9 cm.
A l'intérieur une zone de couleur crème de *Torulopsis Laurentii* qui s'est développé dans la trace du plasmode
Au centre se trouve l'endroit d'ensemencement qui s'est développé une deuxième fois pour donner un plasmode jaune clair d'un diamètre de 4 cm.

PHOTO III. — *Culture de 2 jours.*
Ensemencée le 11 août 1943 sur gélose lavée, pH 6, avec une suspension de levure (le culot de centrifugation lavé avec l'eau stérile). 1 = ensemencement, 2 = levure (culot).
Photographiée le 13 août 1943.

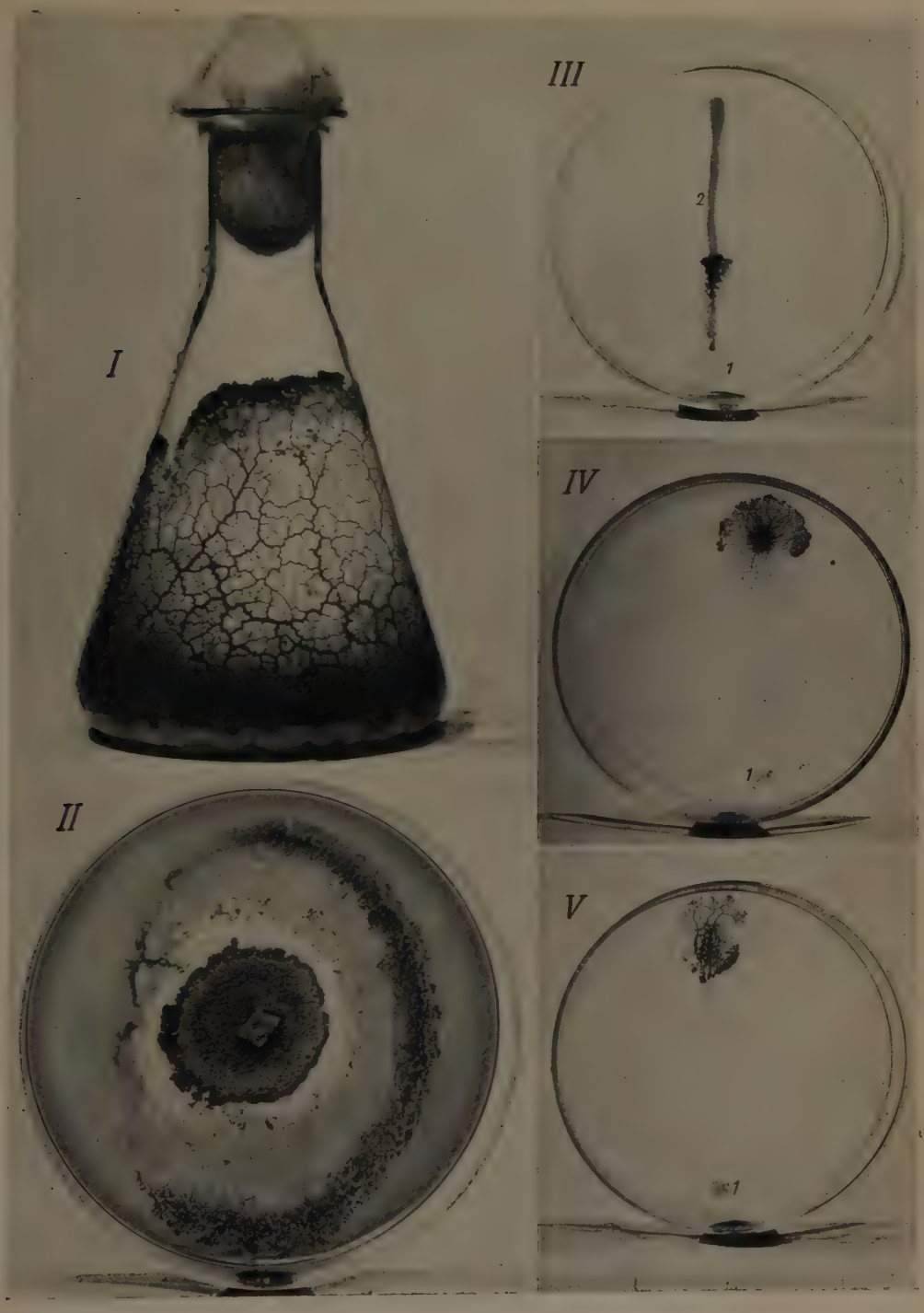
PHOTO IV. — *Culture de 2 jours.*
Ensemencée le 11 août 1943 sur gélose lavée, pH 6, avec une suspension de levure le culot de centrifugation auquel on a ajouté un extrait aqueux). 1 = ensemencement.
Photographiée le 13 août 1943.

PHOTO V. — *Culture de 2 jours.*
Ensemencée le 11 août 1943 sur gélose lavée, pH 6 avec une suspension de levure (le culot de centrifugation et auquel on a ajouté un extrait alcoolique). 1 = ensemencement.
Photographiée le 13 août 1943.

PHOTO VI. — *Culture de 13 jours.*
Ensemencée le 25 novembre 1943 sur gélose lavée, pH 6 avec 10 % de l'extrait alcoolique de levure.
Photographiée le 8 décembre 1943 à travers le fond de la boîte de PÉTRI.
Après *5 jours*, le plasmode s'est dispersé en petits flocons jaunes sur le milieu, après *1 semaine* ils ont pénétré profondément dans le milieu et au bout de *15 jours* il y a 3 grands plasmodes dont le plus grand se trouve sur la gélose et les 2 autres entre la gélose et le verre. Ils sont jaune clair, ont un contour bien développé et une nervation pas très dense.
La superficie est environ 8¾ cm², celle du plasmode ensemencé ½ cm². 1 = ensemencement.
Le plasmode a formé des trous dans la gélose (2).

PHOTO VII. — *Culture de 2 jours.*
Ensemencée le 6 octobre 1943 sur gélose lavée, pH 6, 10 cm³ par tube. Bacto-peptone Difco 0,2 %, Glucose 0,5 %.
Photographiée le 8 octobre 1943.
La nervuration se compose d'un réseau de grosses veines espacées entre lesquelles on distingue un second réseau de fines veines.
Il s'est déplacé rapidement et se trouve à 11 cm de l'endroit ensemencé superficie 3½ cm².
1 = ensemencement, 2 = plasmode.

PHOTO VIII. — La même culture, au bout de 5 jours.
Photographiée le 11 octobre 1943.
Le plasmode s'est divisé en deux parties dont l'une se trouve entre la gélose et le verre et l'autre à la surface de la gélose.
Le plasmode qui se trouve dans le fond du tube est à une distance de 14½ cm du point d'ensemencement. Sa superficie est 5 cm².
Le deuxième plasmode est revenu et se trouve à 8½ cm du point d'ensemencement, son contour est très net et sa nervuration dense.
Superficie 2¾ cm².
1 = ensemencement, 2 = plasmode à la surface, 3 = plasmode entre la gélose et le verre.


I
II
III
2
1
IV
1
V
1

VI
VII
VIII
1
2
3

Botany. — *The inhibition of germination, caused by extracts of seedballs of the sugarbeet (Beta Vulgaris).* By CHR. P. A. DUYM, J. G. KOMEN, A. J. ULTÉE and B. M. VAN DER WEIDE. (Communicated by Prof. V. J. KONINGSBERGER.)

(Communicated at the meeting of April 26, 1947.)

Introduction.

A great number of recent publications deals with inhibiting substances (5, 38, 39); they partly concern the inhibition of the growth (3, 11, 15, 22, 23, 31) and partly that of the germination. To the latter belong those on the inhibition of the germination by extracts of fleshy fruits (10, 15, 17, 18, 19, 24, 26, 28, 32, 37) and by extracts of seeds (or seedballs in the case of *Beta*) (4, 6, 7, 8, 9, 14, 20, 21, 25, 27, 29, 34, 36). Moreover, several authors reported on inhibiting effects of gases and volatile substances from seeds and fruits on germination (2, 6, 16, 36). To this category also belongs an extensive literature on the action of ethene.

We have restricted ourselves to the effect of extracts of the seedballs of *Beta*, following the experiments of FRÖSCHEL (7, 8, 9, 14, 20, 21, 25, 34). JENTYS (14) was the first who prepared extracts of beet seeds and tested these on inhibition. He too gave the first analyses of the inorganic components of these extracts. FRÖSCHEL (1939) published his, almost classical, experiment on germination with unilateral water supply, in which he clearly demonstrated the presence of a germination inhibitor in seeds of *Beta*. The liquid, after passing beet seeds, retarded the germination of seeds of at least 29 other species. This proved that this inhibitor is not specific to the species. In a second report (8) FRÖSCHEL demonstrated the reversability of the effect. From FRÖSCHEL's first paper it was already clear that the inhibiting substance was easily soluble in water. He used this property to prepare water extracts. Other important features are:

1. The inhibition increases with the concentration.
2. The effect is most marked in the light.
3. Adsorption of the inhibitor by the usual adsorbants could not be proved.
4. Animal charcoal adsorbs the brown colour from the extracts; this colour has nothing to do with the inhibition.
5. The inhibiting substance is thermostable and insoluble in alcohol and ether.
6. The occurrence of inhibitors in seeds is widely spread.
7. The inhibitor from barley also decreases the intensity of respiration.

GILLIS (12, 13) made a qualitative and for the greater part quantitative analysis of the inorganic components of beet seed extracts. The occurrence of oxalate ions in rather a high concentration was striking. GILLIS proved an inhibiting effect of a 0.03 normal potassium oxalate solution.

It is generally assumed that inhibiting substances are important in preventing a premature germination of the seed and in securing a normal development of the plant.

Experimental part.

Preparation of the seed extracts. We followed the method of FRÖSCHEL (private communication).

> 1 kg *Beta* seeds are immersed in 4 l distilled water and filtered with suction after standing for 24 hours at room temperature (filtrate about 3 l). Then 2.5 % active charcoal is added, the solution heated to boiling and filtered again. The filtrate is almost colourless. In order to prevent growth of bacteria the extract is sterilized at 120°C for half an hour. This we call the standard solution.

Some tests were made with solutions not treated with active coal, but otherwise prepared in the same way and with solutions not heated at all. A treatment with coal without heating does not give a complete decolourization.

Determination of the rate of germination. Tests were made with *Avena sativa* (Victory oats 1941), *Lepidium sativum* (harvest 1945) and *Papaver orientale*. The seeds were considered germinated when the seedcoats were perforated by the germ.

The seeds were placed on wet filter paper, wrapped around glass plates of 3 mm thickness in Petri-dishes (Ø 11 cm). Each dish contained 100 seeds and 15 ml solution (or distilled water in the controls). *Avena* seeds had previously been soaked for one hour in the solutions in question. The results are given as the mean of the observations of the different authors. The most important tests were repeated under sterile conditions; the results showed no difference with the not sterile ones.

This method is essentially different from that used by FRÖSCHEL, who placed the seeds on paper strips, while one end of these was immersed in the solutions. In this way the liquid is flowing through the paper (proved by the accumulation of coloured substances). Thus the water of the moving solution partly disappears by 1. absorbtion by the germinating seeds and 2. evaporation; and therefore the concentration of the solution increases. The inhibited seeds do not absorb much water and so this increase is mostly due to evaporation. This concentrated solution gives a much stronger inhibiting effect than with our method, but the concentration of the solution on the place of the germinating seeds is not known.

Influence of light on the inhibition. Tests were made to investigate the action of light on the inhibition of germination.

TABLE 1.

% Germinated *Avena* seeds after

	24 *h*	48 *h*
Distilled water; in the light	72	87
Beet seed extract; in the light	15	48
Distilled water; in the dark	82	94
Beet seed extract; prepared and tested in the dark	67	83
Beet seed extract; prepared in the light and tested in the dark	67	87

The inhibiting effect is most marked in the light.

Influence of decolourizing and heating of the extracts. Four types of extracts were used in the following experiments in order to investigate whether the inhibitors are partly removed during the purification and sterilizing of the extracts.

TABLE 2.

% Germinated *Avena* seeds after

	16 *h*	20 *h*
Distilled water	65	88
Beet seed extract, not decolourized and not heated	17	27
„ „ „ , not decolourized and heated	13	23
„ „ „ , decolourized and not heated	16	44
„ „ „ , decolourized and heated	16	41

% Germinated *Lepidium* seeds after

	20 *h*	40 *h*
Distilled water	31	67
Beet seed extract, decolourized and not heated	23	58
„ „ „ , decolourized and heated	21	52

Heating of the solutions had no influence, decolourizing slightly diminished the inhibiting capacity.

On the nature of the inhibiting solution. All extracts were optically inactive. In order to get an impression of the concentration of the decolourized solution we determined:

I. Dry weight in vacuo over conc. H_2SO_4.
II. Dry weight of the residue at 105° C.
III. Dry weight of the residue when ignited.

TABLE 3.

Dry weight in %.

	I	II	III
Beet seed extract, decolourized	0.95	0.87	0.65
Id. after electrodialysis	0.093	0.050	0.020
Not decolourized extracts			
a) according to JENTYS (14)	—	1.27	0.93
b) according to GILLIS (13)	—	1.17	0.67
Extraction of beat seed extract with peroxyde-free ether			
a) at pH 7.1	0.001	—	—
b) after addition of 10 % conc. hydrochloric acid	0.021	—	—
c) after addition of 10 % conc. (40 %) potass. hydroxyde	0.001	—	—

When extracting the acidified solution the oxalic acid partly goes into the ether layer.

TABLE 4.

Test results. % germinated *Avena* seeds after

	24 *h*
Distilled water	81
Beet seed extract	47
Id., after electrodialysis	81
Extract with peroxyde-free ether	
a) at pH 7.1	80
b) after addition of 10 % conc. hydrochloric acid	91
c) after addition of 10 % conc. (40 %) potass. hydroxyde	94
Residue on evaporation dissolved in orig. vol. of water	51
Residue of ignition dissolved in orig. vol. of water	51

These tables show that the inhibition is mainly due to the inorganic compounds of the extract.

Inorganic components of the inhibiting solutions. The most detailed analysis is that of GILLIS (12, 13), but he gives no figures of the content of phosphate and carbonate. This is shown by the fact that the total of his cation-equivalents is higher than that of the anions. Probably organic anions and silicates also play a part.

The results of GILLIS and JENTYS are:

TABLE 5.

Milli-equivalents in 100 ml according to

	JENTYS	GILLIS	mean
Na^+	4.5	4.6	4.6
K^+	4.6	4.4	4.5
NH_4^+	—	0.38	0.38
Mg^{++}	—	0.62	0.62
Ca^{++}	—	0.01	0.01
Cl^-	2.5	4.4	3.5
NO_3^-	—	1.37	1.37
SO_4^{--}	1.19	1.12	1.15
$C_2O_4^{--}$	—	1.85	1.85
HPO_4^{--}	0.74	—	0.74
HCO_3^-	2.4 *	—	2.4

* JENTYS analysed the ashes and this may account for the high amount of carbonate, which may originate from the oxalate and eventually other organic material.

We prepared a solution based on the figures of the mean, adding bicarbonate to balance the anions and cations.

TABLE 6.

Salt solution:	mol per l	g per l
NaCl	0.0350	2.042
Na_2SO_4 10 aq	0.0007	0.225
Na_2HPO_4 1 aq	0.0037	0.658
KNO_3	0.0137	1.387
$K_2C_2O_4$ 1 aq	0.0093	1.712
$KHCO_3$	0.0129	1.291
$MgSO_4$ 7 aq	0.0031	0.763
$(NH_4)_2SO_4$	0.0019	0.252
$CaSO_4$	0.00005	0.007
$NaHCO_3$	0.0021	0.176
$RbNO_3$	trace	trace

The osmotic value can be estimated about 4 atm.

We tested the retardation of the germination of *Avena* seeds by this solution:

TABLE 7.

% Germinated *Avena* seeds after

	17 *h*	20 *h*
Distilled water	68	86
Beet seed extract	20	45
Inorganic salt solution	28	58

Under sterile conditions we found:

TABLE 8.

% Germinated *Avena* seeds after

	17 *h*	21 *h*
Distilled water	54	82
Beet seed extract	26	60
Inorganic salt solution	33	61

The same experiments with *Papaver orientale* instead of with *Avena* gave the following results:

TABLE 9.

% Germinated *Papaver* seeds after

	72 *h*
Distilled water	68
Beet seed extract	39
Inorganic salt solution	49

Consequently the inorganic salts are for the greater part responsible for the inhibition. We then investigated whether the inhibition was due

to specific ions by replacing each ion successively by another one ($Na^+$by K^+, $Mg^{++}$by Na^+, $C_2O_4^{--}$ by HCO_3^- and so on.) All these solutions showed the same effect.

The salt solution tested in the dark showed a much weaker inhibition than in the light, just like beet seed extracts:

TABLE 10.
% Germinated *Avena* seeds after

	21 *h*
Distilled water; tested in the light	79
Beet seed extract, id.	55
Inorg. salt solution, id.	57
Distilled water; tested in the dark	86
Beet seed extract, id.	78
Inorg. salt solution, id.	79

Influence of osmotic pressure.

TABLE 11.
% Germinated *Avena* seeds after

	17 *h*	19 *h*
Distilled water	68	86
Beet seed extract, not decolourized, not heated	19	41
Id., diluted twice	60	74
Id., concentrated 4 × by freezing	1	2
Inorganic salt solution	24	47
Id., diluted twice	62	74
Mannitol (3.3 % in water, isosmotic)	13	24
Id., 1.65 %	32	52

Mannitol has a strong osmotic effect, due to the low permeability of the protoplasm for this substance.

Electrodialysis. In order to prove directly that electrolytes have a great influence, we subjected beet seed extract to electrodialysis with cellophane membranes during two days. The resistance of the solution had then increased from 143 Ohms to 2100 Ohms. The results obtained are shown in table 4.

Volatile substances. Volatile substances in beet seeds were found to have an inhibiting effect on germination, but they are not present any more in decolourized extracts.

The influence of beet seed extract on cell elongation. The method used in measuring cell elongation was the coleoptile-cylindertest of BONNER (1), as modified by VAN SANTEN (30). We cut cylinders from the zone of 4—6 mm from the tip (length of the cylinders 1½—2 mm). Indole-3-acetic

acid in a concentration of 1 mg/l was added, as auxin appeared to be the limiting factor for elongation after about one hour and a half. When the experiments last much longer, food factors begin to play a part.

TABLE 12.
Growth in per cent after

	3 *h*	12 *h*
Distilled water	9.2	14.5
Beet seed extract	1.8	2.8
Id., diluted 1 : 50	4.5	8.8
Distilled water with indole-3-acetic acid	16.4	23.1
Beet seed extract with indole-3-acetic acid	2.1	3.0
Id., diluted 1 : 100 with indole-3-acetic acid	11.7	20.0
Inorganic salt soln. with indole-3-acetic acid	0.8	—
Tap water with indole-3-acetic acid	6.6	13.2

Inorganic salts and even tap water decrease the growth rate of the cylinders.

The influence of the extracts on the respiration of Avena seeds and coleoptile cylinders. Dormant seeds show a weak respiration; on germination the consumption of oxygen rapidly increases. We investigated the respiration of inhibited seeds during the first 4 hours by means of the manometric method of WARBURG. Temperature 25° C.; manometer vessels illuminated with 5000 lux.

TABLE 13.
Comsumption of oxygen in mm^3/15 seeds in 210 min after immersion in the solutions.

Distilled water	63
Beet seed extract	61
Inorg. salt solution	71

These differences are within the experimental error. The same results were obtained with *Avena* coleoptile cylinders. In this case the respiratory quotient was 1.0 both in water and in beet seed extract.

Conclusions. The inhibiting effect on the germination of beet seed extracts is caused for the major part by its inorganic components. This follows from the almost equally strong inhibiting effect of the inorganic salt solution. Moreover the inhibiting power disappears after electrodialysis of the extract by which the ions and small organic molecules are removed.

It is known that the osmotic pressure of the medium influences the germination percentage (19, 24, 28, 35, 37). From this view it is significant that mannitol exerts a strong retardation, as this substance slowly penetrates into the cells and therefore has a high osmotic effect. The inhibition by beet seed extracts is generally attributed to specific substances (8, 14, 36, 39). In this case the effect of specific inhibiting substances can only be slight and secondary. The occurrence of auxins in concentrations that

could inhibit germination is not probable in our heated and decolourized extracts (39). Moreover the inhibition of the growth would shift into an acceleration by diluting the extract. This is not the case.

The identity of the inhibition of the growth and of the germination has never been proved, although it has tacitly been assumed by several authors (e.g. 15). There are at least some marked differences, such as the influence of indole-3-acetic acid, which is slight and inconsistent on germination (e.g. 24a) and pronounced in the cell elongation as measured with BONNER's test method. The coleoptile cylinders are very susceptible to ions and other regulators of permeability.

Remarkable is the strong influence of light on the inhibition, which we cannot explain, but which also appears to be important in the case of inhibition by salt and sugar solutions.

Summary.

An investigation was made on substances in the extracts of seedballs of *Beta*, which inhibited the germination. It is shown that in this case osmotic effects are the more important. Specific inhibiting substances could not be detected.

April 1947. *Botanical Laboratory, Utrecht.*

LITERATURE.

1. J. BONNER, J. Gen. Physiol. **17**, 63 (1933).
2. H. BORRISS, Jahrb. Wiss. Botan. **91**, 93 (1943).
3. P. BOYSEN JENSEN, Planta **31**, 244 (1940—1941).
4. L. G. COX, H. M. MUNGER and E. A. SMITH, Plant Physiol. **20**, 289 (1945).
5. M. EVENARI, Palest. J. Botan. J-ser. II, 1 (1940).
6. M. EVENARI, E. KONIS and S. B. ULLMANN, Chron. Bot. VII (4), 149 (1942).
7. P. FRÖSCHEL, Natuurw. Tijdschr. **21**, 93 (1939).
8. ———, Biol. Jaarb. **7**, 73 (1940).
9. P. FRÖSCHEL and E. TILEMANS, Bières et Boissons **10**, (1940).
10. G. L. FUNKE, Beihefte Botan. Centr. **59**a, 235 (1939).
11. H. FUNKE, Jahrb. Wiss. Botan. **91**, 54 (1943).
12. J. GILLIS, Natuurw. Tijdschr. **21**, 263 (1940).
13. ———, Rec. Trav. Chim. **59**, 526 (1940).
14. E. JENTYS, Bull. Intern. Acad. Polon. Sc. Litt. *B*, Sc. nat. 105, 1922.
15. J. JUEL, Dansk Botan. Ark. **12**, n°. 4 (1946).
16. J. KEIL, Jahrb. Wiss. Botan. **88**, 345 (1939).
17. A. KÖCKEMANN, Ber. deutsch. botan. Ges. **52**, 523 (1934).
18. ———, Beihefte Botan. Centr. **55**a, 191 (1936).
19. E. KONIS, Palest. J. Botan. J-ser. II, 6 (1940).
20. M. VAN LAER and P. FRÖSCHEL, Bières et Boissons **8** (1942).
21. M. VAN LAER, P. FRÖSCHEL and J. VANDERSCHUEREN, Institut National des Industries de Fermentation de Bruxelles, Comm. techn. de l'O.N.E.R.I.F. N°. 9 (1942).
22. P. LARSEN, Planta **30**, 673 (1939—1940).
23. H. LINSER, Planta **31**, 32 (1940—1941).
24. L. S. LITVINOV, Bull. Inst. Recherches Biol. Perm **11**, 171 (1928).
24a. T. L. LOO and Y. W. TANG, Am. J. Botany **32**, 106 (1945).

25. W. MAGNUS, Ber. deutsch. botan. Ges. **38**, 19 (1920).
26. P. B. MEDAWAR, G. M. ROBINSON and R. ROBINSON, Nature **151**, 195 (1943).
27. G. MOSHEOV, Palest. J. Botan. J-ser. I, 86 (1938).
28. H. OPPENHEIMER, Sitz ber. Akad. Wiss. Wien I, **131**, 279 (1923).
29. U. RUGE, Z. Botan. **30**, 12, 34, 529 (1939).
30. A. M. A. VAN SANTEN, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **41**, 513 (1938).
31. H. SÖDING, Z. Botan. **36**, 443 (1940—1941).
32. W. A. TETJUREW, Planta **32**, 211 (1941).
33. K. V. THIMANN and C. L. SCHNEIDER, Am. J. Botany **25**, 270 (1938).
34. B. TOLMAN and M. STOUT, J. Agr. Research **64**, 817 (1940).
35. R. UHVITS, Am. J. Botany **33**, 278 (1946).
36. S. B. ULLMANN, Diss. Jerusalem 1940.
37. R. ULRICH, Compt. Rend. **208**, 1835 (1939).
38. H. VELDSTRA and E. HAVINGA, Rec. Trav. Chim. **62**, 841 (1943).
39. ———, Enzymologia **11**, 373 (1945).

Astronomy. — *Egyptian "Eternal Tables"*. I. By B. L. VAN DER WAERDEN. (Communicated by Prof. A. PANNEKOEK.)

(Communicated at the meeting of March 29, 1947.)

Introduction.

Very little is known about the history of Egyptian astronomy. Were the Egyptians in possession of long observations of movement, periods and stationary points of the planets, so that even the Chaldaeans could learn from them, as Diodoros (I 81) says? Were they able to predict celestial phenomena by graphical or geometrical methods, as Theon of Smyrna (p. 177 HILLER) asserts? Were their astronomy and astrology autochtonous or largely influenced by Greek or Chaldaean (= Babylonian) ideas? Does Egyptian astronomical science go back to old-Egyptian wisdom, or is it a product of Hellenistic times? [1]) What was the nature of the knowledge of the Egyptians about the planets laid down in the "eternal tables", as mentioned by a horoscope [2]) for A D 81?

The best method to decide these questions would be the careful analysis of Egyptian astronomical and astrological texts. Only quite recently discussion of the astronomical texts started, mainly through the work of OTTO NEUGEBAUER [3]) [4]) [5]).

I will discuss three of the most important astronomical texts, published by NEUGEBAUER [5]), concerning the dates of entrance of the planets into the signs of the zodiac, and I will show that *they are calculated entirely by Babylonian methods.*

I believe that this fact, combined with other indications of Babylonian influence such as the pictures of the goatfish (= Capricorn), the archer (= Sagittarius) and the weigher (= Libra) on the famous zodiac of Dendera, will help to throw more light on the dark origins of Hellenistic astronomy and astrology in Egypt.

Egyptian Planetary Texts.

The texts will be designed, as in NEUGEBAUER's paper, by the letters P, S and T.

P is the Berlin Papyrus P 8279, first published by SPIEGELBERG [6]).

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[1]) A thorough discussion of these questions is given by O. NEUGEBAUER: Egyptian planetary texts, Trans. Amer. Philos. Soc. 32 (1942) p. 235.

[2]) Published by KENYON, Greek Papyri in the British Museum, Cat. I (1893) p. 133.

[3]) O. NEUGEBAUER and A. VOLTEN, Ein demotischer astronomischer Papyrus (Pap. Carlsberg 9). Quellen u. Studien Gesch. Math. B 4 (1938) p. 383.

[4]) O. LANGE and O. NEUGEBAUER, Papyrus Carlsberg No. 1, ein hieratisch-demotischer kosmologischer Text, Kgl. Danske Vid. Selsk. Hist.-fil. Skrifter 1, no. 2 (1940).

[5]) O. NEUGEBAUER, Eg. Plan. Texts, Trans. Amer. Philos. Soc. 32 (1942), p. 209.

[6]) W. SPIEGELBERG, Demotische Papyrus aus Berlin, Leipzig 1902.

S denote the "Stobart tablets", published by BRUGSCH [7]).

T is the Teptunis papyrus II 274, published by GREENFELL, HUNT and GOODSPEED [8]).

NEUGEBAUER re-edited these Egyptian Planetary Texts (l.c.) with an excellent astronomical and historical commentary.

He found that dates in P are given according to the Egyptian calendar, in S according to the Alexandrian calendar. P covers the years 14—41 of the reign of Augustus (— 16 till + 11 AD). S consists of 3 parts:

A : Years Vespasian 4 — 10 (A D 71 — 77)
$C_1 + C_2$: Years Trajan 9 — Hadrian 3 (A D 105 — 118)
E : Years Hadrian 11 — 17 (A D 126 — 132).

The Greek papyrus T gives dates of entrance of planets into the signs for the years 10 — 18 Trajan nearly identical with those of S. It contains, furthermore, an interesting additional column giving dates of the new moon and subsequent crescent, calculated according to a simple rule, which can also be found in Papyrus Carlsberg 9, published by NEUGEBAUER and VOLTEN (l.c.).

Comparing the positions given in P and S with modern calculations, NEUGEBAUER found a systematic difference (text minus calculation) of about 4° in the beginning of the reign of Augustus and decreasing with time. This means that the texts use a fixed origin of the zodiac, connected with the fixed stars, just as Babylonian moon and planetary tables do. If modern longitudes are reduced to the ecliptic of 100, the difference becomes 5°; hence the origin of the zodiac in our Egyptian texts coincides with that of the Babylonian ephemerides and observation texts of the latest time [9]).

NEUGEBAUER has made it highly probable that P and S belong to the so-called "eternal tables" mentioned by PTOLEMAIOS [10]) and by a Greek horoscope for A D 81 mentioned before. He could, however, not decide by what means the tables were calculated. He rightly observes that some kind of theory must have been used, for the entrance of a planet into a sign is indicated even if it is invisible. In the main "linear part" of the orbits, where the motion is approximately uniform, he found agreement between text and calculation, but in the retrograde part and immediately before and after there are considerable discrepancies, a fact which excludes

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[7]) H. STOBART, Egyptian Antiquities, Paris and Berlin 1855. — H. BURGSCH, Nouvelles Recherches sur la division de l'année, suivies d'une mémoire sur des observations planétaires, Berlin (F. Schneider) and Paris (P. Duprat) 1856.

[8]) B. P. GREENFELL, A. S. HUNT and E. J. GOODSPEED, The Teptunis Papyri II, Univ. of California Publ. 2, London 1907.

[9]) KUGLER, Sternkunde und Sterndienst in Babel II, p. 520. — B. L. v. D. WAERDEN, Babylonische Planetenrechnung, Eudemus I (1941) p. 47. I found for earlier texts (210—160 B C) a difference of about 3° 30′, for 160—130 B C about 4° 10′, and for 110—60 B C exactly 5° 20′, compared with the ecliptic of —100.

[10]) PTOLEMAIOS, Syntaxis IX 2 (p. 211 Heiberg).

the use of something like epicycles and excenters. He conjectures that the tables are the result of combined observations and calculations.

In the following lines it will be shown that the tables P, S, and T are calculated according to Babylonian methods. The velocities of any planet are assumed to be constant during a definite part of the synodic period and in a definite part of the ecliptic, and to jump suddenly to another constant value if these limits in time or space are exceeded, just as in certain Babylonian procedure texts. As a typical example, I quote the procedure text Rm IV 431 [11]). In this text the ecliptic is divided by the points 2° ♐, 17° ♉, 9° ♒, and 9 ♏ into 4 parts, for which different speeds are assumed. On the "fast arcs" from 2° ♐ to 17° ♉ the velocity of Jupiter is

during	1^M after heliacal rising	15′ per	1^D ($1^M = 30^D$)
„	3^M	8′ per	1^D
„	4^M retrograde	5′ per	1^D
„	3^M	7′40″ per	1^D
„	1^M until heliacal setting	15′ per	1^D
„	1^M until heliacal rising	15′ per	1^D

On the "middle arcs" from 9° ♏ to 2° ♐ and from 17° ♉ to 9° ♒ these velocities are multiplied by $\frac{15}{16}$, and on the "slow arcs" from 9° ♒ to 9° ♏ by $\frac{5}{6}$.

Hence the velocities show the same ratio as 40: 45: 48. The total synodic arc is 36° on the "fast arcs", 33°45′ on the "middle arcs" and 30° on the "slow arcs".

It will be shown, that quite similar schemes are used in our Egyptian "eternal tables"; only the velocities are different from those of the known Babylonian procedure texts.

The purpose of the column of new moons and crescents in T becomes clear too: it allows the reduction of Babylonian dates to the Egyptian or Alexandrian calendar; for the Babylonian month begins with the crescent.

Venus.

The dates for Venus for the year 9 Trajan in text S are:

Month	Day	Sign	Difference	Month	Day	Sign	Difference
1	16	♍	24	7	4	♈	25
2	10	♎	24	7	29	♉	25
3	4	♏	25	8	24	♊	24
3	29	♐	23	9	18	♋	25
4	22	♑	24	10	13	♌	25
5	16	♒	24	11	8	♍	38
6	10	♓	24	12	16	♎	—

As one sees from the differences, the planet moves during the first 11 months with a nearly constant velocity. Only at the end of the year, just

[11]) See KUGLER, Sternkunde I, p. 136, and V. D. WAERDEN, Eudemus I, p. 35. KUGLER calls the procedure texts "Lehrtexte".

before the retrogradation begins, the motion becomes slower. During the uniform "linear part" of the motion the time needed to pass a sign alternates with few exceptions between 24 and 25 days throughout the whole table.

Two explanations for these constant differences are possible: either some of the positions were observed and the others found by linear interpolation, or all of them were calculated by assuming a constant velocity.

To decide between these two possibilities, I have reduced the dates of the next synodic period (year 11/12) to the year 9/10 by subtracting a synodic period (584 days) from the dates and a synodic arc of Venus ($360° + 215\frac{1}{2}°$) from the positions. In the same way, I have reduced the dates of the year 12/13 to the year 9/10 by subtracting 2 synodic periods, etc. for the whole text S. All positions thus obtained are shown as points in diagram 1.

As is seen from this diagram, the linear parts of the 16 synodic periods covered by the text are brought to exact coincidence by our reduction procedure. The deviations seldom exceed 2 days. This proves that all positions are calculated. For if they were partly derived from observation, deviations up to 5 or 6 days ought to occur, owing to the anomalies of the sun and Venus. Observational errors and the difficulty of dividing the ecliptic into 12 exactly equal signs would tend to increase these deviations still more.

Furthermore one sees from the diagram, that about the date month 11 day 13 a sudden change in velocity takes place. After this change, the velocity remains constant again during 1 month at least. The law of the retrograde motion is not quite clear, but in any case the retrograde arc is too large (at least 18°, probably 20° or more), just as in the Babylonian Venus tablet Sp I 548 + 230 [12]). After the retrograde motion the velocity increases again stepwise until it resumes its original value in the linear part.

The highest and second highest velocities, obtained graphically from the diagram, are

480° in 392 or 393 days (linear part)
23° in 30 days (before and after).

One gets much simpler values, if the velocities are calculated not for 30 days but for one Babylonian month. As I have shown in Eudemus I, the fundamental unit of time in Babylonian planetery texts is the mean synodic month of 29,53 days, which is divided for the purpose of easy calculation into 30 artificial "days". Writing 1^M and 1^D for the Babylonian month and artificial day, 1^d for the real day, we find the two velocities

$$480° \text{ in } 400^D, \text{ i.e. } \tfrac{6}{5} \ (° \text{ per } {}^D)$$
$$22\tfrac{1}{2}° \text{ in } 30^D, \text{ i.e. } \tfrac{3}{4} \ (° \text{ per } {}^D)$$

These two values are certain, but the law of motion during the retrograde course and immediately before and after is not quite clear. A reasonable

12) KUGLER, Sternkunde und Sternendienst I, p. 205.

conjecture seems to be the following, represented by the broken line in the diagram:

14^M	4^D	velocity	$1°12'$	per 1^D,	travelled distance	$508°48'$
1^M		„	$45'$	„ „	„ „	$22°30'$
1^M		„	$36'$	„ „	„ „	$18°$
1^M	10^D	„	$30'$	retrograde,	„ „	$-20°$
1^M	10^D	„	$36'$	per 1^D,	„ „	$24°$
1^M		„	$45'$	„ „	„ „	$22°30'$
19^M	24^D	total synodic period,			„ „	$575°48'$

Slight alterations are possible, but in any case the law of motion is quite analogous to those of the Babylonian procedure texts for Jupiter (see Introduction) and Saturn (see hereafter), only in the case of Venus no distinction between different parts of the ecliptic needs to be made, the anomalies of Venus and the sun being small.

Mars.

As yet 4 indications pointed to Babylon:
the fixed origin of the zodiac,
the retrograde arc of Venus of about 20°, which is too large,
the simple values of the velocities in Babylonian units,
the law of motion analogous to Babylonian procedure texts.

We could add that planetary calendars of the same kind as our Egyptian ones, recording the entrance of planets into signs of the ecliptic, exist in cuneiform texts since 200 BC [13]). But the definite proof of the Babylonian origin of the Egyptian tables is given by considering the motion of Mars in text S.

In this case too we have a "linear part" of the motion and an "irregular part", where it is slower or even retrograde. In the linear part the date differences, that is, the times required for traversing the signs of the ecliptic, are as follows (see following page):

It is seen at once, that the following differences occur most frequently:

in	♍	♎	48
in	♏	♐	41, 42 and 43
in	♑	♒	38
in	♓	♈	41
in	♉	♊	46
in	♋	♌	54

[13]) See KUGLER, Sternkunde und Sternendienst I, p. 92—107, II 470—513 and 3. Ergänzungsheft p. 358—374. In the planetary tables for the years 178, 234 and 236 of the Seleucid era (KUGLER II, p. 496, 490 and 474) the velocity of Venus in the linear part of the motion is 30° in 25^D, just as in our Egyptian tables.

Date differences for Mars in different Signs
(~ means retrograde motion)

Text and year		♍ ♎	♏ ♐	♑ ♒	♓ ♈	♉ ♊	♋ ♌
Vespasian year	4/5		41 41	37 34	39 38	44 46	54 55
	6	~ 48	41 41	38 38	40 38	44 41	54 54
A	7/8	48	~ 43	38 38	41 41	46 43	48 53
	9	48 48	43	~		42	
Trajan year	9		41	38 38	40 40	46 42	51 54
	10/11	48	~ 43	38 38	41 41	46 44	48 51
C_1	12/13	48 48	44 52	~	43 43	48 48	53 49
	14/15	45 48	42 42	38 39	~	46 46	54 54
C_2	16/17	43 44	43 42	38 38	41 39	~	52 52
	18/19	46 45	39 31	48 38	41 41	46	~ 51
Hadrian year	1/2	46 46[14]	41[14] 39	36 37	39 40	46 46[15]	60[15]
	3	~ 46	46 41	26 38	40 35	43 46[14]	
Hadrian year	11					46 46	53 48
E	12	49 46	42 42	36 38	41		

In the middle of the linear part, near the conjunction, deviations from these normal values occur frequently, just as if after the heliacal setting or rising the dates were corrected in order to restore the exact synodical period, but near the beginning and the end of the linear part, the normal values prevail.

I shall call these normal values t, taking in the second case (♏ ♐) $t = 42$ as a mean value.

Now, in a Babylonian table for Mars exactly the same division of the ecliptic into 6 parts of 2 signs each occurs. The rule for calculation of the synodical arc of Mars in AO 6481 is as follows[16]):

if the arc begins in ♍ or ♎ it is 360° + 40°, but if the arc of 40° exceeds 30° ♎, then for every degree one half degree is added;

if it begins in ♏ or ♐ the arc is 360° + 60°, but if it exceeds 30° ♐, for every degree one-half degree is added;

if it begins in ♑ or ♒, the arc is 360° + 90°, but if it exceeds 30° ♒, for every degree 15′ are subtracted;

if it begins in ♓ or ♈, the arc is 360° + 67½°, but if it exceeds 30° ♈, for every degree 20′ are subtracted;

if it begins in ♉ or ♊, the arc is 360° + 45°, but if it exceeds 30° ♊, for every degree 20′ are subtracted;

if it begins in ♋ or ♌, the arc is 360° + 30°, but if it exceeds 30° ♌, for every degree 20′ are added.

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14) Following NEUGEBAUER, I have corrected 1 1 ♏ (C_2, year 1) into 1 30, and 12 30 ♋ (year 3) into 12 1.

15) I have corrected 12 14 ♋ (C_2, year 1) into 12 24.

16) KUGLER, Sternkunde und Sternendienst in Babel II, p. 580.

Hence the velocity of Mars is, just as in our Egyptian text, largest in ♑ and ♒, smaller in ♓ and ♈, still smaller in ♏ and ♐, much smaller in ♉ and ♊, still smaller in ♍ and ♎, and smallest in ♋ and ♌. This qualitative correspondence can even be formulated quantitatively: if 360 + u is the synodical arc of Mars according to AO 6481, and t as before the time necessary to traverse a sign of the ecliptic in the linear part of the motion according to S, we have the following corresponding values

Signs	u	360° : u	t
♍ ♎	40°	9	48
♏ ♐	60°	6	42
♑ ♒	90°	4	38
♓ ♈	67° 30′	5½	41
♉ ♊	45°	8	46
♋ ♌	30°	12	54

Hence a linear relation

$$t = 2 \cdot \frac{360}{u} + 30$$

holds. This means that the velocity of Mars in the different signs of the ecliptic is given by

$$\frac{30^\circ}{t} = \frac{30\,u}{720 + 30\,u} = \frac{u}{24 + u} \ (^\circ \text{ per } ^d) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Theoretical deduction of (1).

A relation like (1) could be expected a priori, if the motion of Mars is represented by a scheme like that of the Babylonian procedure texts.

Let us suppose that this scheme was as follows:

In ♋ and ♌, where the velocities are smallest, let v_1 be the velocity of fast motion (valid in the linear part of the course), v_2 that of the following slower motion, — v_3 that of retrograde motion, v_4 that of the slow motion following it. Let, further, τ_2, τ_3, τ_4 be the numbers of days during which the velocities are v_2, — v_3, and v_4.

In ♍ and ♎, let all these velocities be multiplied by a factor c_1, in ♏ and ♐ by c_2, in ♑ and ♒ by c_3, in ♓ and ♈ by c_4, in ♉ and ♊ by c_5, the times τ_2, τ_3, τ_4 remaining unaltered, just as in the procedure texts for Jupiter and Saturn.

Now what is the time necessary to traverse the two signs ♋ and ♌, if the planet enters the sign ♋ with velocity v_1, remaining within these two signs during the times τ_2, τ_3 and τ_4 (with velocities v_2, — v_3 and v_4) and leaving the sign ♌ with velocity v_1 again?

During the time $\tau_2 + \tau_3 + \tau_4$ the travelled distance is $\tau_2 v_2 - \tau_3 v_3 + \tau_4 v_4$.

The remaining distance $60° - \tau_2 v_2 + \tau_3 v_3 - \tau_4 v_4$ is traversed with velocity v_1, hence the time is

$$\tau_2 + \tau_3 + \tau_4 + \frac{60° - \tau_2 v_2 + \tau_3 v_3 - \tau_4 v_4}{v_1} =$$

$$\frac{60}{v_1} + \tau_2\left(1 - \frac{v_2}{v_1}\right) + \tau_3\left(1 + \frac{v_3}{v_1}\right) + \tau_4\left(1 - \frac{v_4}{v_1}\right) = \frac{60}{v_1} + \tau$$

where τ depends only of τ_2, τ_3, τ_4 and the *ratios* of the velocities. τ is the *delay* caused by retardation and retrogradation, for without this delay the time would be $\frac{60}{v_1}$.

Now τ remains the same if the velocities v_1, v_2, v_3, v_4 are multiplied by a common factor c_i. Hence the delay τ is independent of the assumed signs ♉ ♌. Thus we find for the time necessary to traverse any pair of signs like ♍ ♎

$$\frac{60}{c_i v_1} + \tau$$

the factor c_i depending on the chosen signs, but τ being always the same.

By a slight modification of the argument one can even see that the delay τ remains the same if the retrograde arc belongs to two adjacent signs having different velocity factors c_i. The times t_2, t_3, and t_4 must in this case be split up into terms $t_2' + t_2''$, etc., belonging to these signs, but the result remains the same. In such a case, the time necessary to traverse e.g. the four signs ♉ ♌ ♍ ♎ will be

$$\frac{60}{v_1} + \frac{60}{c_i v_1} + \tau$$

where τ is again the delay due to retardation and retrogradation:

$$\tau = \tau_2\left(1 - \frac{v_2}{v_1}\right) + \tau_3\left(1 + \frac{v_3}{v_1}\right) + \tau_4\left(1 - \frac{v_4}{v_1}\right).$$

Now what is the time necessary to traverse the whole ecliptic, beginning with normal velocity v_1 or $c_i v_1$, next proceeding slower, going back, then proceeding forward slowly and fast again? If we call this time T, we have obviously

$$T = \frac{60}{v_1} + \frac{60}{c_1 v_1} + \frac{60}{c_2 v_1} + \frac{60}{c_3 v_1} + \frac{60}{c_4 v_1} + \frac{60}{c_5 v_1} + \tau.$$

Thus it is seen that *T is independent of the sign or signs in which retardation or retrogradation takes place* [17]).

We do not know the exact value of T, because we ignore the exact

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[17]) The time *T* is not identical with the orbital period, which is the time in which Mars in *mean* motion would traverse the ecliptic.

Babylonian law of motion. In our text S, the time T necessary to traverse the whole ecliptic varies between 698^d and 712^d, the mean value being about 707^d. The deviations from this mean value are probably due to casual corrections of the dates, just as the deviations from the normal values of t we previously observed. It turns out that the best agreement is reached by adopting the value

$$T = 1\tfrac{14}{15} \text{ years} = 706^d.$$

Now let v be the sun's velocity. In one synodic period Mars covers a distance $360 + u$, and the sun a distance $720 + u$, hence the synodic period is

$$\frac{720 + u}{v}$$

But the time necessary for Mars to proceed 360°, including the retardated and retrograde part of the motion, is T, hence the time necessary to cover the remaining distance u in the linear part of the motion is

$$\frac{720 + u}{v} - T.$$

So the velocity $\frac{30}{t}$ is

$$\frac{30}{t} = \frac{u}{\frac{720 + u}{v} - T} = \frac{u}{720 - Tv + u} \cdot v. \quad . \quad . \quad . \quad . \quad (2)$$

Now Tv is the arc which the sun covers in $T = \frac{29}{15}$ years, vid

$$Tv = \tfrac{29}{15} \cdot 360° = 720° - 24°$$

hence (2) yields

$$\frac{30}{t} = \frac{u}{24 + u} v$$

which is nearly the same thing as (1), since v is nearly 1 (° per ^{d}).

The exactitude of our determination of the constants in (1) is of course not sufficient to decide which exact values of T and v were adopted. But in any case the form of the relation (1) is just what might be expected from Babylonian planetary theory, and also the order of magnitude of the constants involved agrees with expectation.

Jupiter.

KUGLER has analyzed 3 types of Babylonian Jupiter tables. In the tables of the first kind the ecliptic is divided into 2 parts: from 30° ♏ to 25° ♐ the synodic arc of Jupiter is 36°, and from 25° ♐ to 30° ♏ it is 30°. In the tables of the second kind the ecliptic is divided into 4 parts: from 9° ♋ to 9° ♏ the synodic arc is 30°, from 9° ♏ to 2° ♑ and from 17 ♉ to 9° ♋ it is 33°45′, and from 2° ♑ to 17 ♉ it is 36°, just as in the

procedure text Rm IV 431. The time intervals necessary to cover these distances of 30°, 33°45′ and 36° are nearly $13^{M}12^{D}$, $13^{M}16^{D}$, and $13^{M}18^{D}$ (see Eudemus I, p. 34).

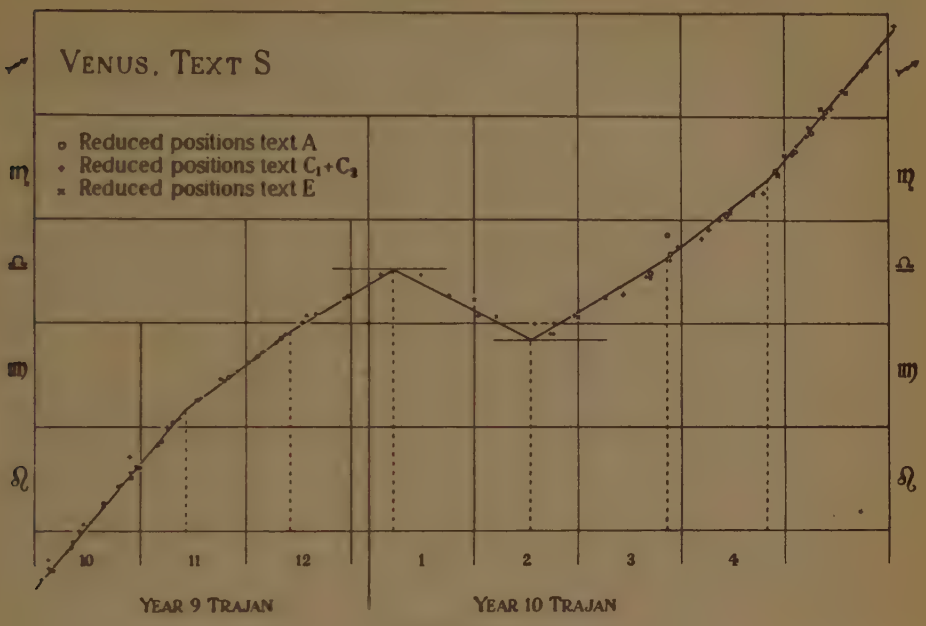

Diagram 1.

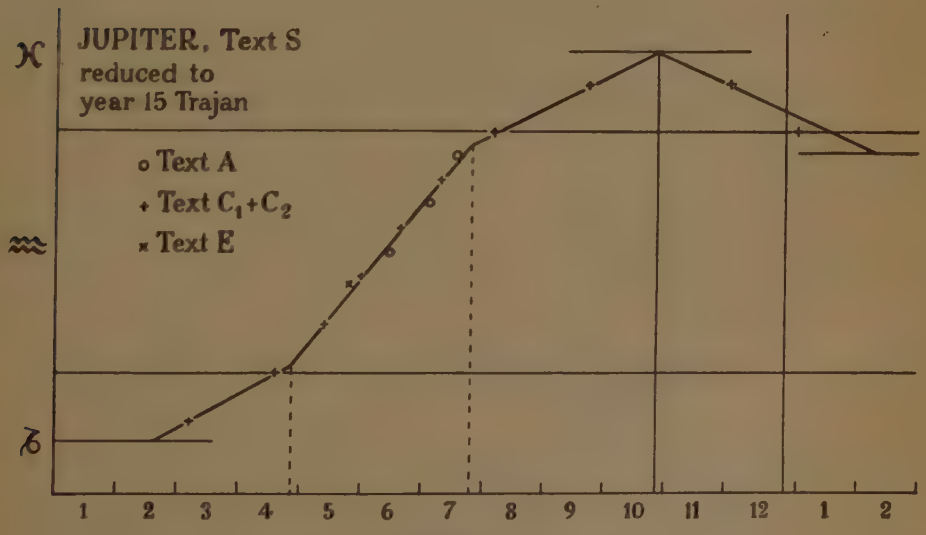

Diagram 2.

If our Egyptian tables are composed according to the same scheme, we should expect that in the "slow part" (♌ ♍ ♎, perhaps also ♉ and ♏) the time necessary to traverse a sign of the ecliptic (30°) should be

$13^{M}12^{D} = 395^{d}$, and the retrograde motion too should be reproduced exactly in the next sign after that lapse of time.

This expectation is fulfilled in both tables P and S, for the dates of entrance into the signs and the time intervals to the next similar entrance are:

Text P	forward	retrograde	forward	Time intervals		
Augustus year 17	2 20 ♍	8 5 ♌	9 29 ♍	395d	399d	[394]
„ 18	3 20 ♎	[9] 9 ♍	[10 28 ♎][18]	397	398	[394]
„ 19	4 22 ♏	10 12 ♎	11 27 ♏	[396]		
„ 20	[5] 23 ♐					
year 26	12 1 ♉			393		
„ 27	12 29 ♌			400		
„ 29	1 29 ♍			400		
„ 30	[3] 4 ♎			400		
„ 31	3 9 ♏			404		
„ 32	4 18 ♐					
year 38	1[1] 14 ♉			393		
„ 39	12 12 ♌			400		
„ 41	1 12 ♍					

Text S	forward	retrograde	forward	Time intervals		
Vespasian year 4			11 24 ♎			387
A „ 5	4 6 ♏	9 16 ♎	12 16 ♏	380		
„ 6	4 21 ♐					
Trajan year 9	12 28 ♍			395		
„ 11	1 22 ♎			394		
C_1 „ 12	2 21 ♏			392		
„ 13	3 18 ♐					
Trajan year 19	10 15 ♉			393		
Hadrian „ 1	11 13 ♌			368, corrected 398		
C_2 „ 2	11[19] 16 ♍					
„ 3	no entrance					
Hadrian year 12	10 3 ♉			393		
„ 13	[11] 1 ♌			398		
E „ 14	12 4 ♍			393		
„ 16	1 1 ♎			396		
„ 17	2 2 ♏			147		
	6 29 ♐					

I cannot explain why several time intervals in P are about 5^{d} too large. Those in S are quite correct: the deviations do not exceed 4^{d}, except twice in A. Even in the signs ♉ and ♏ the correct intervals appear; only in Hadrian year 17 the dates are irregular. In all other cases the signs ♉ and

[18]) Restored by interpolation.

[19]) Presumably an error: should be 12.

♏ are obviously included in the region of slow motion, just as in the Jupiter tables of the first kind.

Next we seek to determine the law of motion in the "fast past" of the ecliptic, consisting at least of the signs ♑ ♒ ♓ ♈ ♉. Assuming a synodic arc of 36° and a synodic period of $18^{M}18^{D} = 401^{d}$ in this region, one can reduce all positions of Jupiter to one synodic period, just as it was done in the case of Venus. I have chosen the year 15 Trajan and reduced all other positions of C in the fast region to it by adding or subtracting multiples of 36° to the positions and of 401^{d} to the dates. From the positions of E in the fast region I first subtracted 5° and from the data 12 years 5^{d} according to Babylonian planetary theory, and likewise I added 15° to the positions of A and 36 years 15^{d} to the data; after this operation these positions too could be reduced to the year 15 Trajan. The result is shown in Diagram 2. The law of motion in the irregular and especially in the retrograde part of the motion is uncertain, but in the linear part a straight line fits very well. As is seen from the diagram, the velocity during 3 months in the neighbourhood of the conjunction is

$$27° \text{ in } 90^{D}, \text{ i.e. } 18' \text{ per } 1^{D}.$$

This is larger than the velocity of 15′ per 1^{D} assumed in Rm IV 431 (see Introduction), but the time of 90^{D} (probably again 1^{M} before heliacal setting, 1^{M} invisible and 1^{M} after rising) is quite the same. The value of 15′ is better than 18′, just as the division of the ecliptic into 4 parts assumed in Rm IV 431 is better than the division into 2 parts used by the more ancient kind of tables. Hence it seems that the Babylonian source of S follows a more primitive system than that of Rm IV 431. Also the retrograde arc seems to be larger (perhaps 12° in the "fast part", 10° in the "slow part" of the ecliptic).

Crystallography. — *Calculation of the stereographic pole figure of the cubic lattice for any given direction* [*HKL*]. I. By W. MAY. (Communicated by Prof. J. M. BURGERS.)

(Communicated at the meeting of March 29, 1947.)

1. *Introduction.*

The method of SCHIEBOLD and SACHS [1]) is often used when the orientation of a single crystal must be determined. A transmission LAUE photograph of the single crystal is taken and from this (generally asymmetric) photograph a stereographic pole figure is prepared. By means of a stereographic net the pole figure is rotated till an important zone lies on the reference circle; this operation brings the projection of the corresponding zone axis in the centre of this circle. By preparing beforehand stereographic pole figures for the more important crystallographic directions of the crystal lattice (standard projections), it is possible to find a correspondence between the rotated projection and one of the standard projections. Indices can then be assigned to every LAUE spot and the axes can be plotted.

On a LAUE photograph nearly always more than one zone can be observed, but it complicates the process too much if for every zone axis a standard projection is prepared. SCHIEBOLD and SACHS, who worked out the method for the cubic face-centered lattice, have limited themselves to 5 standard projections, viz. for the directions [1 1 0], [0 0 1], [1 1 2], [1 3 0], [1 1 1]; one of the corresponding zones is practically always present in the LAUE pattern. The above sequence of crystallographic directions is that of decreasing packing density. Our experience with a large number of LAUE photographs of aluminium single crystals confirms this limitation; the various projections were used according to the following percentage scale: [1 1 0] 60 %, [0 0 1] 21 %, [1 1 2] 11 %, [1 3 0] 7 %, [1 1 1] 1 %.

In order to obtain precise results, it is necessary to construct these standard projections, as reproduction of those given by SCHIEBOLD and SACHS is rather inaccurate (for a radius of 7 cm the error is about 2°), especially when an enlargement is required. The construction can be made in several ways:

a. construction with the aid of a stereographic net, by laying off the calculated angles between the direction of the standard projection and the poles of the planes. It is quite probable that SCHIEBOLD and SACHS have

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[1]) E. SCHIEBOLD und G. SACHS, Z. Krist. **63** (1926) 34.

constructed their projections in this way [2]). It is clear that this method is not very accurate either, as the accuracy of a stereographic net with a radius of 10 cm is about $\frac{1}{2}°$. As the net is also used for the orientation determination, the possible error is doubled and comes to 1°.

b. construction with the aid of descriptive geometry. This method has clearly the drawback that this construction becomes very complicated and that errors will occur frequently. The accuracy will be slightly better than method *a*.

c. calculation of the coordinates of the projection. With this method any desired degree of accuracy can be obtained so that the precision of the standard projection is only limited by the plotting of the calculated distances and does not exceed a few tenths of a degree. In the following sections this calculation is given in detail.

2. *General course of the calculation.*

In fig. 1 the axes X and Y are mutually perpendicular and lie in the plane of projection. In their point of intersection O a perpendicular Z is

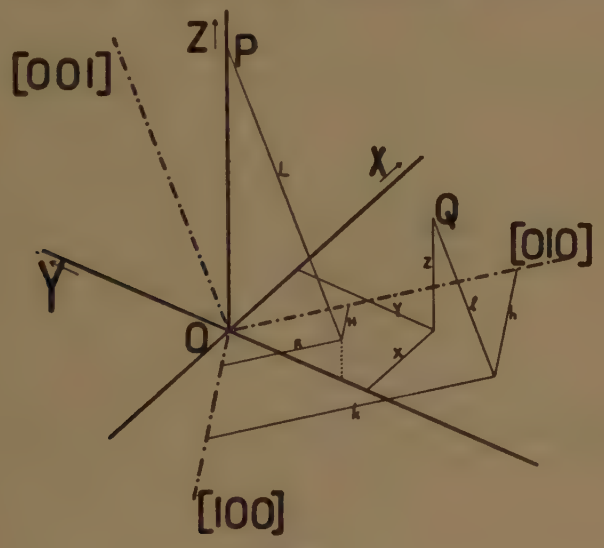

Fig. 1.

erected so that $OXYZ$ is a three-dimensional set of rectangular axes. The cubic lattice is represented by the three axes [1 0 0], [0 1 0] and [0 0 1] and forms a second set. Let the direction $[HKL]$ of the cubic lattice

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[2]) We have not been able to find another method published; see for example:
Internationale Tabellen zur Bestimmung von Kristallstrukturen, Berlin 1935, II, 687.
R. GLOCKER, Materialprüfung mit Röntgenstrahlen, 2. Aufl., Berlin 1936, 364.
F. HALLA und H. MARK, Röntgenographische Untersuchung von Kristallen, Leipzig 1937, 199.
C. S. BARRETT, Structure of Metals, New-York, 1943, 33.
A. TAYLOR, An Introduction to X-Ray Metallography, London 1945, 251.

coincide with the Z axis, $[HKL]$ being the direction for which the standard projection must be calculated. This involves that the pole (HKL) lies in the centre of the projection and therefore the poles of the corresponding zone planes lie on the reference circle. It is further supposed that $[0\,0\,1]$ lies in the OYZ-plane; then the X axis lies in the plane O — $[1\,0\,0]$— $[0\,1\,0]$ [3]). In this way the pole $(0\,0\,1)$ is always situated on the positive side of the Y axis of the projection, as is also the case in the standard projections of SCHIEBOLD and SACHS.

In the cubic lattice the direction $[hkl]$ (that is the line connecting the origin with the point hkl) is perpendicular to the plane (hkl), so that one may say that plane (hkl) is represented by point hkl. Let Q in fig. 1 represent the point hkl; then first the coordinates X, Y and Z of Q must be calculated. Then the point of intersection of the direction $[hkl]$ with the reference sphere (centre O) is stereographically projected upon the plane OXY and the coordinates x and y of the projection are calculated from X, Y, Z.

3. *X, Y, Z as a function of H, K, L, h, k, l.*

The problem of the calculation of the coordinates of a point in a second set of rectangular axes when the coordinates in the first set are given, the

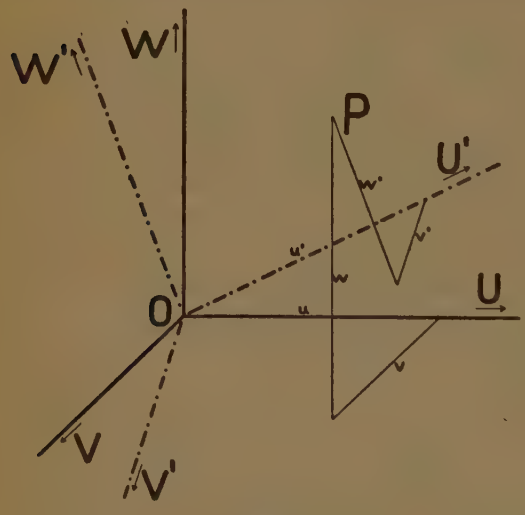

Fig. 2.

TABLE I.

	U	V	W
U'	a_1	b_1	c_1
V'	a_2	b_2	c_2
W'	a_3	b_3	c_3

sets having the same origin and a given mutual position, is solved in elementary analytical geometry. If in table I a_i, b_i and c_i $(i = 1, 2, 3)$ represent the cosines of the angles between the two sets $OUVW$ and

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[3]) This can be easily proved as follows: $OX \perp$ plane OYZ and therefore $OX \perp$ [001]; now [100] and [010] are both $\perp$ [001] and therefore OX, [100] and [010] lie in the same plane, perpendicular to [001].

$OU'V'W'$ (fig. 2), then the transformation equations for the coordinates are:

$$\left.\begin{aligned} u &= a_1 u' + a_2 v' + a_3 w' \\ v &= b_1 u' + b_2 v' + b_3 w' \\ w &= c_1 u' + c_2 v' + c_3 w' \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (1)$$

Let the set $OUVW$ of fig. 2 be set $OXYZ$ of fig. 1 and set $OU'V'W'$ the cubic lattice, then we have:

$$\left.\begin{aligned} u &= -Y;\ v = -X;\ w = +Z \\ u' &= +k;\ v' = +h;\ w' = +l. \end{aligned}\right\} \quad \ldots\ldots \quad (2)$$

The cosines are related to each other by six independent equations:

$$a_i^2 + b_i^2 + c_i^2 = 1 \ (i = 1, 2, 3) \qquad a_i a_j + b_i b_j + c_i c_j = 0 \ (i = 1, 2, 3;\ j = 1, 2, 3;\ i \neq j).$$

From this and the location of the cubic lattice (section 2) they can be calculated (see table II):

TABLE II.

$$\left.\begin{aligned} a_1 &= +\frac{KL}{\sqrt{(H^2+K^2)\cdot\Sigma H^2}};\ b_1 = -\frac{H}{\sqrt{H^2+K^2}};\ c_1 = +\frac{K}{\sqrt{\Sigma H^2}} \\ a_2 &= +\frac{HL}{\sqrt{(H^2+K^2)\cdot\Sigma H^2}};\ b_2 = +\frac{K}{\sqrt{H^2+K^2}};\ c_2 = +\frac{H}{\sqrt{\Sigma H^2}} \\ a_3 &= -\frac{H^2+K^2}{\sqrt{\Sigma H^2}};\ b_3 = 0;\ c_3 = +\frac{L}{\sqrt{\Sigma H^2}} \end{aligned}\right\} \quad (3)$$

Putting (2) and (3) in (1) gives:

$$X = \frac{Hk - Kh}{\sqrt{H^2+K^2}};\quad Y = \frac{H(Hl-Lh) + K(Kl-Lk)}{\sqrt{(H^2+K^2)\cdot\Sigma H^2}};\quad Z = \frac{Hh+Kk+Ll}{\sqrt{\Sigma H^2}} \quad (4)$$

where $\Sigma H^2 = H^2 + K^2 + L^2$.

4. *x and y as a function of H, K, L, h, k, l.*

The point of intersection of the line OQ with the reference sphere (centre O, radius R) is now projected stereographically in the plane of projection OXY. *It will be clear that this is wholly equivalent to projecting the plane* (hkl).

In fig. 3*a* is drawn the plane through the Z axis and point Q and the intersection of this plane with the reference sphere. In fig. 3*b* the projection plane is given. It is easy to see that:

$$OS = R \tan \tfrac{1}{2} \angle ROV = R \sqrt{\frac{1-\cos \angle MQO}{1+\cos \angle MQO}} = R \sqrt{\frac{OQ-MQ}{OQ+MQ}}.$$

$$OS = R\sqrt{\frac{\sqrt{X^2+Y^2+Z^2}-Z}{\sqrt{X^2+Y^2+Z^2}+Z}} = R\frac{\sqrt{X^2+Y^2}}{\sqrt{X^2+Y^2+Z^2}+Z} = R\frac{\sqrt{X^2+Y^2}}{\sqrt{\Sigma h^2}+Z},$$

where $\Sigma h^2 = h^2 + k^2 + l^2$.

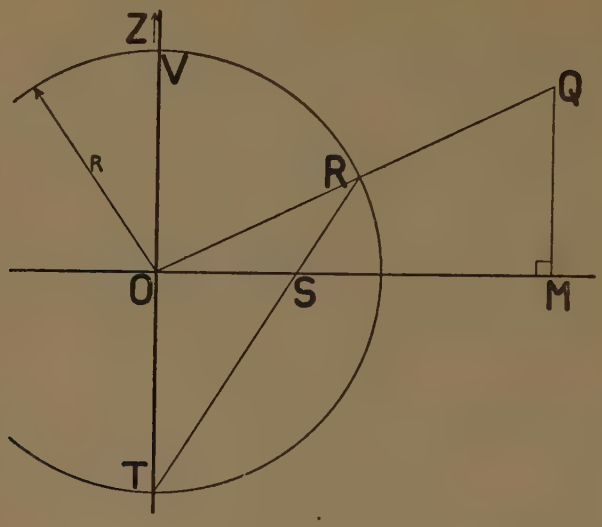

Fig. 3*a*.

Fig. 3*b* shows that:

$$x = \frac{OS}{\sqrt{X^2+Y^2}}\,X \text{ and } y = \frac{OS}{\sqrt{X^2+Y^2}}\,Y.$$

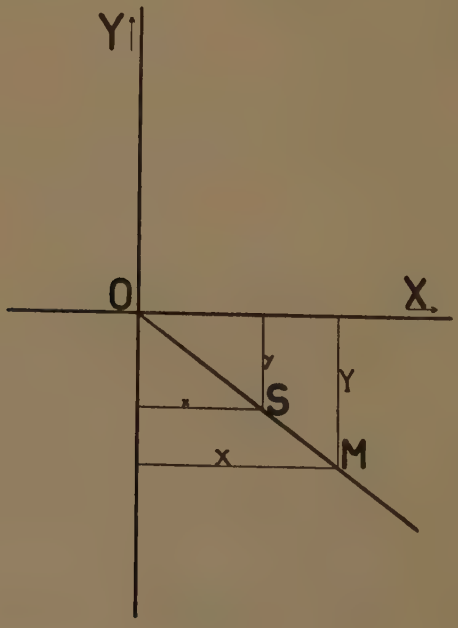

Fig. 3*b*.

With equations (4) this gives:

$$\left.\begin{aligned} x &= \frac{(Hk-Kh)\cdot\sqrt{\Sigma H^2}}{(\sqrt{\Sigma H^2\cdot\Sigma h^2}+Hh+Kk+Ll)\cdot\sqrt{H^2+K^2}}\,R;\\ y &= \frac{H(Hl-Lh)+K(Kl-Lk)}{(\sqrt{\Sigma H^2\cdot\Sigma h^2}+Hh+Kk+Ll)\cdot\sqrt{H^2+K^2}}\,R. \end{aligned}\right\} \quad . \quad . \quad (5)$$

With these equations it is possible to calculate the coordinates of the stereographic projection of every plane in the cubic lattice for any given direction $[HKL]$.

It is possible to extend this method to other symmetry systems, but the difficulties in handling the more and more complicated equations will grow rapidly.

History of Science. — *The Amici-microscopes about 1850 in possession of the University of Utrecht.* By P. H. VAN CITTERT and J. G. VAN CITTERT–EYMERS. (Communicated by Prof. A. J. KLUYVER.)

(Communicated at the meeting of March 29, 1947.)

When about 1800 achromatic microscope objectives were constructed, only relatively weak objectives could be made, so that an important improvement was only attained for the lower magnifications. Indeed, the achromatic objective must necessarily consist of a convex crown lens and a concave flint lens. Now, in order to obtain high magnifications the convex lens must be very much more powerful than in the case of a single monochromatic lens and this leads to practical difficulties in their construction. SELLIGUE and CHEVALIER, soon followed by others, were the first to obtain high magnifications by combining into a system a number of small lenses, each of which had been separately achromatized as well as possible. The advantage of this method was that by simply adding or removing one or more of the lenses, the magnification could be altered very easily. The drawback, however, was the accumulation of the separate spherical aberrations, which in the case of higher magnifications spoiled the resolving power. This explains why the resolving power of the compound microscope, though increased considerably by achromatizing, still remained below that of the simple microscope. It is due to the ingenious AMICI (1786—1863), among others, that these difficulties were overcome. He showed that in order to arrive at a high resolving power the objective must be composed of different parts, each of which separately still can give rise to aberrations, but which are so computed that they neutralize each other's impairing influence. He was also the first to draw attention to the part played by the cover glass and to the great advantage of having at one's disposal a number of eyepieces of different powers, as well as a number of different objectives. He pointed out, moreover, the influence of a larger aperture and the great advantage of immersion.

About 1850 the University of Utrecht was, as far as can be traced, in the possession of three original AMICI-microscopes:

A. the microscope bought in 1835 for *f* 750.—, described by HARTING [1]) in his famous book entitled: Het Mikroskoop, part II, page 85, 1848;

B. the microscope bought in 1836 for *f* 500.— by Prof. VORSSELMAN DE HEER for Prof. G. MOLL on behalf of the Physical Society (Natuurkundig Gezelschap) of Utrecht;

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[1]) Prof. P. HARTING was professor at the University of Utrecht from 1843 to 1882. He was succeeded in 1882 by Prof. A. A. W. HUBRECHT and died in 1885.

P. H. VAN CITTERT and J. G. VAN CITTERT–EYMERS: *The Amici-microscopes about 1850 in possession of the University of Utrecht.*

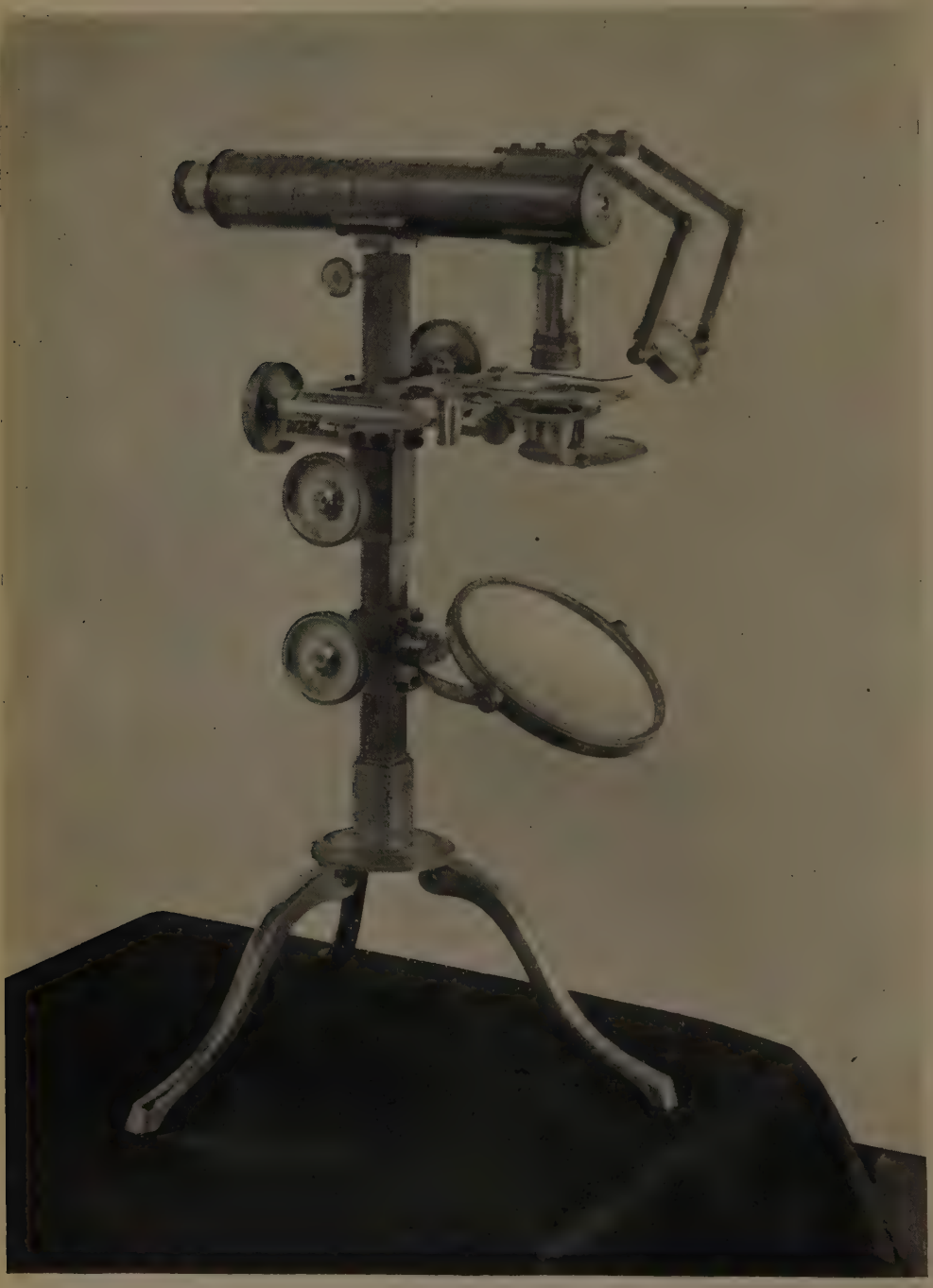

Fig. 1. AMICI microscope, bought in 1836 by the Physical Society of Utrecht,

C. the microscope bought in 1849 for *f* 250. — by Prof. HARTING personally and presented by him to the Zoological Laboratory of Utrecht. This microscope is amply described in the above mentioned book by HARTING, part III, page 205, 1850.

As regards microscope A, HARTING informs us that ten achromatic double lenses belonged to it, out of which various combinations could be formed. Only four of them, called by him 1, 2, 3 and 4, are mentioned in his book. Further, five eyepieces belonged to this microscope, the strongest of which was not used by HARTING, as its magnification was unnecessarily large. Besides, a weak eyepiece of DOLLOND was used, designated by HARTING as no. 1. The AMICI-eyepieces used by him were numbered 2, 3, 4 and 5. From HARTING's description we may conclude that in mechanical respect this microscope probably strongly resembled microscope B, described below. Whether microscope A has been in the possession of the Zoological Laboratory or in that of Prof. HARTING personally cannot be ascertained. It has disappeared without leaving any traces. One of the eyepieces (no. 4), however, has recently been found again (see below).

The mechanical construction of microscope B (and also probably of microscope A) can be seen in fig. 1. It is constructed as follows: on a folding tripod is mounted a brass stand along which a very large concave illuminating mirror and the stage as well, can be made to slide by rack and pinion. The horizontal microscope is attached to the upper end of the stand. AMICI constructed the tube horizontally, convinced that this facilitated the observations: he attained this by placing a rectangular prism above the objective. The stage can be moved in two directions at right angles to each other. These displacements can be measured with the aid of two micrometer screws of which the divisions correspond to 0.0031 mm and to 0.00246 mm (= 0.0001 inch) respectively. Directly under the stage is fitted a cone with a diaphragm-wheel with three apertures. This cone can be rotated out of the optical axis. There is, further, a small ground glass plate capable of the same rotation. A plano-convex condensing lens for the illuminating of opaque objects is attached to the microscope tube, and one of the objectives is fitted with a LIEBERKUEHN mirror [2]). Three different camera lucida belonged to this microscope [3]). Originally five eyepieces and ten objectives belonged to it. The objectives were numbered from 1—8, 10 and 12 and could be combined by two small cylindrical tubes no. 0 and 11 in the following ways:

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[2]) A. LIEBERKUEHN mirror is a small concave mirror fixed to the objective for the purpose of converging light from above onto opaque objects.

[3]) A detailed description can be found in: Descriptive Catalogue of the collection of microscopes in charge of the Utrecht University Museum with an introductory historical survey of the resolving power of the microscope by P. H. VAN CITTERT, pages 68—71, 1934. Edition P. Noordhoff N.V., Groningen.

U 1	U 2	U 3	U 4	U 5	U 6	U 7	U 8
10	10						
	11	0	0	0	0	0	0
	12	1	1	1	1	4	6
			2	2	2	5	7
				2	3	6	8

The only difference between *U* 4 and *U* 5 was that the LIEBERKUEHN mirror (also marked 2) was attached to the lower end of the combination 0—1—2.

According to AMICI the thickness of the cover glass was of no importance when using the combination *U* 1 to *U* 5, *U* 6 required a definite clearly determined thickness, *U* 7 should be used without a cover glass, while with *U* 8 it was necessary to cover the object with a thin plate of mica.

AMICI gives the following magnifications for a distance from the eyepiece to the surface of the table = 33 cm:

TABLE I.

Eyep. / Object.	1 (Am 1)	2	3 (Am 3)	4[1] (Am 4)	5
U 1	38				
U 2	61				
U 3	120				
U 4	247	480	636	1039	
U 6	348	676	896	1464	4685
U 7	305	591	784	1306	4100
U 8	585	1137	1507	2463	7881

At the time when the Descriptive Catalogue, quoted above, was composed, all that was left of these various optical accessories, were only tube 0, objective 1, fitted with the LIEBERKUEHN mirror and the eyepieces 1, 3, 4 and 5. Lately, however, 5 objectives of this microscope viz. 4, 5, 6, 7 and 8, from which can be formed the systems *U* 7 and *U* 8 have been found again in the Zoological Laboratory of Utrecht, together with a large number of objectives belonging to the microscope C, described below.

Of the eyepieces, no. 1 is a very long one: it consists of a plano-convex field-lens and a ditto eyeglass; the plane surfaces of both are turned toward the eye. No. 3 is an Huygenian eyepiece, also fitted with two plano-convex lenses; no. 4 is composed of two plano-convex lenses, one immediately on top of the other, whereas no. 5 consists of a single spherical lens. AMICI

[1]) The Desc. Cat. gave for eyepiece Am 4 the magnifications 1039, 1264, 1406 and 2263, instead of 1039, 1464, 1306 and 2463. This is partly due to the fact that in AMICI's letter the figures 4 of 1464 and 2463 were written so illegibly that they looked like figures 2 and partly to the fact that AMICI gives the number 1406 instead of 1306.

remarks, with some truth, that this eyepiece is practically never used: the magnification is unduly large and the image formed is not sharp. As even the weakest eyepiece (no. 1) is relatively strong, we have added a weaker one belonging to a microscope of HARTNACK [5]) [6]), in order to check the optical power of the various optical combinations. HARTING did the same thing when examining microscope *A* by adding an eyepiece of DOLLOND. We should have liked to use that same eyepiece of DOLLOND, in order to make a direct comparison possible, but the DOLLOND-microscope, to which this eyepiece belonged, is lost to the Utrecht collection (see below). Table II gives the magnification and the resolving power [7]) of the various combinations, eyepiece no. 5 not being used:

TABLE II.

Eyep. / Object.	Magnification at 25 cm distance				Resolving power in mm
	Hck	Am 1	Am 3	Am 4	
U 3	45	100	250	400	1/ 400 [1])
U 7	115	255	640	1020	1/ 800
U 8	210	465	1165	1865	1/1000

[1]) with eyepiece Hck only 1/200 mm.

Microscopes A and B dating practically from the same year, their optical properties could be expected to be very much the same. Indeed, from a comparison of the data of HARTING, those of AMICI and of our measurements it appears that the objective systems no. 1, 2 and 4 of the lost HARTING microscope A and identical with the combinations *U* 5, *U* 6 and *U* 8 of microscope B, which is still in Utrecht; combination no. 3 of microscope A, however, does not fit in with any of the combinations *U* 1—*U* 8 of microscope B. The eyepieces no. 3, 4 and 5, mentioned by HARTING, are identical with the eyepieces 2, 3 and 4 of microscope B, while HARTING states that the eyepiece, called by him no. 6, is impracticable, just as is the case with no. 5 of microscope B.

HARTING gives on page 91 of part II of his book the resolving powers of some combinations of the objectives and the eyepieces. Since, for their determination he used as a criterium their capacity of resolving the meshes of a network, his measurements cannot be compared immediately with ours in which the GRAYSON ruling was used. HARTING's data lead to a resolving power of 1/400 mm for his objective no. 1 and 1/1400 mm for no. 4, whereas the corresponding combinations *U* 3 and *U* 8 give at present a resolving power of 1/400 and 1/1000 mm respectively.

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[5]) Desc. Cat., page 81, Q 3.

[6]) Denoted by us by Hck, as distinctive from the AMICI eyepieces, which we shall call Am 1 ... Am 5.

[7]) All measurements are done with the aid of a "GRAYSON ruling" (Desc. Cat. page 7), with pencils of rays of very slight divergence.

Concerning microscope C HARTING (loc. cit.) informs us that he regarded this microscope, though surpassed by several microscopes in mechanical respect, at the most perfect microscope in optical respect he ever examined. The mechanical design can be seen immediately in fig. 2, to be found on plate 6 of part III of HARTING's book. We mention here that besides a coarse adjustment with rack and pinion, a fine adjustment by means of the screw *l* is possible, that the mirror admits of a sideways motion in order to make oblique illumination possible and that the tube, in contradistinction to microscope A and B is not provided with a rectangular prism immediately above the objective, but is the usual vertical tube, and can be drawn out from 29 to 57 cm; for those observers, however, who prefer to observe horizontally a prism *A* can be mounted between the parts *n* and *f* of the tube. A small tube is mounted under the object table for the screwing in of one of the objective systems to be used as a condensor. According to HARTING no less than 21 achromatic double-lenses belonged to this microscope with which 13 different systems could be formed. Several of these combinations had about equal focal distances and consequently also equal magnifications, but were intended to be used with cover glasses of different thickness (from 0—1.5 mm). Four of these systems gave for instance, when used with the weakest eyepiece, the magnifications × 664, 672, 644 and 650, resp., but should be used with cover glasses of thickness 0, 0.2, 0.25 and 0.33 mm resp. In all, there were three eyepieces, all of a peculiar construction: they consisted of two small sliding tubes *x* and *y* (see fig. 2), each provided with a plano-convex lens, while a diaphragm *z* was mounted in the inner tube. When the tubes were completely pushed in, the system formed a RAMSDEN eyepiece, it could, however, be altered immediately into a Huygenian eyepiece by drawing out the tube. HARTING describes the optical power of this microscope in detail. He compares the resolving power with among others that of an OBERHÄUSER microscope [8]), and concludes that the AMICI microscope surpasses the latter.

When the "Descriptive Catalogue" was composed, microscope C was missing in the Utrecht collection, so that it was not described in the catalogue. What had become of it was totally unknown. When visiting the Science Museum in London one of us, however, was very astonished to find the missing microscope C exhibited there with the inscription: "purchased of the Zoological Laboratory of the Utrecht University through Dr. C. W. HUBRECHT in 1886, presented by Prof. P. HARTING to the University". On making inquiries we learned that a number of microscopes, belonging to the Utrecht University, were sold to the late Mr. CRISP, these microscopes being bought after Mr. CRISP's death by Mr. COURT. Some of these instruments are exhibited in the Science Museum, but are still owned by Mr. COURT. Mr. COURT very kindly gave us some further informations: Among these microscopes of the Zoological Laboratory in Utrecht in

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[8]) Desc. Cat, page 80, Q 2.

charge of Prof. HARTING, which were sold after his death, was a microscope, made by DOLLOND which actually belonged to the Physical Laboratory but happened to be under his care at the time of his death; it was

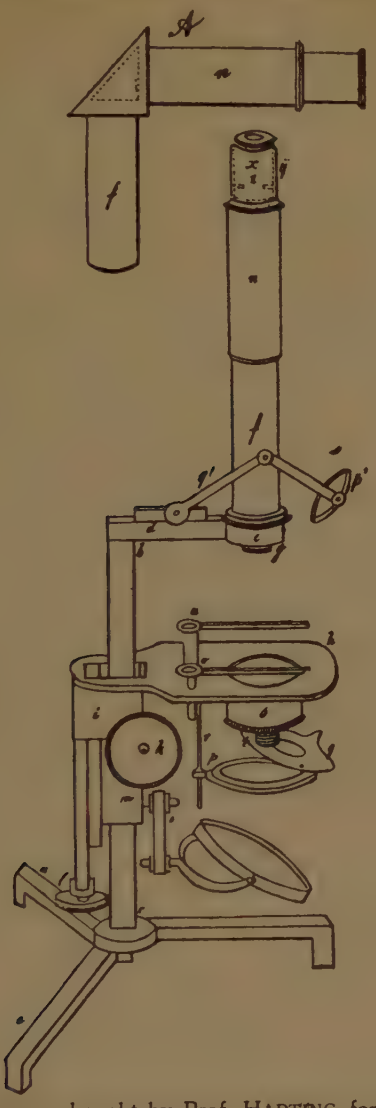

Fig. 2. AMICI microscope, bought by Prof. HARTING for his own use in 1849.

to this microscope that the eyepiece belonged, used by HARTING when examining microscope A. A second microscope bought was the above mentioned AMICI microscope C, which is provided with one system of 3 objectives, marked *A, B, C,* one adjustable eyepiece, a binocular fitting, a double body fitting, a quadriocular fitting and an experimental prism for use with the quadriocular body, which had been found by HARTING to be unsuitable [9]). A third microscope, made by KIPP, was bought, thrown in instead

[9]) These accessories are described in P. HARTING: "De nieuwste verbeteringen van het microscoop", 1858, page 74—112.

of a WOLLASTON microscope, made by HARTING himself (Desc. Cat. page 31, *E* 2), which was the original offer. Even the VAN LEEUWENHOEK microscope (Desc. Cat. page 14, *A* 1), the VAN MUSSCHENBROEK microscope (Desc. Cat. page 17, B 1) and the achromatic objective of BEELDSNIJDER (Desc. Cat. page 63, *L*) were offered for sale, but were fortunately saved for the Utrecht collection and for the Netherlands.

As practically all objectives of the above described AMICI microscope B were missing, it seemed likely that these were all sold to Mr. CRISP with the AMICI microscope C. It was, however, impossible to trace any of these, as Mr. CRISP used to combine all separate objectives and eyepieces to form a special collection, instead of leaving them with the instruments to which they originally belonged.

Recently, however, through the kind intermediation of Dr. BRETSCHNEIDER a small box containing a large number of achromatic lenses and eyepieces was found again in the Utrecht Zoological Laboratory. An accompanying letter from AMICI made it clear that the lenses, contained in the box, were all the missing lenses from the AMICI microscope C, except of course the system *A*—*B*—*C*, which is actually in London. Moreover, however, this box also contained the objectives 4, 5, 6, 7 and 8 from the microscope B and also an eyepiece, which was nearly identical to eyepiece no. 3 of microscope B and must therefore in all probability be identified as the eyepiece no. 4 from microscope A (see above).

According to AMICI's letter the following 20 objectives: *A, B, C, M, N, P, Q, R, Y, Z,* 0, 1, 2, 3, 5, ., . .., ..., ..., ..., belonged to the microscope C, further there were three cylindrical tubes, marked 3′, 4′ and *X*. All these objectives (of coarse except *A, B* and *C*) and the three tubes were found again. The lenses could be combined in the following ways:

	L 1[1]	L 2	L 3	L 4	L 5[2]	L 6	L 7
	A	N	N	N	M	4′	3′
	B	P	P	P	N	2	3
	C	Q		R	P	1	1
					R	0	0
thickness of cover glass in mm		0	0	1 =	1 1/2	1	0—1/4

	L 8	L 9	L 10	L 11	L 12	L 13
	5	5	X	X	Y	Z
	3′	4′	. . . .	. . . .	X	X
	3	2	. . .	. . .	. . .	. . .
	1	1	. .	. .	. .	. .
	0	0	.	.	.	.
thickness of cover glass in mm	1/4—1/2	1	0—1/7	1/5	1/4	1/3

[1] Can also be used without lens A.

[2] M is a correcting negative lens.

The adjustable eyepieces mentioned above are unfortunately missing. According to AMICI the variable distance between the two eyepiece lenses was necessary because the various objectives were not all achromatized in the same way and the adjustment allowed to a certain extent the correction of the differences left. A few testobjects, enclosed between two glass plates, one having a thickness of 1 mm and the other being very thin, belonged originally to the microscope.

As the original microscope and one of its eyepieces were in London and the other eyepieces were missing, we combined the objectives with the HARTNACK eyepiece and with the eyepieces *Am* 1, *Am* 3 and *Am* 4 of the AMICI microscope B in order to be able to measure the magnifications and the resolving powers of the different objective systems. We mounted the lenses in an arbitrary microscope tube, taking as the length of the tube that of the tube of microscope B. The magnifications and the resolving powers measured by us are given in table III. All measurements are done with

TABLE III.

Eyep. / Object.	Magnification for a distance of 25 cm				Resolving power in mm.
	Hck	Am 1	Am 3	Am 4	
L 2	90	200	500	800	1/ 800
L 3	60	130	335	535	1/ 400
L 4	100	220	555	890	1/1000 [1]
L 5	85	190	470	760	1/ 600
L 6	180	400	1000	1600	1/1000 [2]
L 7	180	400	1000	1600	1/1000
L 8	190	420	1055	1690	1/1000
L 9	200	445	1110	1780	1/ 800
L 10	280	620	1555	2490	1/1200
L 11	290	645	1610	2580	1/1200
L 12	270	600	1500	2400	1/1000
L 13	280	620	1555	2485	1/1000

[1] With Hck and Am 1 only 1/800 mm; with an extra cover glass of 1 mm thickness for all magnifications only 1/800 mm.

[2] With Hck only 1/800 mm; with an extra cover glass of 1 mm thickness, however, also 1/1000 mm.

a thickness of the cover glass equal to 0.17 mm, viz. the cover glass of the GRAYSON rulings (refractive index 2,549), but sometimes an extra cover glass was added in order to approximate the thickness mentioned by AMICI.

HARTING states that he was able to resolve with system *L* 2 the 7th group of a NOBERT plate, which corresponds to 1/1100 mm, with *L* 4 and *L* 6 the 8th group (1/1300 mm) and with *L* 11 the 9th group (1/1560 mm); these measurements, however, cannot be compared directly with our measurements, as HARTING used widely diverging beams instead of beams of very

slight divergence as were those, used by us. Therefore our measurements gave results for the resolving power, which were about 1,3 times as low as HARTING's results [10]), and taking this into account the agreements must be considered as striking, especially as about a century lies between these two sets of measurements.

In fig. 3 our results are plotted in the graphs published in the Desc. Cat. page 8 [11]). In these graphs the horizontal line gives the theoretical limit

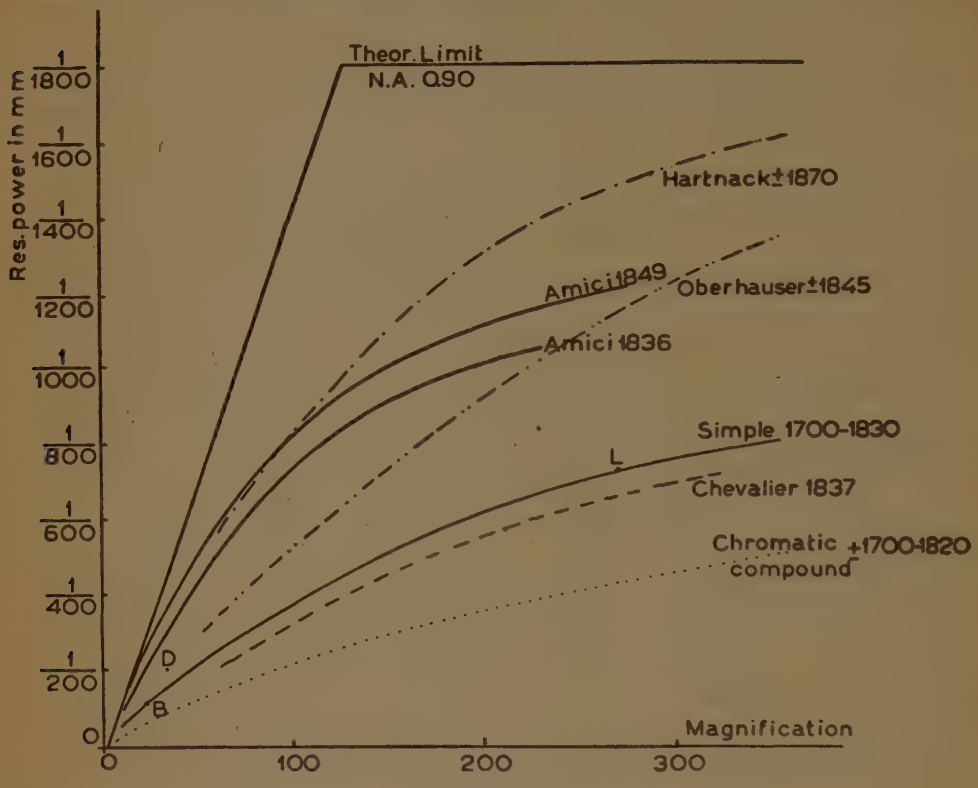

Fig. 3.

for $N.A. = 0{,}9$ and $\lambda = 5000$ Å. The construction of the slanting line is based on the experimental fact that the angular distance of two points must be at least $1'$ in order to be seen separately. How in the course of time these limits have been more and more approximated is clearly to be seen. The single microscopes and also the chromatic compound microscopes have been considered as a group. The points L for the VAN LEEUWENHOEK microscope and D for the VAN DEYL microscopes have been indicated separately (see Desc. Cat. l.c.). This figure shows very clearly how till about 1830 the resolving powers of the compound microscopes fall short by a long way of those of the single microscopes. This was no longer the

[10]) P. H. VAN CITTERT, Proc. Kon. Akad. v. Wetensch., Amsterdam, **39**, 182 (1936).
[11]) See also Ned. Tijds. v. Nat. **2**, 51 (1935).

case when achromatic objectives were made successfully. Yet it is a fact that the VAN LEEUWENHOEK microscope, dated about 1700, is superior to an achromatic microscope made by CHEVALIER in 1837. The lines AMICI 1836 (microscope B) and AMICI 1849 (microscope C) indicate the very large progress, due to AMICI. Whereas HARTING remarks that the AMICI microscope 1849 (C) surpasses the OBERHÄUSER microscope (1845), our measurements show that this is, at least for the lesser magnifications, even the case with the AMICI microscope 1836.

To be sure the work of AMICI has indeed improved the optical power of the microscope by leaps and bounds!

Hormonology. — *Testosterone and luteinization.* By F. J. A. PAESI. (From the Pharmacological Laboratory, University of Leiden, Holland.) (Communicated by Prof. J. VAN DER HOEVE.)

(Communicated at the meeting of April 26, 1947.)

The relations between male hormone and the ripening of the ovary.

In our laboratory (PAESI e.a. 1943, GAARENSTROOM e.a. 1944) and elsewhere too, it was found that testosterone may exercise a favourable influence on the development of the follicle cavity in rats. In immature mice it even led to ovulation (DE JONGH e.a., 1944); in the rat so-called "blood points" have repeatedly been observed by ourselves as well as by others (BAUER 1943, SELYE 1939). We could show, moreover, that treatment with chorionic gonadotrophin leads in the ovary of hypophysectomized rats to the production of an androgenic substance, and that its interstitium, but no other part, becomes hypertrophic (PAESI e.a. 1943). Such a substance was supposed to be present in normal animals too, and to influence the ripening of the follicle by a direct action on the ovary and not by acting through the intermediary of the hypophysis (GAARENSTROOM and DE JONGH 1946).

Somewhat older experiences with regard to a favourable effect of male hormone on the *luteinization* process (HOHLWEG 1937, rat; SALMON 1938a and *b*, rat; NATHANSON e.a. 1938, rat; SELYE 1939, mouse) appear now in a new light: GAARENSTROOM and DE JONGH are of opinion that the favourable influence exercised on the luteinization depends on the more advanced degree of follicle ripening, and that this influence too is therefore referable to a direct influence of the androgenic substance on the ovary, and not to one exercised by way of the hypophysis.

Are the corpora lutea influenced by the androgenic substance?

In connection with the considerations set forth in the preceding paragraph, it interested us to know whether already existing corpora lutea are influenced by male hormone. That oestrogens act in this way, has repeatedly been described.

LAQUEUR and KOETS obtained in 1945 by means of testosterone in *normal* adult rats corpora lutea that showed great similarity to corpora lutea graviditatis. It would be interesting to know whether this effect was obtained with the co-operation of the hypophysis, as in the case of the analogous activity of the oestrogens is highly probable, or without its intermediary, i.e. by a direct influence exercised on the corpus luteum.

In this respect experiments carried out at an earlier date by GAARENSTROOM (1941) are of importance. He injected a number of young rats

with a full gonadotrophic mixture [1]), and found that the ovaries became almost entirely filled with corpora lutea. After the hypophysis and the right ovary had been removed, the animals were during a period of ten days subjected to a treatment with ambinon in combination with various other substances. At the end of the experiment the weight and histological structure of the remaining ovary were compared with the weight and structure of the right one. The decrease in weight of the ovary, that at the end of the experiment still consisted almost entirely of corpora lutea, appeared to be far more effectively retarded by the combination of ambinon and testosterone than by ambinon alone. This result was ascribed to the formation of a larger number of new corpora lutea, for a measurement of the three *largest* corpora lutea showed that the latter had been no better protected against atrophy by ambinon plus testosterone than by ambinon alone. The possibility that testosterone might exercise a preservative effect on the corpora lutea, or even cause an improvement of their condition, was on this account, for the moment, set aside.

Our own experiments.

It seemed desirable to us to place GAARENSTROOM's conclusion on a more solid base. Although we too are of opinion that the largest corpora lutea should be the first to show a measurable difference in the degree of atrophy when the animals have been treated in different ways, we think that the possibility of a selective influence on the *smaller* ones should also be taken into account. Moreover, GAARENSTROOM confined himself to an examination of only *one section out of each group of ten*, which particularly with regard to the smaller corpora lutea may lead to erroneous conclusions. Finally, it seemed advisable to us to *count* the corpora lutea. Our experimental method resembled in the main that of GAARENSTROOM:

A number of adult female rats, weighing from 137 g. to 179 g., were treated during a week with ambinon (twice daily 2.5 R.U.) plus pregnyl (twice daily 2.5 I.U.). In this way the ovaries were intensively luteinized in all animals. On the eighth day of the experiment the hypophysis and the right ovary were removed, and preparations of them were made in the usual way. The animals were then divided in three groups: *a*, *b* and *c*, that were subjected from the 9th to the 18th day to the following treatments:

a) ambinon (twice daily 2.5 R.U.) plus testosterone propionate (twice daily 0.5 mg.),
b) ambinon (twice daily 2.5 R.U.) plus oil,
c) 0,9 per cent NaCl plus oil.

On the 19th day the animals were killed. Preparations were made of the left ovary and the sella turcica. In the latter no rests of the hypophysis were found. The corpora lutea were counted, and in each ovary the four largest and the four smallest ones were selected for measurement: of each of them the two largest, vertically intersecting diameters were measured by the aid of an eyepiece-micrometer. By taking the third power of their mean a measure for the *content* of the corpora lutea was obtained. The four figures obtained for the smaller ones and those obtained for the larger ones were averaged.

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1) Ambinon (hypophysis extract, Organon) plus pregnyl (chorionic gonadotrophin, Organon).

The experiment was repeated with a second group of rats, weighing from 137 to 166 g.

In a third series of experiments the first treatment was ended one day before the hypophysectomy, and the second started two days after the latter. The removal of the right ovary too took place two days after the hypophysectomy. We deviated here from our former course because we wished to find out whether certain phenomena that had been observed in the preceding series should be ascribed to the influence of some gonadotrophic hormone that remained in the circulation. These rests might belong to the amount produced by the animal's own hypophysis or to injected pregnyl and ambinon.

Results.

The results are given in the table. Data pertaining to nine rats that died during the course of the experiments are omitted.

The table confirms, to begin with, two observations made by GAARENSTROOM:

1. In all three series the ovary weight (column C) showed the smallest decrease under the combined influence of ambinon and testosterone, and the largest decrease in the controls.

2. Neither of the treatments influenced in any series the mean size of the largest corpora lutea (column G) in a favourable way.

A number of new facts are also revealed. These will now be discussed.

TABLE.

In the columns C—G the values obtained for the left ovaries are expressed as a percentage of the corresponding values obtained for the right ones. The values themselves are averages; the number of animals from which the latter were taken, are given in column B.

	A	B	C	D	E	F	G
Series	Treatment after hypophysectomy	Number of animals	Ovary weight	Number of corp. lutea	Mean weight of corp. lut. calcul. from. C : D	Mean volume of the 4 smallest corp. lutea	Mean volume of the 4 largest corpora lutea
I	ambinon + testosterone	3	84	122	71	85	62
	ambinon + oil	2	67	115	62	55	67
	0.9 per cent NaCl + oil	3	50	113	52	27	64
II	ambinon + testosterone	2	85	127	67	20	68
	ambinon + oil	2	68	98	68	37	73
	0.9 per cent NaCl + oil	3	62	105	59	21	97
III*	ambinon + testosterone	5	119	137	81	124	83
	ambinon + oil	5	90	106	82	103	70
	0.9 per cent NaCl + oil	5	76	91	83	133	86

* The first and the second treatment separated from each other by an interval of a few days.

A. The counts of the corpora lutea (column *D*) have proved without any doubt that the increase under the combined influence of ambinon and testosterone is larger than under the influence of ambinon alone. Testosterone therefore must have exercised a favourable effect on the increase Whether ambinon alone gives an increase, remains uncertain.

In the third series, in which the right ovary was removed three days after the end of the first treatment, no new corpora lutea were found in the controls. In the second series their presence was dubious. in the first one conspicuous. In the latter the appearance of new corpora lutea wil have to be ascribed to the presence of gonadotrophic substances produced in the previous period by the animal's own hypophysis or else introduced in its circulation during that period by means of the injections.

B. By dividing the relative ovary weights (column *C*) by the number of corpora lutea (column *D*) values are obtained that may serve as a measure for the average weight of the corpora lutea (column *E*). When these values are found to be equal in groups that have been treated differently, the difference in the ovary weight must be due to an increase in the number of corpora lutea, and then any influence that might be interpreted as a retardation of the atrophy ("preservation") of the existing corpora lutea, seems excluded. This doubtless applies to the *third series*. In view of the fact that the ovaries, before as well as after the second treatment, were almost entirely luteinized, we are of opinion that neither ambinon alone nor ambinon plus testosterone are able to preserve the existing corpora lutea when, as in the third series the hormonal influences from the preceding period must have completely disappeared. It is for the rest hardly conceivable that they would be able to achieve this when a small quantity of hypophysis hormone or some pregnyl (first and second series of experiments) would have remained in the circulation; signs that seem to point in this direction, should be regarded with a good deal of distrust. However, before considering them more closely, we want to say a few words with regard to the corpora lutea belonging to the smallest size class:

C. In the third series of experiments the smallest corpora lutea did not atrophy; it looks on the contrary more as if they increased in size. This is in contradiction to the behaviour of the largest ones, and also in sharp contrast to that of the smallest corpora lutea in the two other series. This cannot be accidental, and must have its cause in the three days that elapsed between the last injection with the gonadotrophic mixture and the removal of the right ovary; it is to be remembered that in two of these three days the hypophysis had also been absent. In these few days the atrophy must have taken place, for in the period of the second treatment there was no further sign of it. The exact nature of the second treatment is here of no importance. The differences in the degree of atrophy of the smallest corpora lutea observed in the first two series, in connection with the special nature of the second treatment, which will presently be discussed, must therefore

already have been established in the very first stage of the latter. This points to a particularly high vulnerability of the very small corpora lutea.

This is to us a motive for a partial revision of the views that we have elsewhere (PAESI, 1946) brought forward. At that time we had observed that the corpora lutea obtained in young mice by a treatment with ambinon plus pregnyl were *larger and showed less tendency to degenerate* than those obtained by a treatment with ambinon alone. As under the influence of ambinon plus pregnyl more oestrogen is formed than with ambinon alone, we supposed at that time that this oestrogen would have exercised, in the generally accepted way, a positively trophic influence on the corpora lutea. This may be right, but the results of the experiments described above show that another possibility should also be considered: if the *smallest* corpora lutea atrophy more rapidly than the larger ones, they will probably degenerate also *in a larger number.*

The particularly high vulnerability of the smaller corpora lutea points to a flaw in the luteinisation process of the latter. As the histological examination reveals that the theca of the smallest corpora lutea does not take part in the luteinization or is at least not over its whole extent concerned in this process, the vascularisation of these corpora lutea falls behind that of the larger ones, and in this difference, we believe, the vulnerability of the smaller ones may have its ground. The inadequacy of the blood supply may be responsible for the tendency to atrophy. It can, of course, not be regarded as accidental that corpora lutea in which these anomalies occur, remain small.

In the experiment with young mice to which reference was made above, the increase in the power of resistance of the corpora lutea under the influence of pregnyl would not have been due entirely to oestrogen production, but also to a more complete participation of the theca in the luteinization.

D. After this consideration of the behaviour of the smallest corpora lutea, we will now return to our main object, and try to find out what the *first two* series of experiments may teach with regard to an influence that might have been exercised on the corpora lutea themselves. We will begin with the influence exercised by ambinon alone.

The table shows that the differences in the ovary weight cannot be entirely due to differences in the number of corpora lutea. The possibility that the corpora lutea are better preserved, should therefore be taken into consideration. On account of the data derived from the behaviour of the smallest corpora lutea, this possibility cannot be excluded, for under the influence of ambinon the degree of atrophy was in both series in the smallest corpora lutea smaller than in the corresponding controls. However, with regard to the considerable differences between the values in column *F* that refer to groups of animals treated in the same way, the interpretation must be careful.

A second possibility is that the *new* corpora lutea in the ambinon group and in the controls are of different size, for such a difference would also cause a difference in the average weight of the corpora lutea. However, to exercise such an influence, the *number* of new corpora lutea should not

be too *small*. As a complete disappearance of corpora lutea in the course of the second treatment need not be taken into account, we may estimate the production of new ones by diminishing the figures given in column *D* by 100. The number of new corpora lutea now appears to be but *small*, and it is therefore improbable that differences in size between these and the already existing corpora lutea might be responsible for the large difference in the average weight of *all* corpora lutea given in column *E*.

As a third possibility the follicle ripening should be considered. It is true that the ovaries at the beginning as well as at the end of the second treatment give the impression of being entirely luteinized, but a certain degree of follicle development not ending in luteinization may nevertheless have occurred, and although this may have been insignificant with regard to the total size of the ovary, it may have been of some importance in comparison with the new corpora lutea. When the average weight of the corpora lutea is calculated in the way described above, it will undergo a fictitious increase when follicles, perhaps large in number, although small in size, begin to ripen. This may have happened in our experiments, for that ambinon may be able to bring about some ripening of new follicles, cannot be denied, *and the number of corpora lutea did not increase under the influence of ambinon.*

E. We will now examine the possibility of an influence exercised on the corpora lutea themselves by comparing the influence of the combination ambinon-*testosterone* with that of ambinon alone. The answer is in this case less dubious than in the previous one.

The variability in the behaviour of the smallest corpora lutea is so large, that the atrophy of the smallest corpora lutea cannot be used as a base for discussion. We will have to build our conclusions therefore on the figures given in the columns *C*, *D* and *E*.

A comparison of the figures of column *E* (the average weight of the corpora lutea obtained by dividing the figures of column *C* by those of column *D*) suggests that the corpora lutea of the animals that had received both ambinon and testosterone are in the first series larger thans those of the animals treated with ambinon alone, but that such a difference is not found in the second series. In view of the entirely identical treatment that the animals in these two series had received, this discrepancy is difficult to explain. It remains entirely incomprehensible when we assume that the corpora lutea are in the first series better preserved by the addition of testosterone, and it is not clear either when we ascribe it to a selective formation of large corpora lutea, against which moreover the same objection may be raised as has been done under *D*.

It is, however, not improbable that on account of minor differences in the treatment that escaped control, a number of newly ripened follicles may in the second series have succeeded in becoming luteinized, whereas a similar number in the first series just failed to reach this stage. Support for this supposition is found in column *D*: the number of new corpora lutea

formed under the influence of testosterone is in the first series small and in the second comparatively large. When it is right, the differences between the values of the first series in column *E* must rest on differences in follicle ripening, and then the real average weight of the corpora lutea may, as the values of the second series suggest, have been identical.

Testosterone therefore would not have helped to preserve the corpora lutea, but its action would have been confined to a stimulation of the production of new ones. An improvement in the condition of the corpora lutea by means of testosterone (LAQUEUR and KOETS) may be obtained in animals whose hypophysis has not been removed, but in this case it will have to be effected through the intermediary of the hypophysis.

Summary.

1. The weight of previously luteinized ovaries of hypophysectomized rats is better preserved by a treatment with pituitary gonadotrophin than without the latter, and still better when in addition testosterone is administered.

2. Pituitary gonadotrophin alone does not give a well-marked increase in the number of corpora lutea, but in co-operation with testosterone the increase is undeniable.

3. The corpora lutea may be better preserved when pituitary gonadotrophin is given; moreover the latter probably causes a modest follicle development.

4. Testosterone apparently does not preserve the corpora lutea. The retardation of the atrophy of the ovary may be due entirely to the formation either of new follicles or of new corpora lutea.

5. Arguments were brought forward, pointing to an especially short duration of life of the smallest corpora lutea.

REFERENCES.

BAUER, J., Ztschr. f. d. ges. exp. Med. **112**, 155 (1943).

GAARENSTROOM, J. H., Acta brev. neerl. **11**, 147 (1941).

GAARENSTROOM, J. H., S. E. DE JONGH and F. J. A. PAESI, Versl. Ned. Akad. v. Wetensch., Afd. Natuurk. **53**, 71 (1944).

GAARENSTROOM, J. H. and S. E. DE JONGH. On the Effect of Gonadotropic and Sex Hormones on the Gonads of Rats, Elsevier Publ. Comp. 1946.

HOHLWEG, W., Klin. Wschr. **16**, 586 (1937).

JONGH, S. E. DE, J. H. GAARENSTROOM and F. J. A. PAESI, Versl. Ned. Akad. v. Wetensch., afd. Natuurk. **53**, 100 (1944).

LAQUEUR, G. L. and P. KOETS, Proc. Soc. Exp. Biol. and Med. **60**, 239, 1945.

NATHANSON, IRA, CL. C. FRANSEEN and A. R. SWEENEY, Proc. Soc. Exp. Biol. and Med. **39**, 385, 1938.

PAESI, F. J. A. and J. H. GAARENSTROOM, Versl. Ned. Akad. v. Wetensch., afd. Natuurk. **52**, 592 (1943).

PAESI, F. J. A., Een Analyse van eenige Werkingen van Gonadotrope Hormonen op het Ovarium, Thesis Leiden 1946.

SALMON, U. J., Proc. Soc. Exp. Biol. and Med. **38**, 352, 1938 *a*.

————, Endocrinology **23**, 779, 1938 *b*.

SELYE, H., J. Endocrinology **1**, 208, 1939.

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